

Research Article

Omar Bazighifan*, and Sameh Askar

On the oscillation of nonlinear delay differential equations and their applications

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Abstract: The oscillation of nonlinear differential equations is used in many applications of mathematical physics, biological and medical physics, engineering, aviation, complex networks, sociophysics and econophysics. The goal of this study is to create some new oscillation criteria for fourth-order differential equations with delay and advanced terms

$$(a_1(x)(w''''(x))^n)' + \sum_{j=1}^r \beta_j(x)w^k(y_j(x)) = 0,$$

and

$$(a_1(x)(w''''(x))^n)' + a_2(x)h(w''''(x)) + \beta(x)f(w(y(x))) = 0.$$

The method is based on the use of the comparison technique and Riccati method to obtain these criteria. These conditions complement and extend some of the results published on this topic. Two examples are provided to prove the efficiency of the main results.

Keywords: Riccati method, advanced term, oscillation, fourth-order, delay differential equations, damped

1 Introduction

Nonlinear differential equations with delay term contribute to some applications in applications of physics, acoustics, biological and medical physics, engineering, aviation, complex networks, sociophysics and econophysics, see refs [1,2].

* **Corresponding author: Omar Bazighifan**, Section of Mathematics, International Telematic University Uninettuno, CorsoVittorio Emanuele II, 39, 00186 Rome, Italy, e-mail: o.bazighifan@gmail.com

Sameh Askar: Department of Statistics and Operations Research, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia

Nowadays, the questions regarding the study of oscillation criteria of differential equations have become an important area of research [3,4].

The authors in refs [5–7] discuss some oscillation theories of differential equations of second-order and they used different techniques to obtain the qualitative properties of these equations. In refs [8,9] the authors discussed some oscillation criteria for higher-order differential equations and used comparison techniques with equations of different orders. Agarwal *et al.* [10], Chatzarakis and Li [11], and Bazighifan *et al.* [12] obtained many criteria for oscillation and qualitative properties of differential equations of different orders. Conversely, there has been a great research interest about the study of fractional differential equations and numerical solutions to these equations [13–15] and fractional-ordered differential systems, see refs [16–18].

Park *et al.* [19] establish some qualitative properties of the equation:

$$(a_1(x)(w^{(n-1)}(x))^n)' + \beta(x)w^k(y(x)) = 0, \quad (1)$$

where $k \leq n$ and under condition

$$\int_{x_0}^{\infty} a_1^{-1/n}(s)ds = \infty.$$

In ref. [20], Zhang *et al.* obtained some oscillatory conditions for (1) under

$$\int_{x_0}^{\infty} a_1^{-1/n}(s)ds < \infty. \quad (2)$$

Baculikova *et al.* [21] used the comparison technique to find the parameters of the oscillation of the equation:

$$[a_1(x)(z^{(r-1)}(x))^n]' + \beta(x)f(z(y(x))) = 0.$$

Moreover, Moaaz and Muhib [22,23] presented some oscillation theorems of (1). Also, Zhang *et al.* [24] establish oscillation conditions for (1), where $n = k$.

Zhang *et al.* [25] obtained some oscillatory theorems by the following equation:

$$[a_1(x)(w''''(x))^n]' + \beta(x)f(w(y(x))) = 0,$$

where $n = 4$. Bazighifan [26] obtained some oscillatory conditions by the following equation:

$$[a_1(x)(w'''(x))^n]' + \beta(x)w^k(\gamma(x)) = 0.$$

The authors in ref. [27] use the comparison technique to find oscillation theorems of (4), where $n = k = 1$ and under the condition (6).

