Research Article

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On certain functional equation in prime rings

https://doi.org/10.1515/math-2022-0002
received October 7, 2021; accepted January 10, 2022

Abstract: The purpose of this paper is to prove the following result. Let R be prime ring of characteristic different from two and three, and let $F : R \to R$ be an additive mapping satisfying the relation $F(x^3) = F(x^2)x - xF(x)x + xF(x^2)$ for all $x \in R$. In this case, F is of the form $4F(x) = D(x) + qx + xq$ for all $x \in R$, where $D : R \to R$ is a derivation, and q is some fixed element from the symmetric Martindale ring of quotients of R.

Keywords: prime ring, derivation, Jordan derivation, two-sided centralizer, functional equation

MSC 2020: 16R60, 16W25, 39B05

Introduction

Throughout, R will represent an associative ring with center $Z(R)$. Given an integer $n > 1$, a ring R is said to be n-torsion free, if for $x \in R$, $nx = 0$ implies $x = 0$. The commutator $xy - yx$ will be denoted by $[x, y]$. A ring R is prime if for $a, b \in R$, $aRb = (0)$ implies that either $a = 0$ or $b = 0$ and is semiprime in case $aRa = (0)$ implies $a = 0$. We denote by Q_{mr}, Q_s, and C the maximal Martindale right ring of quotients, symmetric Martindale ring of quotients, and extended centroid of a semiprime ring R, respectively (see [1], Chapter 2). An additive mapping $D : R \to R$ is called a derivation if $D(xy) = D(x)y + xD(y)$ holds for all pairs $x, y \in R$ and is called a Jordan derivation in case $D(x^2) = D(x)x + xD(x)$ is fulfilled for all $x \in R$. A derivation $D : R \to R$ is inner in case D is of the form $D(x) = [a, x]$ for all $x \in R$ and some fixed $a \in R$. Every derivation is Jordan derivation. The converse is in general not true. A classical result of Herstein [2] asserts that any Jordan derivation on a prime ring with characteristic different from two is a derivation. A brief proof of Herstein theorem can be found in [3]. In [4], one can find a generalization of Herstein theorem. Cusack [5] generalized Herstein theorem to 2-torsion free semiprime rings (see [6] for an alternative proof). Herstein theorem has been fairly generalized by Beidar et al. [7]. For results related to Herstein theorem, we refer to [8–11]. We proceed with the following result proved by Brešar [12] (see [13] for a generalization).

Theorem 1. *Let R be a 2-torsion free semiprime ring and let $D : R \to R$ be an additive mapping satisfying the relation.*

$$D(xyx) = D(x)yx + xD(y)x + xyD(x) \tag{1}$$

for all pairs $x, y \in R$. In this case, D is a derivation.

An additive mapping satisfying the relation (1) on an arbitrary ring is called a Jordan triple derivation. It is easy to prove that any Jordan derivation on a 2-torsion free ring is a Jordan triple derivation, which means that Theorem 1 generalizes Cusack's generalization of Herstein theorem.

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Motivated by Theorem 1, Vukman et al. [14] have proved the following result (see [15] for a generalization).

Theorem 2. *Let R be a 2-torsion free semiprime ring and let $F : R \to R$ be an additive mapping satisfying the relation*

$$T(xyx) = T(x)yx - xT(y)x + xyT(x) \tag{2}$$

for all pairs $x, y \in R$. In this case, F is of the form $2T(x) = qx + xq$, where $q \in Q_s(R)$ is some fixed element.

We proceed with the following functional equation:

$$F(xyx) = F(xy)x - xF(y)x + xF(yx), \tag{3}$$

which appears naturally in the proof of Theorem 2 in [16]. One can easily prove that in case we have an additive mapping $F : R \to R$, where R is 2-torsion free semiprime ring, satisfying the relation (3) for all pairs $x, y \in R$, then F is of the form $2F(x) = D(x) + ax + xa$, where $D : R \to R$ is a derivation and $a \in R$ some fixed element (see [16] for the details). In [16], one can find the following conjecture.

Conjecture 3. *Let R be a 2-torsion free semiprime ring and let $F : R \to R$ be an additive mapping satisfying the relation (3) for all pairs $x, y \in R$. In this case, F is of the form $2F(x) = D(x) + qx + xq$ for all $x \in R$, where $D : R \to R$ is a derivation and $q \in Q_s(R)$ some fixed element.*

By our knowledge, the aforementioned conjecture is still an open question. The substitution $y = x$ in (1), (2), and (3) gives

$$D(x^3) = D(x)x^2 + xD(x)x + x^2D(x), \tag{4}$$

$$F(x^3) = F(x)x^2 - xF(x)x + x^2F(x) \tag{5}$$

and

$$F(x^3) = F(x^2)x - xF(x)x + xF(x^2). \tag{6}$$

The relation (4) has been considered in [7] (actually, much more general situation has been considered). A result related to (5) can be found in [15]. It is our aim in this paper to prove the following result, which is related to the aforementioned conjecture.

Theorem 4. *Let R be a prime ring of characteristic different from two and three, and let $F : R \to R$ be an additive mapping satisfying the relation*

$$F(x^3) = F(x^2)x - xF(x)x + xF(x^2) \tag{7}$$

for all $x \in R$. In this case, F is of the form $4F(x) = D(x) + qx + xq$, where $D : R \to R$ is a derivation, and $q \in Q_s(R)$ is some fixed element.

Main results

As the main tool in this paper, we use the theory of functional identities (Brešar-Beidar-Chebotar theory). The theory of functional identities considers set-theoretic maps on rings that satisfy some identical relations. When treating such relations, one usually concludes that the form of the mappings involved can be described, unless the ring is very special. We refer the reader to [17] for the introductory account on the theory of functional identities, where Brešar presents this theory and its applications to a wider audience and to [18] for the full treatment of this theory.

Let R be an algebra over a commutative ring ϕ and let

$$p(x_1, x_2, x_3) = \sum_{\pi\in\mathbb{S}_3} x_{\pi(1)}x_{\pi(2)}x_{\pi(3)} \tag{8}$$

be a fixed multilinear polynomial in noncommuting indeterminates x_i over ϕ. Here, $\mathbb{S}_3$ stands for the symmetric group of order 3. Let $\mathcal{L}$ be a subset of R closed under p, i.e., $p(\bar{x}_3) \in \mathcal{L}$ for all $x_1, x_2, x_3 \in \mathcal{L}$, where $\bar{x}_3 = (x_1, x_2, x_3)$. We shall consider a mapping $D : \mathcal{L} \to R$ satisfying

$$F(p(\bar{x}_3)) = \sum_{\pi\in\mathbb{S}_3} (F(x_{\pi(1)}x_{\pi(2)})x_{\pi(3)} - x_{\pi(1)}F(x_{\pi(2)})x_{\pi(3)} + x_{\pi(1)}F(x_{\pi(2)}x_{\pi(3)})) \tag{9}$$

for all $x_1, x_2, x_3 \in \mathcal{L}$. Let us mention that the idea of considering the expression $[p(\bar{x}_3), p(\bar{y}_3)]$ in its proof is taken from [19]. For the proof of Theorem 4, we need Theorem 5, which might be of independent interest.

Theorem 5. *Let $\mathcal{L}$ be a 6-free Lie subring of R closed under p. If $T : \mathcal{L} \to R$ is an additive mapping satisfying* (9), *then there exists $q \in R$ such that $4F(x) = D(x) + xq + qx$ for all $x \in \mathcal{L}$.*

Proof. For any $a \in R$ and $\bar{x}_3 \in \mathcal{L}^3$, we have

$$[p(\bar{x}_3), a] = p([x_1, a], x_2, x_3) + p(x_1, [x_2, a], x_3) + p(x_1, x_2, [x_3, a]).$$

Thus,

$$F[p(\bar{x}_3), a] = F(p([x_1, a], x_2, x_3)) + F(p(x_1, [x_2, a], x_3)) + F(p(x_1, x_2, [x_3, a])). \tag{10}$$

By using (10), it follows that

$$\begin{aligned}
F[p(\bar{x}_3), a] = {} & \sum_{\pi\in\mathbb{S}_3} F([x_{\pi(1)}, a]x_{\pi(2)})x_{\pi(3)} - \sum_{\pi\in\mathbb{S}_3} [x_{\pi(1)}, a]F(x_{\pi(2)})x_{\pi(3)} + \sum_{\pi\in\mathbb{S}_3} [x_{\pi(1)}, a]F(x_{\pi(2)}x_{\pi(3)}) \\
& + \sum_{\pi\in\mathbb{S}_3} F(x_{\pi(1)}[x_{\pi(2)}, a])x_{\pi(3)} - \sum_{\pi\in\mathbb{S}_3} x_{\pi(1)}F([x_{\pi(2)}, a])x_{\pi(3)} + \sum_{\pi\in\mathbb{S}_3} x_{\pi(1)}F([x_{\pi(2)}, a]x_{\pi(3)}) \\
& + \sum_{\pi\in\mathbb{S}_3} F(x_{\pi(1)}x_{\pi(2)})[x_{\pi(3)}, a] - \sum_{\pi\in\mathbb{S}_3} x_{\pi(1)}F(x_{\pi(2)})[x_{\pi(3)}, a] + \sum_{\pi\in\mathbb{S}_3} x_{\pi(1)}F(x_{\pi(2)}[x_{\pi(3)}, a]) \\
= {} & \sum_{\pi\in\mathbb{S}_3} F([x_{\pi(1)}x_{\pi(2)}, a])x_{\pi(3)} - \sum_{\pi\in\mathbb{S}_3} [x_{\pi(1)}, a]F(x_{\pi(2)})x_{\pi(3)} + \sum_{\pi\in\mathbb{S}_3} [x_{\pi(1)}, a]F(x_{\pi(2)}x_{\pi(3)}) \\
& - \sum_{\pi\in\mathbb{S}_3} x_{\pi(1)}F([x_{\pi(2)}, a])x_{\pi(3)} + \sum_{\pi\in\mathbb{S}_3} x_{\pi(1)}F([x_{\pi(2)}x_{\pi(3)}, a]) + \sum_{\pi\in\mathbb{S}_3} F(x_{\pi(1)}x_{\pi(2)})[x_{\pi(3)}, a] \\
& - \sum_{\pi\in\mathbb{S}_3} x_{\pi(1)}F(x_{\pi(2)})[x_{\pi(3)}, a].
\end{aligned}$$

In particular,

$$\begin{aligned}
F[p(\bar{x}_3), p(\bar{y}_3)] = {} & \sum_{\pi\in\mathbb{S}_3} F([x_{\pi(1)}x_{\pi(2)}, p(\bar{y}_3)])x_{\pi(3)} - \sum_{\pi\in\mathbb{S}_3} [x_{\pi(1)}, p(\bar{y}_3)]F(x_{\pi(2)})x_{\pi(3)} \\
& + \sum_{\pi\in\mathbb{S}_3} [x_{\pi(1)}, p(\bar{y}_3)]F(x_{\pi(2)}x_{\pi(3)}) - \sum_{\pi\in\mathbb{S}_3} x_{\pi(1)}F([x_{\pi(2)}, p(\bar{y}_3)])x_{\pi(3)} \\
& + \sum_{\pi\in\mathbb{S}_3} x_{\pi(1)}F([x_{\pi(2)}x_{\pi(3)}, p(\bar{y}_3)]) + \sum_{\pi\in\mathbb{S}_3} F(x_{\pi(1)}x_{\pi(2)})[x_{\pi(3)}, p(\bar{y}_3)] \\
& - \sum_{\pi\in\mathbb{S}_3} x_{\pi(1)}F(x_{\pi(2)})[x_{\pi(3)}, p(\bar{y}_3)]
\end{aligned} \tag{11}$$

for all $\bar{x}_3, \bar{y}_3 \in \mathcal{L}^3$. For $i = 1, 2$, we have

$$\begin{aligned}
F[x_{\pi(i)}x_{\pi(i+1)}, p(\bar{y}_3)] = {} & -F[p(\bar{y}_3), x_{\pi(i)}x_{\pi(i+1)}] \\
= {} & \sum_{\sigma\in\mathbb{S}_3} F([x_{\pi(i)}x_{\pi(i+1)}, y_{\sigma(1)}y_{\sigma(2)}])y_{\sigma(3)} - \sum_{\sigma\in\mathbb{S}_3} [x_{\pi(i)}x_{\pi(i+1)}, y_{\sigma(1)}]F(y_{\sigma(2)})y_{\sigma(3)} \\
& + \sum_{\sigma\in\mathbb{S}_3} [x_{\pi(i)}x_{\pi(i+1)}, y_{\sigma(1)}]F(y_{\sigma(2)}y_{\sigma(3)}) - \sum_{\sigma\in\mathbb{S}_3} y_{\sigma(1)}F([x_{\pi(i)}x_{\pi(i+1)}, y_{\sigma(2)}])y_{\sigma(3)}
\end{aligned} \tag{12}$$

$$+\sum_{\sigma\in\mathbb{S}_3} y_{\sigma(1)}F([x_{\pi(i)}x_{\pi(i+1)}, y_{\sigma(2)}y_{\sigma(3)}]) + \sum_{\sigma\in\mathbb{S}_3} F(y_{\sigma(1)}y_{\sigma(2)})[x_{\pi(i)}x_{\pi(i+1)}, y_{\sigma(3)}]$$
$$-\sum_{\sigma\in\mathbb{S}_3} y_{\sigma(1)}F(y_{\sigma(2)})[x_{\pi(i)}x_{\pi(i+1)}, y_{\sigma(3)}] \tag{12}$$

and

$$\begin{aligned}
F[x_{\pi(2)}, p(\bar{y}_3)] &= -F[p(\bar{y}_3), x_{\pi(2)}]\\
&= \sum_{\sigma\in\mathbb{S}_3} F([x_{\pi(2)}, y_{\sigma(1)}y_{\sigma(2)}])y_{\sigma(3)} - \sum_{\sigma\in\mathbb{S}_3}[x_{\pi(2)}, y_{\sigma(1)}]F(y_{\sigma(2)})y_{\sigma(3)}\\
&\quad + \sum_{\sigma\in\mathbb{S}_3}[x_{\pi(2)}, y_{\sigma(1)}]F(y_{\sigma(2)}y_{\sigma(3)}) - \sum_{\sigma\in\mathbb{S}_3} y_{\sigma(1)}F([x_{\pi(2)}, y_{\sigma(2)}])y_{\sigma(3)}\\
&\quad + \sum_{\sigma\in\mathbb{S}_3} y_{\sigma(1)}F([x_{\pi(2)}, y_{\sigma(2)}y_{\sigma(3)}]) + \sum_{\sigma\in\mathbb{S}_3} F(y_{\sigma(1)}y_{\sigma(2)})[x_{\pi(2)}, y_{\sigma(3)}]\\
&\quad - \sum_{\sigma\in\mathbb{S}_3} y_{\sigma(1)}F(y_{\sigma(2)})[x_{\pi(2)}, y_{\sigma(3)}]
\end{aligned}$$

for all $\bar{y}_3 \in \mathcal{L}^3$. Therefore, (11) can be written as follows:

$$\begin{aligned}
F[p(\bar{x}_3), p(\bar{y}_3)] &= \sum_{\pi\in\mathbb{S}_3}\sum_{\sigma\in\mathbb{S}_3} F([x_{\pi(1)}x_{\pi(2)}, y_{\sigma(1)}y_{\sigma(2)}])y_{\sigma(3)}x_{\pi(3)} - \sum_{\pi\in\mathbb{S}_3}\sum_{\sigma\in\mathbb{S}_3}[x_{\pi(1)}x_{\pi(2)}, y_{\sigma(1)}]F(y_{\sigma(2)})y_{\sigma(3)}x_{\pi(3)}\\
&\quad + \sum_{\pi\in\mathbb{S}_3}\sum_{\sigma\in\mathbb{S}_3}[x_{\pi(1)}x_{\pi(2)}, y_{\sigma(1)}]F(y_{\sigma(2)}y_{\sigma(3)})x_{\pi(3)} - \sum_{\pi\in\mathbb{S}_3}\sum_{\sigma\in\mathbb{S}_3} y_{\sigma(1)}F([x_{\pi(1)}x_{\pi(2)}, y_{\sigma(2)}])y_{\sigma(3)}x_{\pi(3)}\\
&\quad + \sum_{\pi\in\mathbb{S}_3}\sum_{\sigma\in\mathbb{S}_3} y_{\sigma(1)}F([x_{\pi(1)}x_{\pi(2)}, y_{\sigma(2)}y_{\sigma(3)}])x_{\pi(3)} + \sum_{\pi\in\mathbb{S}_3}\sum_{\sigma\in\mathbb{S}_3} F(y_{\sigma(1)}y_{\sigma(2)})[x_{\pi(1)}x_{\pi(2)}, y_{\sigma(3)}]x_{\pi(3)}\\
&\quad - \sum_{\pi\in\mathbb{S}_3}\sum_{\sigma\in\mathbb{S}_3} y_{\sigma(1)}F(y_{\sigma(2)})[x_{\pi(1)}x_{\pi(2)}, y_{\sigma(3)}]x_{\pi(3)} - \sum_{\pi\in\mathbb{S}_3}\sum_{\sigma\in\mathbb{S}_3}[x_{\pi(1)}, y_{\sigma(1)}y_{\sigma(2)}y_{\sigma(3)}]F(x_{\pi(2)})x_{\pi(3)}\\
&\quad + \sum_{\pi\in\mathbb{S}_3}\sum_{\sigma\in\mathbb{S}_3}[x_{\pi(1)}, y_{\sigma(1)}y_{\sigma(2)}y_{\sigma(3)}]F(x_{\pi(2)}x_{\pi(3)}) - \sum_{\pi\in\mathbb{S}_3}\sum_{\sigma\in\mathbb{S}_3} x_{\pi(1)}F([x_{\pi(2)}, y_{\sigma(1)}y_{\sigma(2)}])y_{\sigma(3)}x_{\pi(3)}\\
&\quad + \sum_{\pi\in\mathbb{S}_3}\sum_{\sigma\in\mathbb{S}_3} x_{\pi(1)}[x_{\pi(2)}, y_{\sigma(1)}]F(y_{\sigma(2)})y_{\sigma(3)}x_{\pi(3)} - \sum_{\pi\in\mathbb{S}_3}\sum_{\sigma\in\mathbb{S}_3} x_{\pi(1)}[x_{\pi(2)}, y_{\sigma(1)}]F(y_{\sigma(2)}y_{\sigma(3)})x_{\pi(3)}\\
&\quad + \sum_{\pi\in\mathbb{S}_3}\sum_{\sigma\in\mathbb{S}_3} x_{\pi(1)}y_{\sigma(1)}F([x_{\pi(2)}, y_{\sigma(2)}])y_{\sigma(3)}x_{\pi(3)} - \sum_{\pi\in\mathbb{S}_3}\sum_{\sigma\in\mathbb{S}_3} x_{\pi(1)}y_{\sigma(1)}F([x_{\pi(2)}, y_{\sigma(2)}y_{\sigma(3)}])x_{\pi(3)}\\
&\quad - \sum_{\pi\in\mathbb{S}_3}\sum_{\sigma\in\mathbb{S}_3} x_{\pi(1)}F(y_{\sigma(1)}y_{\sigma(2)})[x_{\pi(2)}, y_{\sigma(3)}]x_{\pi(3)} + \sum_{\pi\in\mathbb{S}_3}\sum_{\sigma\in\mathbb{S}_3} x_{\pi(1)}y_{\sigma(1)}F(y_{\sigma(2)})[x_{\pi(2)}, y_{\sigma(3)}]x_{\pi(3)}\\
&\quad + \sum_{\pi\in\mathbb{S}_3}\sum_{\sigma\in\mathbb{S}_3} x_{\pi(1)}F([x_{\pi(2)}x_{\pi(3)}, y_{\sigma(1)}y_{\sigma(2)}])y_{\sigma(3)} - \sum_{\pi\in\mathbb{S}_3}\sum_{\sigma\in\mathbb{S}_3} x_{\pi(1)}[x_{\pi(2)}x_{\pi(3)}, y_{\sigma(1)}]F(y_{\sigma(2)})y_{\sigma(3)}\\
&\quad + \sum_{\pi\in\mathbb{S}_3}\sum_{\sigma\in\mathbb{S}_3} x_{\pi(1)}[x_{\pi(2)}x_{\pi(3)}, y_{\sigma(1)}]F(y_{\sigma(2)}y_{\sigma(3)}) - \sum_{\pi\in\mathbb{S}_3}\sum_{\sigma\in\mathbb{S}_3} x_{\pi(1)}y_{\sigma(1)}F([x_{\pi(2)}x_{\pi(3)}, y_{\sigma(2)}])y_{\sigma(3)}\\
&\quad + \sum_{\pi\in\mathbb{S}_3}\sum_{\sigma\in\mathbb{S}_3} x_{\pi(1)}y_{\sigma(1)}F([x_{\pi(2)}x_{\pi(3)}, y_{\sigma(2)}y_{\sigma(3)}]) + \sum_{\pi\in\mathbb{S}_3}\sum_{\sigma\in\mathbb{S}_3} x_{\pi(1)}F(y_{\sigma(1)}y_{\sigma(2)})[x_{\pi(2)}x_{\pi(3)}, y_{\sigma(3)}]\\
&\quad - \sum_{\pi\in\mathbb{S}_3}\sum_{\sigma\in\mathbb{S}_3} x_{\pi(1)}y_{\sigma(1)}F(y_{\sigma(2)})[x_{\pi(2)}x_{\pi(3)}, y_{\sigma(3)}] + \sum_{\pi\in\mathbb{S}_3}\sum_{\sigma\in\mathbb{S}_3} F(x_{\pi(1)}x_{\pi(2)})[x_{\pi(3)}, y_{\sigma(1)}y_{\sigma(2)}y_{\sigma(3)}]\\
&\quad - \sum_{\pi\in\mathbb{S}_3}\sum_{\sigma\in\mathbb{S}_3} x_{\pi(1)}F(x_{\pi(2)})[x_{\pi(3)}, y_{\sigma(1)}y_{\sigma(2)}y_{\sigma(3)}]
\end{aligned}$$