Motivated by the results we presented earlier, the oscillation conditions for the following equations are studied:

$$(a_1(x)(w'''(x))^n)' + \sum_{j=1}^r \beta_j(x)w^k(\gamma_j(x)) = 0 \tag{3}$$

and

$$(a_1(x)(w'''(x))^n)' + a_2(x)h(w'''(x)) + \beta(x)f(w(\gamma(x))) = 0. \tag{4}$$

Throughout, the following hypotheses have been assumed:

- (H1) $a_1, a_2, \beta \in C([x_0, \infty), [0, \infty))$, $\gamma_j(x) \in C([x_0, \infty), \mathbb{R})$, n and k are quotient of odd positive integers,
- (H2) $a_1'(x) + a_2(x) \geq 0$, $a_1(x) > 0$, $\beta_j(x) > 0$, $\gamma_j(x) \leq x$, $\lim_{x \rightarrow \infty} \gamma_j(x) = \infty$, $j = 1, 2, \dots, r$,
- (H3) $h, f \in (\mathbb{R}, \mathbb{R})$, $h(u) \geq b_h u^n > 0$, $f(u) \geq b_f u^n > 0$ for $u \neq 0$ and b_h, b_f are constants. Moreover, equation (3) has been studied under the condition:

$$\int_{x_0}^{\infty} \frac{1}{a_1^{1/n}(s)} ds = \infty, \tag{5}$$

and equation (4) has been studied under the conditions $\gamma(x) \geq x$ and

$$\int_{x_0}^{\infty} \left[\frac{1}{a_1(s)} \exp \left(- \int_{x_0}^s \frac{a_2(u)}{a_1(u)} du \right) \right]^{1/n} ds < \infty. \tag{6}$$

Definition 1.1. [12] A solution of (3) and (4) is said to be non-oscillatory if it is positive or negative, ultimately; otherwise, it is said to be oscillatory.

Definition 1.2. [12] Equations (3) and (4) are said oscillatory if each of their solutions is oscillatory.

The motivation of studying this article is to obtain new oscillatory properties and to continue the previous works [20,27].

In this article, the Riccati method was used, which depends on reducing the order of the equation. The

comparison technique with a first-order equation has also been relied upon, so these methods give more accurate criteria. These conditions expand and complement some of the previously published results.

2 Oscillation criteria

Here are some lemmas we need to prove the results:

Lemma 2.1. [28] Let $w \in C^m([x_0, \infty), (0, \infty))$ and $w^{(m)}(x)$ is of a fixed sign, on $[x_0, \infty)$ such that, for all $x \geq x_1$,

$$w^{(m-1)}(x)w^{(m)}(x) \leq 0.$$

If we have $\lim_{x \rightarrow \infty} w(x) \neq 0$ and $\lambda \in (0, 1)$, then

$$w(x) \geq \frac{\lambda}{(m-1)!} x^{m-1} |w^{(m-1)}(x)|.$$

Lemma 2.2. [29] If $w^{(r)}(x) > 0$, $r = 0, 1, \dots, r$, and $w^{(r+1)}(x) < 0$, then

$$\frac{w(x)}{x^r/r!} \geq \frac{w'(x)}{x^{r-1}/(r-1)!}.$$

Lemma 2.3. ([3], Lemma 2.2.2) Let $w \in C^m([x_0, \infty), (0, \infty))$ and $w^{(m-1)}(x)w^{(m)}(x) \leq 0$, then

$$w(\theta x) \geq M \theta^{m-1} w^{(m-1)}(x),$$

for $\theta \in (0, 1)$, there exists $M > 0$ and for all sufficient large x .