for all $\bar{x}_3, \bar{y}_3 \in \mathcal{L}^3$. On the other hand, by using $[p(\bar{x}_3), p(\bar{y}_3)] = -[p(\bar{y}_3), p(\bar{x}_3)]$, we obtain from aforementioned identity

$$\begin{aligned}
F[p(\bar{x}_3), p(\bar{y}_3)] &= \sum_{\pi\in\mathbb{S}_3}\sum_{\sigma\in\mathbb{S}_3} F([x_{\pi(1)}x_{\pi(2)}, y_{\sigma(1)}y_{\sigma(2)}])x_{\pi(3)}y_{\sigma(3)} - \sum_{\pi\in\mathbb{S}_3}\sum_{\sigma\in\mathbb{S}_3}[x_{\pi(1)}, y_{\sigma(1)}y_{\sigma(2)}]F(x_{\pi(2)})x_{\pi(3)}y_{\sigma(3)}\\
&\quad + \sum_{\pi\in\mathbb{S}_3}\sum_{\sigma\in\mathbb{S}_3}[x_{\pi(1)}, y_{\sigma(1)}y_{\sigma(2)}]F(x_{\pi(2)}x_{\pi(3)})y_{\sigma(3)} - \sum_{\pi\in\mathbb{S}_3}\sum_{\sigma\in\mathbb{S}_3} x_{\pi(1)}F([x_{\pi(2)}, y_{\sigma(1)}y_{\sigma(2)}])x_{\pi(3)}y_{\sigma(3)}\\
&\quad + \sum_{\pi\in\mathbb{S}_3}\sum_{\sigma\in\mathbb{S}_3} x_{\pi(1)}F([x_{\pi(2)}x_{\pi(3)}, y_{\sigma(1)}y_{\sigma(2)}])y_{\sigma(3)} + \sum_{\pi\in\mathbb{S}_3}\sum_{\sigma\in\mathbb{S}_3} F(x_{\pi(1)}x_{\pi(2)})[x_{\pi(3)}, y_{\sigma(1)}y_{\sigma(2)}]y_{\sigma(3)}
\end{aligned}$$

$$
\begin{aligned}
&- \sum_{\pi\in\mathbb{S}_3}\sum_{\sigma\in\mathbb{S}_3} x_{\pi(1)}F(x_{\pi(2)})[x_{\pi(3)}, y_{\sigma(1)}y_{\sigma(2)}]y_{\sigma(3)} - \sum_{\pi\in\mathbb{S}_3}\sum_{\sigma\in\mathbb{S}_3} [x_{\pi(1)}x_{\pi(2)}x_{\pi(3)}, y_{\sigma(1)}]F(y_{\sigma(2)})y_{\sigma(3)}\\
&+ \sum_{\pi\in\mathbb{S}_3}\sum_{\sigma\in\mathbb{S}_3} [x_{\pi(1)}x_{\pi(2)}x_{\pi(3)}, y_{\sigma(1)}]F(y_{\sigma(2)}y_{\sigma(3)}) - \sum_{\pi\in\mathbb{S}_3}\sum_{\sigma\in\mathbb{S}_3} y_{\sigma(1)}F([x_{\pi(1)}x_{\pi(2)}, y_{\sigma(2)}])x_{\pi(3)}y_{\sigma(3)}\\
&+ \sum_{\pi\in\mathbb{S}_3}\sum_{\sigma\in\mathbb{S}_3} y_{\sigma(1)}[x_{\pi(1)}, y_{\sigma(2)}]F(x_{\pi(2)})x_{\pi(3)}y_{\sigma(3)} - \sum_{\pi\in\mathbb{S}_3}\sum_{\sigma\in\mathbb{S}_3} y_{\sigma(1)}[x_{\pi(1)}, y_{\sigma(2)}]F(x_{\pi(2)}x_{\pi(3)})y_{\sigma(3)}\\
&+ \sum_{\pi\in\mathbb{S}_3}\sum_{\sigma\in\mathbb{S}_3} y_{\sigma(1)}x_{\pi(1)}F([x_{\pi(2)}, y_{\sigma(2)}])x_{\pi(3)}y_{\sigma(3)} - \sum_{\pi\in\mathbb{S}_3}\sum_{\sigma\in\mathbb{S}_3} y_{\sigma(1)}x_{\pi(1)}F([x_{\pi(2)}x_{\pi(3)}, y_{\sigma(2)}])y_{\sigma(3)}\\
&- \sum_{\pi\in\mathbb{S}_3}\sum_{\sigma\in\mathbb{S}_3} y_{\sigma(1)}F(x_{\pi(1)}x_{\pi(2)})[x_{\pi(3)}, y_{\sigma(2)}]y_{\sigma(3)} + \sum_{\pi\in\mathbb{S}_3}\sum_{\sigma\in\mathbb{S}_3} y_{\sigma(1)}x_{\pi(1)}F(x_{\pi(2)})[x_{\pi(3)}, y_{\sigma(2)}]y_{\sigma(3)}\\
&+ \sum_{\pi\in\mathbb{S}_3}\sum_{\sigma\in\mathbb{S}_3} y_{\sigma(1)}F([x_{\pi(1)}x_{\pi(2)}, y_{\sigma(2)}y_{\sigma(3)}])x_{\pi(3)} - \sum_{\pi\in\mathbb{S}_3}\sum_{\sigma\in\mathbb{S}_3} y_{\sigma(1)}[x_{\pi(1)}, y_{\sigma(2)}y_{\sigma(3)}]F(x_{\pi(2)})x_{\pi(3)}\\
&+ \sum_{\pi\in\mathbb{S}_3}\sum_{\sigma\in\mathbb{S}_3} y_{\sigma(1)}[x_{\pi(1)}, y_{\sigma(2)}y_{\sigma(3)}]F(x_{\pi(2)}x_{\pi(3)}) - \sum_{\pi\in\mathbb{S}_3}\sum_{\sigma\in\mathbb{S}_3} y_{\sigma(1)}x_{\pi(1)}F([x_{\pi(2)}, y_{\sigma(2)}y_{\sigma(3)}])x_{\pi(3)}\\
&+ \sum_{\pi\in\mathbb{S}_3}\sum_{\sigma\in\mathbb{S}_3} y_{\sigma(1)}x_{\pi(1)}F([x_{\pi(2)}x_{\pi(3)}, y_{\sigma(2)}y_{\sigma(3)}]) + \sum_{\pi\in\mathbb{S}_3}\sum_{\sigma\in\mathbb{S}_3} y_{\sigma(1)}F(x_{\pi(1)}x_{\pi(2)})[x_{\pi(3)}, y_{\sigma(2)}y_{\sigma(3)}]\\
&- \sum_{\pi\in\mathbb{S}_3}\sum_{\sigma\in\mathbb{S}_3} y_{\sigma(1)}x_{\pi(1)}F(x_{\pi(2)})[x_{\pi(3)}, y_{\sigma(2)}y_{\sigma(3)}] + \sum_{\pi\in\mathbb{S}_3}\sum_{\sigma\in\mathbb{S}_3} F(y_{\sigma(1)}y_{\sigma(2)})[x_{\pi(1)}x_{\pi(2)}x_{\pi(3)}, y_{\sigma(3)}]\\
&- \sum_{\pi\in\mathbb{S}_3}\sum_{\sigma\in\mathbb{S}_3} y_{\sigma(1)}F(y_{\sigma(2)})[x_{\pi(1)}x_{\pi(2)}x_{\pi(3)}, y_{\sigma(3)}]
\end{aligned}
$$

for all $\bar{x}_3, \bar{y}_3 \in \mathcal{L}^3$. By comparing so obtained identities, we arrive at

$$
\begin{aligned}
0 = &\sum_{\pi\in\mathbb{S}_3}\sum_{\sigma\in\mathbb{S}_3} F([x_{\pi(1)}x_{\pi(2)}, y_{\sigma(1)}y_{\sigma(2)}])y_{\sigma(3)}x_{\pi(3)} - F(x_{\pi(1)}x_{\pi(2)})y_{\sigma(1)}y_{\sigma(2)}y_{\sigma(3)}x_{\pi(3)}\\
&+ F(y_{\sigma(1)}y_{\sigma(2)})x_{\pi(1)}x_{\pi(2)}y_{\sigma(3)}x_{\pi(3)} - y_{\sigma(1)}F(y_{\sigma(2)})x_{\pi(1)}x_{\pi(2)}y_{\sigma(3)}x_{\pi(3)}\\
&- x_{\pi(1)}F(y_{\sigma(1)}y_{\sigma(2)})x_{\pi(2)}y_{\sigma(3)}x_{\pi(3)} - y_{\sigma(1)}x_{\pi(1)}F([y_{\sigma(2)}y_{\sigma(3)}, x_{\pi(2)}])x_{\pi(3)}\\
&+ x_{\pi(1)}y_{\sigma(1)}F(y_{\sigma(2)})x_{\pi(2)}y_{\sigma(3)}x_{\pi(3)} + x_{\pi(1)}F(x_{\pi(2)})y_{\sigma(1)}y_{\sigma(2)}y_{\sigma(3)}x_{\pi(3)}\\
&+ y_{\sigma(1)}F(x_{\pi(1)}x_{\pi(2)})y_{\sigma(2)}y_{\sigma(3)}x_{\pi(3)} - y_{\sigma(1)}x_{\pi(1)}F(x_{\pi(2)})y_{\sigma(2)}y_{\sigma(3)}x_{\pi(3)}\\
&+ \sum_{\pi\in\mathbb{S}_3}\sum_{\sigma\in\mathbb{S}_3} F([y_{\sigma(1)}y_{\sigma(2)}, x_{\pi(1)}x_{\pi(2)}])x_{\pi(3)}y_{\sigma(3)} - F(y_{\sigma(1)}y_{\sigma(2)})x_{\pi(1)}x_{\pi(2)}x_{\pi(3)}y_{\sigma(3)}\\
&+ F(x_{\pi(1)}x_{\pi(2)})y_{\sigma(1)}y_{\sigma(2)}x_{\pi(3)}y_{\sigma(3)} - x_{\pi(1)}F(x_{\pi(2)})y_{\sigma(1)}y_{\sigma(2)}x_{\pi(3)}y_{\sigma(3)}\\
&- y_{\sigma(1)}F(x_{\pi(1)}x_{\pi(2)})y_{\sigma(2)}x_{\pi(3)}y_{\sigma(3)} - x_{\pi(1)}y_{\sigma(1)}F([x_{\pi(2)}x_{\pi(3)}, y_{\sigma(2)}])y_{\sigma(3)}\\
&+ y_{\sigma(1)}x_{\pi(1)}F(x_{\pi(2)})y_{\sigma(2)}x_{\pi(3)}y_{\sigma(3)} + y_{\sigma(1)}F(y_{\sigma(2)})x_{\pi(1)}x_{\pi(2)}x_{\pi(3)}y_{\sigma(3)}\\
&+ x_{\pi(1)}F(y_{\sigma(1)}y_{\sigma(2)})x_{\pi(2)}x_{\pi(3)}y_{\sigma(3)} - x_{\pi(1)}y_{\sigma(1)}F(y_{\sigma(2)})x_{\pi(2)}x_{\pi(3)}y_{\sigma(3)}\\
&+ \sum_{\pi\in\mathbb{S}_3}\sum_{\sigma\in\mathbb{S}_3} x_{\pi(1)}y_{\sigma(1)}F([x_{\pi(2)}x_{\pi(3)}, y_{\sigma(2)}y_{\sigma(3)}])\\
&+ x_{\pi(1)}y_{\sigma(1)}x_{\pi(2)}x_{\pi(3)}F(y_{\sigma(2)})y_{\sigma(3)} - x_{\pi(1)}y_{\sigma(1)}x_{\pi(2)}x_{\pi(3)}F(y_{\sigma(2)}y_{\sigma(3)})\\
&+ x_{\pi(1)}y_{\sigma(1)}y_{\sigma(2)}y_{\sigma(3)}F(x_{\pi(2)}x_{\pi(3)}) - x_{\pi(1)}y_{\sigma(1)}x_{\pi(2)}F(y_{\sigma(2)})y_{\sigma(3)}x_{\pi(3)}\\
&+ x_{\pi(1)}y_{\sigma(1)}x_{\pi(2)}F(y_{\sigma(2)}y_{\sigma(3)})x_{\pi(3)} - x_{\pi(1)}y_{\sigma(1)}y_{\sigma(2)}y_{\sigma(3)}F(x_{\pi(2)})x_{\pi(3)}\\
&- x_{\pi(1)}F([x_{\pi(2)}, y_{\sigma(1)}y_{\sigma(2)}])y_{\sigma(3)}x_{\pi(3)} + x_{\pi(1)}y_{\sigma(1)}F([x_{\pi(2)}, y_{\sigma(2)}])y_{\sigma(3)}x_{\pi(3)}\\
&- x_{\pi(1)}y_{\sigma(1)}F([x_{\pi(2)}, y_{\sigma(2)}y_{\sigma(3)}])x_{\pi(3)} - x_{\pi(1)}F([y_{\sigma(1)}y_{\sigma(2)}, x_{\pi(2)}])x_{\pi(3)}y_{\sigma(3)}\\
&+ x_{\pi(1)}y_{\sigma(1)}y_{\sigma(2)}F(x_{\pi(2)})x_{\pi(3)}y_{\sigma(3)} - x_{\pi(1)}y_{\sigma(1)}y_{\sigma(2)}F(x_{\pi(2)}x_{\pi(3)})y_{\sigma(3)}\\
&+ \sum_{\pi\in\mathbb{S}_3}\sum_{\sigma\in\mathbb{S}_3} y_{\sigma(1)}x_{\pi(1)}F([y_{\sigma(2)}y_{\sigma(3)}, x_{\pi(2)}x_{\pi(3)}])\\
&+ y_{\sigma(1)}x_{\pi(1)}y_{\sigma(2)}y_{\sigma(3)}F(x_{\pi(2)})x_{\pi(3)} - y_{\sigma(1)}x_{\pi(1)}y_{\sigma(2)}y_{\sigma(3)}F(x_{\pi(2)}x_{\pi(3)})\\
&+ y_{\sigma(1)}x_{\pi(1)}x_{\pi(2)}x_{\pi(3)}F(y_{\sigma(2)}y_{\sigma(3)}) - y_{\sigma(1)}x_{\pi(1)}y_{\sigma(2)}F(x_{\pi(2)})x_{\pi(3)}y_{\sigma(3)}\\
&+ y_{\sigma(1)}x_{\pi(1)}y_{\sigma(2)}F(x_{\pi(2)}x_{\pi(3)})y_{\sigma(3)} - y_{\sigma(1)}x_{\pi(1)}x_{\pi(2)}x_{\pi(3)}F(y_{\sigma(2)})y_{\sigma(3)}
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
&- y_{\sigma(1)}F([y_{\sigma(2)}, x_{\pi(1)}x_{\pi(2)}])x_{\pi(3)}y_{\sigma(3)} + y_{\sigma(1)}x_{\pi(1)}F([y_{\sigma(2)}, x_{\pi(2)}])x_{\pi(3)}y_{\sigma(3)} \\
&- y_{\sigma(1)}x_{\pi(1)}F([y_{\sigma(2)}, x_{\pi(2)}x_{\pi(3)}])y_{\sigma(3)} - y_{\sigma(1)}F([x_{\pi(1)}x_{\pi(2)}, y_{\sigma(2)}])y_{\sigma(3)}x_{\pi(3)} \\
&- y_{\sigma(1)}x_{\pi(1)}x_{\pi(2)}F(y_{\sigma(2)}y_{\sigma(3)})x_{\pi(3)} + y_{\sigma(1)}x_{\pi(1)}x_{\pi(2)}F(y_{\sigma(2)})y_{\sigma(3)}x_{\pi(3)}
\end{aligned} \tag{13}
$$

for all $\bar{x}_3, \bar{y}_3 \in \mathcal{L}^3$. By using the theory of functional identities, we can conclude that

$$
\begin{aligned}
&\sum_{\pi\in\mathbb{S}_2}\sum_{\sigma\in\mathbb{S}_2} F([x_{\pi(1)}x_{\pi(2)}, y_{\sigma(1)}y_{\sigma(2)}]) - F(x_{\pi(1)}x_{\pi(2)})y_{\sigma(1)}y_{\sigma(2)} + F(y_{\sigma(1)}y_{\sigma(2)})x_{\pi(1)}x_{\pi(2)} \\
&- y_{\sigma(1)}F(y_{\sigma(2)})x_{\pi(1)}x_{\pi(2)} + x_{\pi(1)}F(x_{\pi(2)})y_{\sigma(1)}y_{\sigma(2)} \\
&\quad = x_1p_1(x_2, y_1, y_2) + x_2p_2(x_1, y_1, y_2) + y_1p_3(x_1, x_2, y_2) + y_2p_4(x_1, x_2, y_1) + \lambda_p(x_1, x_2, y_1, y_2)
\end{aligned} \tag{14}
$$

and

$$
\begin{aligned}
&\sum_{\pi\in\mathbb{S}_2}\sum_{\sigma\in\mathbb{S}_2} F([x_{\pi(2)}x_{\pi(3)}, y_{\sigma(2)}y_{\sigma(3)}]) - x_{\pi(2)}x_{\pi(3)}F(y_{\sigma(2)}y_{\sigma(3)}) + y_{\sigma(2)}y_{\sigma(3)}F(x_{\pi(2)}x_{\pi(3)}) + x_{\pi(2)}x_{\pi(3)}F(y_{\sigma(2)})y_{\sigma(3)} \\
&- y_{\sigma(2)}y_{\sigma(3)}F(x_{\pi(2)})x_{\pi(3)} = q_1(x_3, y_2, y_3)x_2 + q_2(x_2, y_2, y_3)x_3 + q_3(x_2, x_3, y_3)y_2 + q_4(x_2, x_3, y_2)y_3 + \lambda_q(x_2, x_3, y_2, y_3).
\end{aligned} \tag{15}
$$

We also have

$$
\begin{aligned}
&\sum_{\pi\in\mathbb{S}_2}\sum_{\sigma\in\mathbb{S}_2} F([y_{\sigma(1)}y_{\sigma(2)}, x_{\pi(1)}x_{\pi(2)}]) - F(y_{\sigma(1)}y_{\sigma(2)})x_{\pi(1)}x_{\pi(2)} + F(x_{\pi(1)}x_{\pi(2)})y_{\sigma(1)}y_{\sigma(2)} \\
&\quad - x_{\pi(1)}F(x_{\pi(2)})y_{\sigma(1)}y_{\sigma(2)} + y_{\sigma(1)}F(y_{\sigma(2)})x_{\pi(1)}x_{\pi(2)} \\
&\quad = x_1p_1'(x_2, y_1, y_2) + x_2p_2'(x_1, y_1, y_2) + y_1p_3'(x_1, x_2, y_2) + y_2p_4'(x_1, x_2, y_1) + \lambda_{p'}(x_1, x_2, y_1, y_2)
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{\pi\in\mathbb{S}_2}\sum_{\sigma\in\mathbb{S}_2} F([y_{\sigma(2)}y_{\sigma(3)}, x_{\pi(2)}x_{\pi(3)}]) - y_{\sigma(2)}y_{\sigma(3)}F(x_{\pi(2)}x_{\pi(3)}) + x_{\pi(2)}x_{\pi(3)}F(y_{\sigma(2)}y_{\sigma(3)}) \\
&\quad - x_{\pi(2)}x_{\pi(3)}F(y_{\sigma(2)})y_{\sigma(3)} + y_{\sigma(2)}y_{\sigma(3)}F(x_{\pi(2)})x_{\pi(3)} \\
&\quad = q_1'(x_3, y_2, y_3)x_2 + q_2'(x_2, y_2, y_3)x_3 + q_3'(x_2, x_3, y_3)y_2 + q_4'(x_2, x_3, y_2)y_3 + \lambda_{q'}(x_2, x_3, y_2, y_3)
\end{aligned}
$$

for all $\bar{x}_3, \bar{y}_3 \in \mathcal{L}^3$ and $p_i, p_i', q_i, q_i' : \mathcal{L}^3 \to R$, $i = 1, 2, 3, 4$ and $\lambda_p, \lambda_{p'}, \lambda_q, \lambda_{q'} : \mathcal{L}^4 \to C(\mathcal{L})$. By comparing identities (14) and (15), we arrive at

$$
\begin{aligned}
&\sum_{\pi\in\mathbb{S}_2}\sum_{\sigma\in\mathbb{S}_2} (-F(x_{\pi(1)}x_{\pi(2)})y_{\sigma(1)}y_{\sigma(2)} + F(y_{\sigma(1)}y_{\sigma(2)})x_{\pi(1)}x_{\pi(2)} - y_{\sigma(1)}F(y_{\sigma(2)})x_{\pi(1)}x_{\pi(2)} + x_{\pi(1)}F(x_{\pi(2)})y_{\sigma(1)}y_{\sigma(2)} \\
&+ x_{\pi(1)}x_{\pi(2)}F(y_{\sigma(1)}y_{\sigma(2)}) - y_{\sigma(1)}y_{\sigma(2)}F(x_{\pi(1)}x_{\pi(2)}) + y_{\sigma(1)}y_{\sigma(2)}F(x_{\pi(1)})x_{\pi(2)} - x_{\pi(1)}x_{\pi(2)}F(y_{\sigma(1)})y_{\sigma(2)}) \\
&= x_{\pi(1)}p_1(x_{\pi(2)}, y_{\sigma(1)}, y_{\sigma(2)}) + x_{\pi(2)}p_2(x_{\pi(1)}, y_{\sigma(1)}, y_{\sigma(2)}) + y_{\sigma(1)}p_3(x_{\pi(1)}, x_{\pi(2)}, y_{\sigma(2)}) \\
&\quad + y_{\sigma(2)}p_4(x_{\pi(1)}, x_{\pi(2)}, y_{\sigma(1)}) + \lambda_1(x_{\pi(1)}, x_{\pi(2)}, y_{\sigma(1)}, y_{\sigma(2)}) - q_1(x_{\pi(2)}, y_{\sigma(1)}, y_{\sigma(2)})x_{\pi(1)} \\
&\quad - q_2(x_{\pi(1)}, y_{\sigma(1)}, y_{\sigma(2)})x_{\pi(2)} - q_3(x_{\pi(1)}, x_{\pi(2)}, y_{\sigma(2)})y_{\sigma(1)} - q_4(x_{\pi(1)}, x_{\pi(2)}, y_{\sigma(1)})y_{\sigma(2)} - \mu_1(x_{\pi(1)}, x_{\pi(2)}, y_{\sigma(1)}, y_{\sigma(2)})
\end{aligned} \tag{16}
$$