Lemma 2.4. [30] Let $F > 0$. Then

$$Eu - Fu^{(r+1)/r} \leq \frac{r^r}{(r+1)^{r+1}} E^{r+1} F^{-r}. \tag{7}$$

For convenience, we write the following notations:

$$R(x) := \int_x^{\infty} \left(\frac{1}{a_1(x)} \int_x^{\infty} \sum_{j=1}^r \beta_j(s) ds \right)^{1/n} dx,$$

$$\bar{R}(x) := \mu_2^{k/n} \int_x^{\infty} \left(\frac{1}{a_1(x)} \int_x^{\infty} \sum_{j=1}^r \beta_j(s) \left(\frac{\gamma_j(s)}{s} \right)^k ds \right)^{1/n} dx,$$

$$\zeta_{x_0}(x) := \exp \left(\int_{x_0}^x \frac{a_2(x)}{a_1(x)} dx \right)$$

and

$$\widehat{R}(x) := \mu_2^{k/n} \int_x^\infty \left(\frac{1}{a_1(x)\zeta_{x_0}(x)} \int_x^\infty \zeta_{x_0}(s) \sum_{j=1}^r \beta_j(s) \left(\frac{y_j(s)}{s} \right)^k ds \right)^{1/n} dx,$$

where $\mu_2 \in (0, 1)$.

$$\begin{aligned} \gamma(x_0, x) &= \exp \left(\int_{x_0}^x \frac{a_2(u)}{a_1(u)} du \right), \\ \delta(x) &= \int_x^\infty \frac{ds}{(a_1(s)\gamma(x_0, s))^{1/n}}, \\ \chi(x) &= \frac{\zeta'(x)}{\zeta(x)} - \frac{b_n a_2(x)}{a_1(x)}, \\ \sigma(x) &= \frac{1}{\gamma^{1/n}(x_0, x)} - \frac{\delta(x)a_2(x)a_1^{1-n/n}(x)}{n} \end{aligned}$$

and

$$\tilde{\sigma}(x) = \frac{a_2(x)}{a_1(x)} + \frac{n^{(n+1)}\zeta(x)\sigma^{n+1}(x)\gamma(x_0, x)}{\delta(x)a_1^{1/n}(x)}.$$

Lemma 2.5. *Let w is an eventually positive solution of (3), then $w' > 0$ and $w''' > 0$.*

Theorem 2.6. *If*

$$y'(x) + \frac{\lambda^k \sum_{j=1}^r \beta_j(x) y_j^{3k}(x)}{6^k a_1^{k/n}(y_j(x))} y^{k/n}(y_j(x)) = 0 \quad (8) \quad \text{and}$$

is oscillatory, then (3) is oscillatory.

Proof. Suppose that (3) has a nonoscillatory solution in $[x_0, \infty)$. Then $w(x) > 0$ and $w(y_j(x)) > 0$ for $x \geq x_1$. Let

$$y(x) := a_1(x)(w'''(x))^n > 0 \quad [\text{by Lemma 2.5}].$$

Then from (3), we obtain

$$y'(x) + \sum_{j=1}^r \beta_j(x) w^k(y_j(x)) = 0. \quad (9)$$

$\lim_{x \rightarrow \infty} w(x) \neq 0$. By Lemma 2.1, we see

$$w^k(y_j(x)) \geq \frac{\lambda^k}{6^k} y_j^{3k}(x) (w'''(y_j(x)))^k, \quad (10)$$

for all $\lambda \in (0, 1)$. By (9) and (10), we see that

$$y'(x) + \frac{\lambda^k}{6^k} \sum_{j=1}^r \beta_j(x) y_j^{3k}(x) (w'''(y_j(x)))^k \leq 0.$$

So, we obtain $y(x) > 0$ and

$$y'(x) + \frac{\lambda^k \sum_{j=1}^r \beta_j(x) y_j^{3k}(x)}{6^k a_1^{k/n}(y_j(x))} y^{k/n}(y_j(x)) \leq 0.$$

From ([31], Theorem 1), equation (8) is nonoscillatory. This is a contradiction. Proof completed. \square

Corollary 2.7. *If $n = k$ and*

$$\liminf_{x \rightarrow \infty} \int_{y(x)}^x \frac{\lambda^k \sum_{j=1}^r \beta_j(s) y_j^{3k}(s)}{6^k a_1^{k/n}(y_j(s))} ds > \frac{1}{e}, \quad (11)$$

then (3) is oscillatory.