for all $\bar{x}_3, \bar{y}_3 \in \mathcal{L}^3$ and $p_i, p_i', q_i, q_i' : \mathcal{L}^3 \to R$, $i = 1, 2, 3, 4$ and $\lambda_p, \lambda_{p'}, \lambda_q, \lambda_{q'} : \mathcal{L}^4 \to C(\mathcal{L})$. By using equation (16) and the theory of functional identities, it follows that

$$
\begin{aligned}
&\sum_{\sigma\in\mathbb{S}_2} (F(y_{\sigma(1)}y_{\sigma(2)})x_{\pi(1)} - y_{\sigma(1)}F(y_{\sigma(2)})x_{\pi(1)}) + q_2(x_{\pi(1)}, y_{\sigma(1)}, y_{\sigma(2)}) \\
&\quad = x_{\pi(1)}m_1(y_{\sigma(1)}, y_{\sigma(2)}) + y_{\sigma(1)}m_2(x_{\pi(1)}, y_{\sigma(2)}) + y_{\sigma(2)}m_3(x_{\pi(1)}, y_{\sigma(1)}) + \lambda_m(x_{\pi(1)}, y_{\sigma(1)}, y_{\sigma(2)})
\end{aligned}
$$

for all $\bar{x}_3, \bar{y}_3 \in \mathcal{L}^3$, $q_2 : \mathcal{L}^3 \to R$, $m_i : \mathcal{L}^2 \to R$, $i = 1, 2, 3$ and $\lambda_m : \mathcal{L}^3 \to C(\mathcal{L})$. Now setting $x_{\pi(1)}, y_{\sigma(1)}, y_{\sigma(2)} = x$ in the aforementioned equation, we obtain

$$
2F(x^2)x - 2xF(x)x + q_2(x, x, x) = xm_1(x, x) + xm_2(x, x) + xm_3(x, x) + \lambda_m(x, x, x) \tag{17}
$$

for all $x \in R$.

On the other hand, using equation (16) and the theory of functional identities, we arrive at

$$\sum_{\sigma\in\mathbb{S}_2}(x_{\pi(1)}F(y_{\sigma(1)}y_{\sigma(2)})-x_{\pi(1)}F(y_{\sigma(1)})y_{\sigma(2)})-p_2(x_{\pi(1)},y_{\sigma(1)},y_{\sigma(2)})$$
$$= n_1'(y_{\sigma(1)},y_{\sigma(2)})x_{\pi(1)}+n_2'(x_{\pi(1)},y_{\sigma(2)})y_{\sigma(1)}+n_3'(x_{\pi(1)},y_{\sigma(1)})y_{\sigma(2)}+\lambda_n'(x_{\pi(1)},y_{\sigma(1)},y_{\sigma(2)})$$

$\bar{x}_3,\bar{y}_3\in\mathcal{L}^3$, $p_2:\mathcal{L}^3\to R$, $n_i':\mathcal{L}^2\to R$, $i=1,2,3$ and $\lambda_n':\mathcal{L}^3\to C(\mathcal{L})$.

Now setting $x_{\pi(1)},y_{\sigma(1)},y_{\sigma(2)}=x$, we obtain

$$2xF(x^2)-2xF(x)x-p_2(x,x,x)=n_1'(x,x)x+n_2'(x,x)x+n_3'(x,x)x+\lambda_n'(x,x,x) \tag{18}$$

for all $x\in R$. Equation (13) can now be rewritten as follows:

$$\begin{aligned}
0=\sum_{\pi\in\mathbb{S}_3}\sum_{\sigma\in\mathbb{S}_3}&(x_{\pi(1)}p_1(x_{\pi(2)},y_{\sigma(1)},y_{\sigma(2)})y_{\sigma(3)}x_{\pi(3)}+x_{\pi(2)}p_2(x_{\pi(1)},y_{\sigma(1)},y_{\sigma(2)})y_{\sigma(3)}x_{\pi(3)}\\
&+y_{\sigma(1)}p_3(x_{\pi(1)},x_{\pi(2)},y_{\sigma(2)})y_{\sigma(3)}x_{\pi(3)}+y_{\sigma(2)}p_4(x_{\pi(1)},x_{\pi(2)},y_{\sigma(1)})y_{\sigma(3)}x_{\pi(3)}\\
&+\lambda_p(x_{\pi(1)},x_{\pi(2)},y_{\sigma(1)},y_{\sigma(2)})y_{\sigma(3)}x_{\pi(3)}+x_{\pi(1)}y_{\sigma(1)}F(y_{\sigma(2)})x_{\pi(2)}y_{\sigma(3)}x_{\pi(3)}\\
&-x_{\pi(1)}F(y_{\sigma(1)}y_{\sigma(2)})x_{\pi(2)}y_{\sigma(3)}x_{\pi(3)}-y_{\sigma(1)}x_{\pi(1)}F([y_{\sigma(2)}y_{\sigma(3)},x_{\pi(2)}])x_{\pi(3)}\\
&+y_{\sigma(1)}F(x_{\pi(1)}x_{\pi(2)})y_{\sigma(2)}y_{\sigma(3)}x_{\pi(3)}-y_{\sigma(1)}x_{\pi(1)}F(x_{\pi(2)})y_{\sigma(2)}y_{\sigma(3)}x_{\pi(3)}\\
&+\sum_{\pi\in\mathbb{S}_3}\sum_{\sigma\in\mathbb{S}_3}(x_{\pi(1)}p_1'(x_{\pi(2)},y_{\sigma(1)},y_{\sigma(2)})x_{\pi(3)}y_{\sigma(3)}+x_{\pi(2)}p_2'(x_{\pi(1)},y_{\sigma(1)},y_{\sigma(2)})x_{\pi(3)}y_{\sigma(3)}\\
&+y_{\sigma(1)}p_3'(x_{\pi(1)},x_{\pi(2)},y_{\sigma(2)})x_{\pi(3)}y_{\sigma(3)}+y_{\sigma(2)}p_4'(x_{\pi(1)},x_{\pi(2)},y_{\sigma(1)})x_{\pi(3)}y_{\sigma(3)})\\
&+\lambda_{p'}(x_{\pi(1)},x_{\pi(2)},y_{\sigma(1)},y_{\sigma(2)})x_{\pi(3)}y_{\sigma(3)}+y_{\sigma(1)}x_{\pi(1)}F(x_{\pi(2)})y_{\sigma(2)}x_{\pi(3)}y_{\sigma(3)}\\
&-y_{\sigma(1)}F(x_{\pi(1)}x_{\pi(2)})y_{\sigma(2)}x_{\pi(3)}y_{\sigma(3)}-x_{\pi(1)}y_{\sigma(1)}F([x_{\pi(2)}x_{\pi(3)},y_{\sigma(2)}])y_{\sigma(3)}\\
&+x_{\pi(1)}F(y_{\sigma(1)}y_{\sigma(2)})x_{\pi(2)}x_{\pi(3)}y_{\sigma(3)}-x_{\pi(1)}y_{\sigma(1)}F(y_{\sigma(2)})x_{\pi(2)}x_{\pi(3)}y_{\sigma(3)})\\
&+\sum_{\pi\in\mathbb{S}_3}\sum_{\sigma\in\mathbb{S}_3}(x_{\pi(1)}y_{\sigma(1)}q_1(x_{\pi(3)},y_{\sigma(2)},y_{\sigma(3)})x_{\pi(2)}\\
&+x_{\pi(1)}y_{\sigma(1)}q_2(x_{\pi(2)},y_{\sigma(2)},y_{\sigma(3)})x_{\pi(3)}+x_{\pi(1)}y_{\sigma(1)}q_3(x_{\pi(2)},x_{\pi(3)},y_{\sigma(3)})y_{\sigma(2)}\\
&+x_{\pi(1)}y_{\sigma(1)}q_4(x_{\pi(2)},x_{\pi(3)},y_{\sigma(2)})y_{\sigma(3)}+x_{\pi(1)}y_{\sigma(1)}\lambda_q(x_{\pi(2)},x_{\pi(3)},y_{\sigma(2)},y_{\sigma(3)})\\
&-x_{\pi(1)}y_{\sigma(1)}x_{\pi(2)}F(y_{\sigma(2)})y_{\sigma(3)}x_{\pi(3)}+x_{\pi(1)}y_{\sigma(1)}x_{\pi(2)}F(y_{\sigma(2)}y_{\sigma(3)})x_{\pi(3)}\\
&-x_{\pi(1)}F([x_{\pi(2)},y_{\sigma(1)}y_{\sigma(2)}])y_{\sigma(3)}x_{\pi(3)}+x_{\pi(1)}y_{\sigma(1)}F([x_{\pi(2)},y_{\sigma(2)}])y_{\sigma(3)}x_{\pi(3)}\\
&-x_{\pi(1)}y_{\sigma(1)}F([x_{\pi(2)},y_{\sigma(2)}y_{\sigma(3)}])x_{\pi(3)}-x_{\pi(1)}F([y_{\sigma(1)}y_{\sigma(2)},x_{\pi(2)}])x_{\pi(3)}y_{\sigma(3)}\\
&+x_{\pi(1)}y_{\sigma(1)}y_{\sigma(2)}F(x_{\pi(2)})x_{\pi(3)}y_{\sigma(3)}-x_{\pi(1)}y_{\sigma(1)}y_{\sigma(2)}F(x_{\pi(2)}x_{\pi(3)})y_{\sigma(3)})\\
&+\sum_{\pi\in\mathbb{S}_3}\sum_{\sigma\in\mathbb{S}_3}(y_{\sigma(1)}x_{\pi(1)}q_1'(x_{\pi(3)},y_{\sigma(2)},y_{\sigma(3)})x_{\pi(2)}\\
&+y_{\sigma(1)}x_{\pi(1)}q_2'(x_{\pi(2)},y_{\sigma(2)},y_{\sigma(3)})x_{\pi(3)}+y_{\sigma(1)}x_{\pi(1)}q_3'(x_{\pi(2)},x_{\pi(3)},y_{\sigma(3)})y_{\sigma(2)}\\
&+y_{\sigma(1)}x_{\pi(1)}q_4'(x_{\pi(2)},x_{\pi(3)},y_{\sigma(2)})y_{\sigma(3)}+y_{\sigma(1)}x_{\pi(1)}\lambda_{q'}(x_{\pi(2)},x_{\pi(3)},y_{\sigma(2)},y_{\sigma(3)})\\
&-y_{\sigma(1)}x_{\pi(1)}y_{\sigma(2)}F(x_{\pi(2)})x_{\pi(3)}y_{\sigma(3)}+y_{\sigma(1)}x_{\pi(1)}y_{\sigma(2)}F(x_{\pi(2)}x_{\pi(3)})y_{\sigma(3)}\\
&-y_{\sigma(1)}F([y_{\sigma(2)},x_{\pi(1)}x_{\pi(2)}])x_{\pi(3)}y_{\sigma(3)}+y_{\sigma(1)}x_{\pi(1)}F([y_{\sigma(2)},x_{\pi(2)}])x_{\pi(3)}y_{\sigma(3)}\\
&-y_{\sigma(1)}x_{\pi(1)}F([y_{\sigma(2)},x_{\pi(2)}x_{\pi(3)}])y_{\sigma(3)}-y_{\sigma(1)}F([x_{\pi(1)}x_{\pi(2)},y_{\sigma(2)}])y_{\sigma(3)}x_{\pi(3)}\\
&-y_{\sigma(1)}x_{\pi(1)}x_{\pi(2)}F(y_{\sigma(2)}y_{\sigma(3)})x_{\pi(3)}+y_{\sigma(1)}x_{\pi(1)}x_{\pi(2)}F(y_{\sigma(2)})y_{\sigma(3)}x_{\pi(3)})
\end{aligned}$$

for all $\bar{x}_3,\bar{y}_3\in\mathcal{L}^3$, $p_i,p_i',q_i,q_i':\mathcal{L}^3\to R$, $i=1,2,3,4$ and $\lambda_p,\lambda_{p'},\lambda_q,\lambda_{q'}:\mathcal{L}^4\to C(\mathcal{L})$. By using the theory of functional identities and exposing everything from the same side (left), the aforementioned equation can now be rewritten as follows:

$$\begin{aligned}
0=\sum_{\pi\in\mathbb{S}_3}\sum_{\sigma\in\mathbb{S}_3}&x_{\pi(1)}(p_1(x_{\pi(2)},y_{\sigma(1)},y_{\sigma(2)})y_{\sigma(3)}x_{\pi(3)}-F(y_{\sigma(1)}y_{\sigma(2)})x_{\pi(2)}y_{\sigma(3)}x_{\pi(3)}\\
&+y_{\sigma(1)}F(y_{\sigma(2)})x_{\pi(2)}y_{\sigma(3)}x_{\pi(3)}+p_1'(x_{\pi(2)},y_{\sigma(1)},y_{\sigma(2)})x_{\pi(3)}y_{\sigma(3)}-y_{\sigma(1)}F([x_{\pi(2)}x_{\pi(3)},y_{\sigma(2)}])y_{\sigma(3)}\\
&+F(y_{\sigma(1)}y_{\sigma(2)})x_{\pi(2)}x_{\pi(3)}y_{\sigma(3)}-y_{\sigma(1)}F(y_{\sigma(2)})x_{\pi(2)}x_{\pi(3)}y_{\sigma(3)}+y_{\sigma(1)}q_1(x_{\pi(3)},y_{\sigma(2)},y_{\sigma(3)})x_{\pi(2)}
\end{aligned} \tag{19}$$

$$
\begin{aligned}
&+ y_{\sigma(1)}q_2(x_{\pi(2)}, y_{\sigma(2)}, y_{\sigma(3)})x_{\pi(3)} + y_{\sigma(1)}q_3(x_{\pi(2)}, x_{\pi(3)}, y_{\sigma(3)})y_{\sigma(2)} + y_{\sigma(1)}q_4(x_{\pi(2)}, x_{\pi(3)}, y_{\sigma(2)})y_{\sigma(3)} \\
&+ y_{\sigma(1)}\lambda_q(x_{\pi(2)}, x_{\pi(3)}, y_{\sigma(2)}, y_{\sigma(3)}) - y_{\sigma(1)}x_{\pi(2)}F(y_{\sigma(2)})y_{\sigma(3)}x_{\pi(3)} + y_{\sigma(1)}x_{\pi(2)}F(y_{\sigma(2)}y_{\sigma(3)})x_{\pi(3)} \\
&- F([x_{\pi(2)}, y_{\sigma(1)}y_{\sigma(2)}])y_{\sigma(3)}x_{\pi(3)} + y_{\sigma(1)}F([x_{\pi(2)}, y_{\sigma(2)}])y_{\sigma(3)}x_{\pi(3)} - y_{\sigma(1)}F([x_{\pi(2)}, y_{\sigma(2)}y_{\sigma(3)}])x_{\pi(3)} \\
&- F([y_{\sigma(1)}y_{\sigma(2)}, x_{\pi(2)}])x_{\pi(3)}y_{\sigma(3)} + y_{\sigma(1)}y_{\sigma(2)}F(x_{\pi(2)})x_{\pi(3)}y_{\sigma(3)} - y_{\sigma(1)}y_{\sigma(2)}F(x_{\pi(2)}x_{\pi(3)})y_{\sigma(3)}) \\
&+ \sum_{\pi\in\mathbb{S}_3}\sum_{\sigma\in\mathbb{S}_3} x_{\pi(2)}(p_2(x_{\pi(1)}, y_{\sigma(1)}, y_{\sigma(2)})y_{\sigma(3)}x_{\pi(3)} + p_2'(x_{\pi(1)}, y_{\sigma(1)}, y_{\sigma(2)})x_{\pi(3)}y_{\sigma(3)}) \\
&+ \sum_{\pi\in\mathbb{S}_3}\sum_{\sigma\in\mathbb{S}_3} x_{\pi(3)}(y_{\sigma(3)}\lambda_{p'}(x_{\pi(1)}, x_{\pi(2)}, y_{\sigma(1)}, y_{\sigma(2)})) \\
&+ \sum_{\pi\in\mathbb{S}_3}\sum_{\sigma\in\mathbb{S}_3} y_{\sigma(1)}(p_3(x_{\pi(1)}, x_{\pi(2)}, y_{\sigma(2)})y_{\sigma(3)}x_{\pi(3)} - x_{\pi(1)}F([y_{\sigma(2)}y_{\sigma(3)}, x_{\pi(2)}])x_{\pi(3)} \\
&+ F(x_{\pi(1)}x_{\pi(2)})y_{\sigma(2)}y_{\sigma(3)}x_{\pi(3)} - x_{\pi(1)}F(x_{\pi(2)})y_{\sigma(2)}y_{\sigma(3)}x_{\pi(3)} + p_3'(x_{\pi(1)}, x_{\pi(2)}, y_{\sigma(2)})x_{\pi(3)}y_{\sigma(3)} \\
&- F(x_{\pi(1)}x_{\pi(2)})y_{\sigma(2)}x_{\pi(3)}y_{\sigma(3)} + x_{\pi(1)}F(x_{\pi(2)})y_{\sigma(2)}x_{\pi(3)}y_{\sigma(3)} + x_{\pi(1)}q_1'(x_{\pi(3)}, y_{\sigma(2)}, y_{\sigma(3)})x_{\pi(2)} \\
&+ x_{\pi(1)}q_2'(x_{\pi(2)}, y_{\sigma(2)}, y_{\sigma(3)})x_{\pi(3)} + x_{\pi(1)}q_3'(x_{\pi(2)}, x_{\pi(3)}, y_{\sigma(3)})y_{\sigma(2)} + x_{\pi(1)}q_4'(x_{\pi(2)}, x_{\pi(3)}, y_{\sigma(2)})y_{\sigma(3)} \\
&+ x_{\pi(1)}\lambda_{q'}(x_{\pi(2)}, x_{\pi(3)}, y_{\sigma(2)}, y_{\sigma(3)}) - x_{\pi(1)}y_{\sigma(2)}F(x_{\pi(2)})x_{\pi(3)}y_{\sigma(3)} + x_{\pi(1)}y_{\sigma(2)}F(x_{\pi(2)}x_{\pi(3)})y_{\sigma(3)} \\
&- F([y_{\sigma(2)}, x_{\pi(1)}x_{\pi(2)}])x_{\pi(3)}y_{\sigma(3)} + x_{\pi(1)}F([y_{\sigma(2)}, x_{\pi(2)}])x_{\pi(3)}y_{\sigma(3)} - x_{\pi(1)}F([y_{\sigma(2)}, x_{\pi(2)}x_{\pi(3)}])y_{\sigma(3)} \\
&- F([x_{\pi(1)}x_{\pi(2)}, y_{\sigma(2)}])y_{\sigma(3)}x_{\pi(3)} - x_{\pi(1)}x_{\pi(2)}F(y_{\sigma(2)}y_{\sigma(3)})x_{\pi(3)} + x_{\pi(1)}x_{\pi(2)}F(y_{\sigma(2)})y_{\sigma(3)}x_{\pi(3)}) \\
&+ \sum_{\pi\in\mathbb{S}_3}\sum_{\sigma\in\mathbb{S}_3} y_{\sigma(2)}(p_4(x_{\pi(1)}, x_{\pi(2)}, y_{\sigma(1)})y_{\sigma(3)}x_{\pi(3)} + p_4'(x_{\pi(1)}, x_{\pi(2)}, y_{\sigma(1)})x_{\pi(3)}y_{\sigma(3)}) \\
&+ \sum_{\pi\in\mathbb{S}_3}\sum_{\sigma\in\mathbb{S}_3} y_{\sigma(3)}(x_{\pi(3)}\lambda_p(x_{\pi(1)}, x_{\pi(2)}, y_{\sigma(1)}, y_{\sigma(2)}))
\end{aligned}
\tag{19}
$$

for all $\bar{x}_3, \bar{y}_3 \in \mathcal{L}^3$, $p_i, p_i', q_i, q_i' : \mathcal{L}^3 \to R$, $i = 1, 2, 3, 4$ and $\lambda_p, \lambda_{p'}, \lambda_q, \lambda_{q'} : \mathcal{L}^4 \to C(\mathcal{L})$. Now by using the theory of functional identities and exposing x_2 from the left side, we obtain

$$0 = \sum_{\pi\in\mathbb{S}_2}\sum_{\sigma\in\mathbb{S}_3} p_2(x_{\pi(1)}, y_{\sigma(1)}, y_{\sigma(2)})y_{\sigma(3)}x_{\pi(3)} + p_2'(x_{\pi(1)}, y_{\sigma(1)}, y_{\sigma(2)})x_{\pi(3)}y_{\sigma(3)}$$

for all $\bar{x}_3, \bar{y}_3 \in \mathcal{L}^3$ and $p_2, p_2' : \mathcal{L}^3 \to R$. Again by using the theory of functional identities and exposing everything from the right side, we obtain

$$0 = \sum_{\sigma\in\mathbb{S}_2} p_2(x_{\pi(1)}, y_{\sigma(1)}, y_{\sigma(2)})$$

and

$$0 = \sum_{\sigma\in\mathbb{S}_2} p_2'(x_{\pi(1)}, y_{\sigma(1)}, y_{\sigma(2)}).$$