Lemma 2.8. *If*

$$\int_{x_0}^\infty \left(M^{k-n} \pi(x) \sum_{j=1}^r \beta_j(x) \frac{y_j^{3n}(x)}{x^{3n}} - \frac{2^n}{(n+1)^{n+1}} \frac{a_1(x)(\pi'(x))^{n+1}}{\mu^n x^{2n} \pi^n(x)} \right) ds = \infty, \quad (12)$$

then $w'' < 0$.

Proof. Let $w''(x) > 0$. From Lemmas 2.1 and 2.2, we find

$$\frac{w(y_j(x))}{w(x)} \geq \frac{y_j^3(x)}{x^3} \quad (13)$$

$$w'(x) \geq \frac{\mu}{2} x^2 w'''(x). \quad (14)$$

Let

$$z_1(x) := \pi(x) \frac{a_1(x)(w'''(x))^n}{w^n(x)} > 0. \quad (15)$$

From (13)–(15), we find

$$\begin{aligned} z_1'(x) &\leq \frac{\pi'(x)}{\pi(x)} z_2(x) - \pi(x) \sum_{j=1}^r \beta_j(x) \frac{y_j^{3n}(x)}{x^{3n}} w^{k-n}(y_j(x)) \\ &\quad - \frac{n\mu}{2} \frac{x^2}{\pi^{1/n}(x) a_1^{1/n}(x)} z_1^{1+1/n}(x). \end{aligned} \quad (16)$$

Since $w'(x) > 0$. From Lemma 2.4 with $E = \pi'/\pi$, $F = n\mu x^2 / (2a_1^{1/n}(x)\pi^{1/n}(x))$, and $u = z_1$, we see that

$$\begin{aligned} z_1'(x) &\leq -M^{k-n} \pi(x) \sum_{j=1}^r \beta_j(x) \frac{y_j^{3n}(x)}{x^{3n}} \\ &\quad + \frac{2^n}{(n+1)^{n+1}} \frac{a_1(x)(\pi'(x))^{n+1}}{\mu^n x^{2n} \pi^n(x)}. \end{aligned}$$

This implies that

$$\int_{x_1}^x \left(M^{k-n} \pi(x) \sum_{j=1}^r \beta_j(x) \frac{y_j^{3n}(x)}{x^{3n}} - \frac{2^n}{(n+1)^{n+1}} \frac{a_1(x)(\pi'(x))^{n+1}}{\mu^n x^{2n} \pi^n(x)} \right) ds \leq z_1(x_1),$$

This is a contradiction (12). Proof completed. \square

Theorem 2.9. *If*

$$u''(x) + M^{k-n} \tilde{R}(x)u(x) = 0 \tag{17}$$

is oscillatory, then (3) is oscillatory.

Proof. From Theorem 2.6 and by Lemmas 2.1 and 2.5, we find

$$w'(x) > 0, \quad w''(x) < 0 \text{ and } w'''(x) > 0. \tag{18}$$

Integrating (3), we find

$$\begin{aligned} a_1(b)(w'''(b))^n &= a_1(x)(w'''(x))^n - \int_x^b \sum_{j=1}^r \beta_j(s) w^k(y_j(s)) ds. \end{aligned} \tag{19}$$

By Lemma 3 in ref [30] with (18), we obtain

$$\frac{w(y_j(x))}{w(x)} \geq \lambda \frac{y_j(x)}{x},$$

which with (19) gives

$$\begin{aligned} a_1(b)(w'''(b))^n - a_1(x)(w'''(x))^n &+ \lambda^k \int_x^b \sum_{j=1}^r \beta_j(s) \left(\frac{y_j(s)}{s} \right)^k w^k(s) ds \leq 0. \end{aligned}$$

By $w' > 0$, we see

$$\begin{aligned} a_1(b)(w'''(b))^n - a_1(x)(w'''(x))^n &+ \lambda^k w^k(x) \int_x^b \sum_{j=1}^r \beta_j(s) \left(\frac{y_j(s)}{s} \right)^k ds \leq 0. \end{aligned} \tag{20}$$

Taking $b \rightarrow \infty$, we find

$$-a_1(x)(w'''(x))^n + \lambda^k w^k(x) \int_x^\infty \sum_{j=1}^r \beta_j(s) \left(\frac{y_j(s)}{s} \right)^k ds \leq 0,$$

that is,

$$w'''(x) \geq \frac{\lambda^{k/n}}{a_1^{1/n}(x)} w^{k/n}(x) \left(\int_x^\infty \sum_{j=1}^r \beta_j(s) \left(\frac{y_j(s)}{s} \right)^k ds \right)^{1/n}.$$

Integrating from x to ∞ , we see

$$-w''(x) \geq \lambda^{k/n} w^{k/n}(x) \int_x^\infty \left(\frac{1}{a_1(x)} \int_x^\infty \sum_{j=1}^r \beta_j(s) \left(\frac{y_j(s)}{s} \right)^k ds \right)^{1/n} dx.$$

Hence,

$$w''(x) \leq -\tilde{R}(x)w^{k/n}(x). \tag{21}$$

Let

$$z_2(x) = \frac{w'(x)}{w(x)},$$

then $z_2(x) > 0$ for $x \geq x_1$, and

$$z_2'(x) = \frac{w''(x)}{w(x)} - \left(\frac{w'(x)}{w(x)} \right)^2.$$

From (21), we find

$$z_2'(x) \leq -\tilde{R}(x) \frac{w^{k/n}(x)}{w(x)} - z_2^2(x). \tag{22}$$

Since $w'(x) > 0$. Thus, (22) becomes

$$z_2'(x) + z_2^2(x) + M^{k-n} \tilde{R}(x) \leq 0, \tag{23}$$

From [32], equation (17) is nonoscillatory, and this is a contradiction. Proof completed. \square

Theorem 2.10. *Suppose that $k \geq n$ and*

$$\left(\frac{1}{y_j'(x)} u'(x) \right)' + M^{k/n-1} R(x)u(x) = 0 \tag{24}$$

is oscillatory, and then (3) is oscillatory.

Proof. Suppose that (12) and (19) hold. So, we see $y_j'(x) \geq 0$ and $w'(x) \geq 0$.

$$\begin{aligned} a_1(b)(w'''(b))^n - a_1(x)(w'''(x))^n &+ w^k(y_j(x)) \int_x^b \sum_{j=1}^r \beta_j(s) ds \leq 0. \end{aligned} \tag{25}$$

Thus, (18) becomes

$$w''(x) \leq -R(x)w^{k/n}(y_j(x)). \tag{26}$$

Let

$$z_3(x) = \frac{w'(x)}{w(y_j(x))}, \tag{27}$$

then $z_3(x) > 0$ for $x \geq x_1$, and

$$z_3'(x) = \frac{w''(x)}{w(y_j(x))} - \frac{w'(x)}{w^2(y_j(x))}w'(y_j(x))y_j'(x) \leq \frac{w''(x)}{w(y_j(x))} - y_j'(x) \left(\frac{w'(x)}{w(y_j(x))} \right)^2.$$

From (26) and (27), we find

$$z_3'(x) + M^{k/n-1}R(x) + y_j'(x)z_3^2(x) \leq 0. \tag{28}$$

From [32], (24) is nonoscillatory, and this is a contradiction. This completes the proof. \square

Lemma 2.11. ([24], Theorem 2.1) *Suppose that w is an eventually positive solution of (4). Then, there exist two possible cases:*

- (N₁) $w(x) > 0, w'(x) > 0, w'''(x) > 0, w^{(4)}(x) < 0;$
- (N₂) $w(x) > 0, w''(x) > 0, w'''(x) < 0.$

for $x \geq x_1$.