Therefore,

$$0 = 2p_2(x, x, x) = 2p_2'(x, x, x)$$

for all $x \in \mathcal{L}$ and $p_2, p_2' : \mathcal{L}^3 \to R$. Equation (18) can now be rewritten as follows:

$$2xF(x^2) - 2xF(x)x = n_1'(x, x)x + n_2'(x, x)x + n_3'(x, x)x + \lambda_n'(x, x, x) \tag{20}$$

for all $x \in \mathcal{L}$, $n_i' : \mathcal{L}^2 \to R$, $i = 1, 2, 3$ and $\lambda_n' : \mathcal{L}^3 \to C(\mathcal{L})$. Now using the theory of functional identities and exposing x_1 in (19) from the left side,

$$
\begin{aligned}
0 = \sum_{\pi\in\mathbb{S}_2}\sum_{\sigma\in\mathbb{S}_3} &(p_1(x_{\pi(2)}, y_{\sigma(1)}, y_{\sigma(2)})y_{\sigma(3)}x_{\pi(3)} - F(y_{\sigma(1)}y_{\sigma(2)})x_{\pi(2)}y_{\sigma(3)}x_{\pi(3)} + y_{\sigma(1)}F(y_{\sigma(2)})x_{\pi(2)}y_{\sigma(3)}x_{\pi(3)} \\
&+ p_1'(x_{\pi(2)}, y_{\sigma(1)}, y_{\sigma(2)})x_{\pi(3)}y_{\sigma(3)} - y_{\sigma(1)}F([x_{\pi(2)}x_{\pi(3)}, y_{\sigma(2)}])y_{\sigma(3)} + F(y_{\sigma(1)}y_{\sigma(2)})x_{\pi(2)}x_{\pi(3)}y_{\sigma(3)} \\
&- y_{\sigma(1)}F(y_{\sigma(2)})x_{\pi(2)}x_{\pi(3)}y_{\sigma(3)} + y_{\sigma(1)}q_1(x_{\pi(3)}, y_{\sigma(2)}, y_{\sigma(3)})x_{\pi(2)} + y_{\sigma(1)}q_2(x_{\pi(2)}, y_{\sigma(2)}, y_{\sigma(3)})x_{\pi(3)} \\
&+ y_{\sigma(1)}q_3(x_{\pi(2)}, x_{\pi(3)}, y_{\sigma(3)})y_{\sigma(2)} + y_{\sigma(1)}q_4(x_{\pi(2)}, x_{\pi(3)}, y_{\sigma(2)})y_{\sigma(3)} + y_{\sigma(1)}\lambda_q(x_{\pi(2)}, x_{\pi(3)}, y_{\sigma(2)}, y_{\sigma(3)}) \\
&+ y_{\sigma(1)}x_{\pi(2)}F(y_{\sigma(2)}y_{\sigma(3)})x_{\pi(3)} - y_{\sigma(1)}x_{\pi(2)}F(y_{\sigma(2)})y_{\sigma(3)}x_{\pi(3)} - F([x_{\pi(2)}, y_{\sigma(1)}y_{\sigma(2)}])y_{\sigma(3)}x_{\pi(3)}
\end{aligned}
$$

$$+ y_{\sigma(1)}F([x_{\pi(2)}, y_{\sigma(2)}])y_{\sigma(3)}x_{\pi(3)} - y_{\sigma(1)}F([x_{\pi(2)}, y_{\sigma(2)}y_{\sigma(3)}])x_{\pi(3)} - F([y_{\sigma(1)}y_{\sigma(2)}, x_{\pi(2)}])x_{\pi(3)}y_{\sigma(3)}$$
$$+ y_{\sigma(1)}y_{\sigma(2)}F(x_{\pi(2)})x_{\pi(3)}y_{\sigma(3)} - y_{\sigma(1)}y_{\sigma(2)}F(x_{\pi(2)}x_{\pi(3)})y_{\sigma(3)})$$

for all $\bar{x}_3, \bar{y}_3 \in \mathcal{L}^3$ and $p_i, p_i', q_i, q_i' : \mathcal{L}^3 \to R$, $i = 1, 2, 3, 4$ and $\lambda_p, \lambda_{p'}, \lambda_q, \lambda_{q'} : \mathcal{L}^4 \to C(\mathcal{L})$. The aforementioned equation can now be rewritten as (everything exposing from the right side) follows:

$$\begin{aligned}
0 = {} & \sum_{\pi\in\mathbb{S}_2}\sum_{\sigma\in\mathbb{S}_3}(y_{\sigma(1)}q_1(x_{\pi(3)}, y_{\sigma(2)}, y_{\sigma(3)}))x_{\pi(2)} + \sum_{\pi\in\mathbb{S}_2}\sum_{\sigma\in\mathbb{S}_3}(y_{\sigma(1)}q_2(x_{\pi(2)}, y_{\sigma(2)}, y_{\sigma(3)}) + p_1(x_{\pi(2)}, y_{\sigma(1)}, y_{\sigma(2)})y_{\sigma(3)} \\
& - F(y_{\sigma(1)}y_{\sigma(2)})x_{\pi(2)}y_{\sigma(3)} + y_{\sigma(1)}F(y_{\sigma(2)})x_{\pi(2)}y_{\sigma(3)} - F([x_{\pi(2)}, y_{\sigma(1)}y_{\sigma(2)}])y_{\sigma(3)} - y_{\sigma(1)}x_{\pi(2)}F(y_{\sigma(2)})y_{\sigma(3)} \\
& + y_{\sigma(1)}x_{\pi(2)}F(y_{\sigma(2)}y_{\sigma(3)}) + y_{\sigma(1)}F([x_{\pi(2)}, y_{\sigma(2)}])y_{\sigma(3)} - y_{\sigma(1)}F([x_{\pi(2)}, y_{\sigma(2)}y_{\sigma(3)}]))x_{\pi(3)} \\
& + \sum_{\pi\in\mathbb{S}_2}\sum_{\sigma\in\mathbb{S}_3}(\lambda_q(x_{\pi(2)}, x_{\pi(3)}, y_{\sigma(2)}, y_{\sigma(3)}))y_{\sigma(1)} + \sum_{\pi\in\mathbb{S}_2}\sum_{\sigma\in\mathbb{S}_3}(y_{\sigma(1)}q_3(x_{\pi(2)}, x_{\pi(3)}, y_{\sigma(3)}))y_{\sigma(2)} \\
& + \sum_{\pi\in\mathbb{S}_2}\sum_{\sigma\in\mathbb{S}_3}(p_1'(x_{\pi(2)}, y_{\sigma(1)}, y_{\sigma(2)})x_{\pi(3)} - F([y_{\sigma(1)}y_{\sigma(2)}, x_{\pi(2)}])x_{\pi(3)} - y_{\sigma(1)}F([x_{\pi(2)}x_{\pi(3)}, y_{\sigma(2)}]) \\
& + F(y_{\sigma(1)}y_{\sigma(2)})x_{\pi(2)}x_{\pi(3)} - y_{\sigma(1)}F(y_{\sigma(2)})x_{\pi(2)}x_{\pi(3)} + y_{\sigma(1)}q_4(x_{\pi(2)}, x_{\pi(3)}, y_{\sigma(2)}) + y_{\sigma(1)}y_{\sigma(2)}F(x_{\pi(2)})x_{\pi(3)} \\
& - y_{\sigma(1)}y_{\sigma(2)}F(x_{\pi(2)}x_{\pi(3)}))y_{\sigma(3)}
\end{aligned}$$

for all $\bar{x}_3, \bar{y}_3 \in \mathcal{L}^3$ and $p_i, p_i', q_i, q_i' : \mathcal{L}^3 \to R$, $i = 1, 2, 3, 4$ and $\lambda_p, \lambda_{p'}, \lambda_q, \lambda_{q'} : \mathcal{L}^4 \to C(\mathcal{L})$. Now by using the theory of functional identities, we obtain from the aforementioned equation:

$$q_1(x, x, x) = q_3(x, x, x) = 0$$

for all $x \in \mathcal{L}$ and $q_1, q_3 : \mathcal{L}^3 \to R$. Equation (17) can now be rewritten as follows:

$$2F(x^2)x - 2xF(x)x = xm_1(x, x) + xm_2(x, x) + xm_3(x, x) + \lambda_m(x, x, x) \tag{21}$$

for all $x \in \mathcal{L}$, $m_i : \mathcal{L}^2 \to R$, $i = 1, 2, 3$ and $\lambda_m : \mathcal{L}^3 \to C(\mathcal{L})$. Since $\mathcal{L}$ is 6-free, after a finite number of steps using equations (20) and (21), we arrive at

$$2\sum(F(xy) - xF(y)) = xf(y) + yg(x) + \lambda(x, y)$$

$$2\sum(F(x)y - F(xy)) = h(y)x + k(x)y + \mu(x, y)$$

for all $x, y \in R$, $f, g, h, k : \mathcal{L} \to R$ and $\lambda, \mu : \mathcal{L}^2 \to C(\mathcal{L})$. Therefore, we obtain

$$2F(xy) + 2F(yx) - 2xF(y) - 2yF(x) = xf(y) + yg(x) + \lambda(x, y) \tag{22}$$

and

$$2F(x)y + 2F(y)x - 2F(xy) - 2F(yx) = h(y)x + k(x)y + \mu(x, y). \tag{23}$$

For all, $x, y \in R$, $f, g, h, k : \mathcal{L} \to R$ and $\lambda, \mu : \mathcal{L}^2 \to C(\mathcal{L})$. Replacing the roles of denotations x and y in (22) and comparing so obtained identities leads to $0 = xf(y) + yg(x) - yf(x) - xg(y) + \lambda(x, y) - \lambda(y, x)$, which yields $f(x) = g(x)$ and $\lambda(x, y) = \lambda(y, x)$ for all $x, y \in \mathcal{L}$, $f, g : \mathcal{L} \to R$ and $\lambda : \mathcal{L}^2 \to C(\mathcal{L})$. Putting x for y in (22) leads to

$$4F(x^2) = 4xF(x) + 2xf(x) + \lambda(x, x). \tag{24}$$

Using the same arguments, it follows from (23) that $h(x) = k(x)$ and $\mu(x, y) = \mu(y, x)$ for all $x, y \in \mathcal{L}$, $h, k : \mathcal{L} \to R$ and $\mu : \mathcal{L}^2 \to C(\mathcal{L})$. Therefore,

$$4F(x^2) = 4F(x)x - 2k(x)x - \mu(x, x).$$

Comparing the aforementioned relations gives

$$0 = x(4F(x) + 2f(x)) + (-4(x) + 2k(x))x + \lambda(x, x) + \mu(x, x).$$

Hence, there exists $r \in R$ and $\lambda : \mathcal{L} \to C(\mathcal{L})$ such that

$$4F(x) + 2f(x) = rx + \lambda(x).$$

Considering $2f(x) = -4F(x) + rx + \lambda(x)$ in (24) gives

$$4F(x^2) = xrx + x\lambda(x) + \lambda(x,x). \tag{25}$$

Replacing y for x and x for x^2 in (22) gives

$$4F(x^3) = 2x^2F(x) + 2xF(x^2) + x^2f(x) + xf(x^2) + \lambda(x^2,x).$$

Using (7) in the aforementioned relation leads to

$$4F(x^2)x - 4xF(x)x + 4xF(x^2) = 2x^2F(x) + 2xF(x^2) + x^2f(x) + xf(x^2) + \lambda(x^2,x).$$

Using (24) in the aforementioned relation leads to

$$4xf(x)x + 3x\lambda(x,x) = 2xf(x^2) + 2\lambda(x^2,x).$$

Considering $2f(x) = -4F(x) + rx + \lambda(x)$ and using (25) in the aforementioned relation gives

$$0 = -8xF(x)x + xrx^2 + x^2rx + 3x^2\lambda(x) + 4x\lambda(x,x) - x\lambda(x^2) - 2\lambda(x^2,x).$$

The complete linearization of this relation and using the theory of functional identities leads to

$$0 = -8F(x)x + rx^2 + xrx + 3x\lambda(x) + 4\lambda(x,x) - \lambda(x^2)$$

and

$$0 = -8F(x) + rx + xr + 3\lambda(x).$$

Therefore,

$$8F(x) = rx + xr + 3\lambda(x). \tag{26}$$

By substituting the aforementioned equation in (7), we obtain

$$0 = -3\lambda(x^3) - 6x\lambda(x^2) - 3x^2\lambda(x).$$

Since $\mathcal{L}$ is a 6-free subset of R, the aforementioned identity implies $\lambda(x^3) = 0$, $\lambda(x^2) = 0$, $\lambda(x) = 0$ for all $x \in R$. From equation (26), we obtain

$$8F(x^2) = rx^2 + x^2r. \tag{27}$$

Right (left) multiplication of the relation (26) by x gives, respectively,

$$8F(x)x = rx^2 + xrx \tag{28}$$

and

$$8xF(x) = xrx + x^2r. \tag{29}$$

The relations (27)–(29) imply that the additive mapping F satisfies the relation

$$8F(x^2) = 8F(x)x + 8xF(x) - 2xrx.$$

The aforementioned equation can now be rewritten as follows:

$$4F(x^2) = 4F(x)x + 4xF(x) - xrx$$

and

$$4F(x^2) = 4F(x)x + 4xF(x) - 2xqx, \tag{30}$$

where $r = 2q$. Let us now introduce the mapping $D : R \to R$ by

$$D(x) = 4F(x) - qx - xq. \tag{31}$$

Obviously, the mapping D is additive. It is our aim to prove that D is a Jordan derivation. Putting x^2 for x in the aforementioned relation, we obtain

$$D(x^2) = 4F(x^2) - qx^2 - x^2q,$$

which gives, after considering the relation (30), the relation

$$D(x^2) = 4F(x)x + 4xF(x) - 2xqx - qx^2 - x^2q. \tag{32}$$

Right (left) multiplication of the relation (31) by x gives, respectively,

$$D(x)x = 4F(x)x - qx^2 - xqx \tag{33}$$

and

$$xD(x) = 4xF(x) - xqx - x^2q. \tag{34}$$

The relations (32)–(34) imply that the additive mapping D satisfies the relation

$$D(x^2) = D(x)x + xD(x)$$

for all $x \in R$. In other words, D is a Jordan derivation on R. According to Herstein theorem, one can conclude that D is a derivation, which completes the proof of the theorem. □

We are now in the position to prove Theorem 4.

Proof of Theorem 4. The complete linearization of (7) gives us (9). First, suppose that R is not a PI ring (satisfying the standard polynomial identity of degree less than 6). According to Theorem 5, then there exists $q \in R$ such that $4F(x) = D(x) + xq + qx$ for all $x \in \mathcal{L}$.

Assume now that R is a PI ring. It is well known that in this case R has a nonzero center (see [20]). Let c be a nonzero central element. Picking any $x \in R$ and setting $x_1 = x_2 = cx$ and $x_3 = x$ in (9), we obtain

$$3F(c^2x^3) = F(c^2x^2)x + 2cF(cx^2)x - c^2xF(x)x - 2cxF(cx)x + xF(c^2x^2) + 2cxF(cx^2).$$

Next, setting $x_1 = x_2 = c$ and $x_3 = x^3$ in (9), we obtain

$$3F(c^2x^3) = F(c^2)x^3 + 4cF(cx^3) - cx^3F(c) - cF(c)x^3 - c^2F(x^2)x + c^2xF(x)x - c^2xF(x^2) + x^3F(c^2)$$

for all $x \in R$. Comparing both identities, we obtain

$$0 = F(c^2)x^3 + 4cF(cx^3) - cx^3F(c) - cF(c)x^3 - c^2F(x^2)x + c^2xF(x)x - c^2xF(x^2) + x^3F(c^2).$$

Next, setting $x_1 = x_2 = x$ and $x_3 = c^2$ in (9), we obtain

$$3F(c^2x^2) = 2c^2F(x^2) + 2F(c^2x)x - c^2F(x)x - xF(c^2)x - c^2xF(x) + 2xF(c^2x). \tag{35}$$

Next, setting $x_1 = x_2 = x$ and $x_3 = c$ in (9), we obtain

$$3F(cx^2) = 2cF(x^2) + 2F(cx)x - cF(x)x - xF(c)x - cxF(x) + 2xF(cx). \tag{36}$$

In case $x = c$, we arrive at $F(c^3) = 2cF(c^2) - c^2F(c)$. Next, setting $x_1 = x_2 = c$ and $x_3 = x$ in (9), we obtain

$$3F(c^2x) = F(c^2)x + 4cF(cx) - cxF(c) - c^2F(x) - cF(c)x + xF(c^2). \tag{37}$$

Setting $x_1 = x_2 = c$ and $x_3 = x^2$ in (9), we obtain

$$3F(c^2x^2) = F(c^2)x^2 + 4cF(cx^2) - cx^2F(c) - c^2F(x^2) - cF(c)x^2 + x^2F(c^2).$$

From the aforementioned equation and (36), we obtain

$$\begin{aligned}9F(c^2x^2) = {} & 3F(c^2)x^2 + 8c^2F(x^2) + 8cF(cx)x - 4c^2F(x)x - 4cxF(c)x - 4c^2xF(x) + 8cxF(cx) \\ & - 3cx^2F(c) - 3c^2F(x^2) - 3cF(c)x^2 + 3x^2F(c^2).\end{aligned}$$

Comparing aforementioned equation with (35) and using (37), we obtain

$$0 = c^2F(x^2) - c^2F(x)x - c^2xF(x) - F(c^2)x^2 - x^2F(c^2) + cF(c)x^2 + xF(c^2)x + cx^2F(c). \tag{38}$$

Complete linearization of the the aforementioned equation and setting $x_1 = c$ and $x_2 = x$, we obtain

$$0 = 2c^2F(cx) + c^2F(c)x - 2c^3F(x) + c^2xF(c) - cxF(c^2) - cF(c^2)x.$$

Setting $c = c^2$ in (38) we obtain

$$0 = c^4F(x^2) - c^4F(x)x - c^4xF(x) - F(c^4)x^2 - x^2F(c^4) + c^2F(c^2)x^2 + xF(c^4)x + c^2x^2F(c^2).$$

On other hand, multiplying equation (38) with c^2, we obtain

$$0 = c^4F(x^2) - c^4F(x)x - c^4xF(x) - c^2F(c^2)x^2 - c^2x^2F(c^2) + c^3F(c)x^2 + c^2xF(c^2)x + c^3x^2F(c).$$

Comparing the aforementioned two equations and using $F(c^4) = 3c^2F(c^2) - 2c^3F(c)$, we obtain

$$0 = -F(c^2)x^2 - x^2F(c^2) + cF(c)x^2 + cx^2F(c) + 2xF(c^2)x - 2cxF(c)x. \tag{39}$$

On the other hand, adding the same two equations together, we obtain

$$0 = 2c^2F(x^2) - 2c^2F(x)x - 2c^2xF(x) - 3F(c^2)x^2 - 3x^2F(c^2) + 3cF(c)x^2 + 3cx^2F(c) + 4xF(c^2)x - 2cxF(c)x.$$

The aforementioned equation can be rewritten as follows:

$$\begin{aligned}0 = {} & 2c^2F(x^2) - 2c^2F(x)x - 2c^2xF(x) - 3F(c^2)x^2 - 3x^2F(c^2) + 3cF(c)x^2 + 3cx^2F(c) + 6xF(c^2)x \\ & - 2xF(c^2)x - 6cxF(c)x + 4cxF(c)x.\end{aligned}$$

Using the aforementioned equation and (39), we obtain

$$0 = 2c^2F(x^2) - 2c^2F(x)x - 2c^2xF(x) + 4cxF(c)x - 2xF(c^2)x.$$

The aforementioned equation can now be rewritten as follows:

$$c^2F(x^2) = c^2F(x)x + c^2xF(x) + xF(c^2)x - 2cxF(c)x$$

and

$$4c^2F(x^2) = 4c^2F(x)x + 4c^2xF(x) - 2x(2F(c^2) - 4cF(c))x.$$

Setting $2F(c^2) - 4cF(c) = q$, we arrive at

$$4c^2F(x^2) = 4c^2F(x)x + 4c^2xF(x) - 2xqx. \tag{40}$$

If $q = 0$ than $F(x)$ is derivation, on the other hand, if $q \neq 0$, we can conclude as follows. Let us now introduce the mapping $D : R \to R$ by

$$D(x) = 4F(x) - qx - xq. \tag{41}$$

Obviously, the mapping D is additive. It is our aim to prove that D is a Jordan derivation. Putting x^2 for x in the aforementioned relation, we obtain

$$c^2D(x^2) = 4c^2F(x^2) - c^2qx^2 - c^2x^2q,$$

which gives, after considering the relation (40), the relation

$$c^2D(x^2) = 4c^2F(x)x + 4c^2xF(x) - 2c^2xqx - c^2qx^2 - c^2x^2q. \tag{42}$$

Right (left) multiplication of the relation (41) by x gives, respectively,

$$D(x)x = 4F(x)x - qx^2 - xqx \tag{43}$$

and

$$xD(x) = 4xF(x) - xqx - x^2q. \tag{44}$$

The relations (42)–(44) imply that the additive mapping D satisfies the relation

$$c^2D(x^2) = c^2D(x)x + c^2xD(x)$$

for all $x \in R$, whence it follows $D(x^2) = D(x)x + xD(x)$ for all $x \in R$. In other words, D is a Jordan derivation on R. According to Herstein theorem, one can conclude that D is a derivation, which completes the proof of the theorem. □

Acknowledgements: The author wishes to thank the referees for their useful comments.

Conflict of interest: The authors state no conflict of interest.

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