Lemma 2.12. *Let $w(x) > 0$ and (N₁) holds. If*

$$z_4(x) = \zeta(x) \frac{a_1(x)(w''''(x))^n}{w^n(x/2)} > 0, \tag{29}$$

where $\zeta \in C^1([x_0, \infty), (0, \infty))$ and $M > 0$ is a constant, then

$$z_4'(x) \leq -b_f \zeta(x) \beta(x) + \chi(x)z_4(x) - \frac{nMt^2}{2(a_1(x)\zeta(x))^{1/n}} z_4^{n+1}(x). \tag{30}$$

Proof. Suppose that $w(x) > 0$ and using Lemma 2.11, we get (N₁) holds. From Lemma 2.3, we see

$$w'(x/2) \geq Mt^2 w'''(x). \tag{31}$$

From $z_4(x)$, we obtain

$$z_4'(x) = \zeta'(x) \frac{a_1(x)(w''''(x))^n}{w^n(x/2)} + \zeta(x) \frac{(a_1(w''''(x)))'(x)}{w^n(x/2)} - n\zeta(x) \frac{w'(x/2)a_1(x)(w''''(x))^n}{2w^{n+1}(x/2)}.$$

Using (29) and (31), we find

$$z_4'(x) \leq \frac{\zeta'(x)}{\zeta(x)} z_4(x) + \zeta(x) \frac{(a_1(w''''(x)))'(x)}{w^n(x/2)} - nMt^2 \zeta(x) \frac{a_1(x)(w''''(x))^{n+1}(x)}{2w^{n+1}(x/2)}.$$

From (4), we see

$$z_4'(x) \leq \frac{\zeta'(x)}{\zeta(x)} z_4(x) - b_h a_2(x) \frac{z_4(x)}{a_1(x)} - b_f \zeta(x) \beta(x) \frac{w^n(y(x))}{w^n(x/2)} - nMt^2 \frac{z_4^{n+1}(x)}{2(\zeta(x)a_1(x))^{1/n}} \leq -b_f \zeta(x) \beta(x) + \left(\frac{\zeta'(x)}{\zeta(x)} - b_h \frac{a_2(x)}{a_1(x)} \right) z_4(x) - nMt^2 \frac{z_4^{n+1}(x)}{2(\zeta(x)a_1(x))^{1/n}}.$$

Hence, we find

$$z_4'(x) \leq -b_f \zeta(x) \beta(x) + \chi(x)z_4(x) - nMt^2 \frac{z_4^{n+1}(x)}{2(\zeta(x)a_1(x))^{1/n}}.$$

The proof is complete. \square

Lemma 2.13. *Let $b_f > 1$ is a constant and (N₂) holds. If*

$$z_5(x) = -\frac{a_1(x)(-w''''(x))^n}{(w''''(x))^n} < 0, \tag{32}$$

then

$$z_5'(x) \leq \frac{b_h a_2(x)}{a_1(x) \delta^n(x) \gamma(x_0, x)} - b_f \beta(x) \left(\frac{\mu}{2} \gamma^2(x) \right)^n - n \frac{z_5^{n+1}(x)}{a_1^n(x)}. \tag{33}$$

Proof. Let that $w(x) > 0$ and (N₂) holds. Since

$$\begin{aligned} & (-a_1(x)(-w''''(x))^n \gamma(x_0, x))' \\ &= (-a_1(x)(-w''''(x))^n)' \gamma(x_0, x) \\ &+ (-a_1(x)(-w''''(x))^n) \gamma'(x_0, x) \frac{a_2(x)}{a_1(x)} \\ &= (-1)^{n+1} (-a_2(x)h(w''''(x)) - \beta(x)f(w(y(x)))) \gamma(x_0, x) \\ &- a_2(x)(-w''''(x))^n \gamma(x_0, x) \\ &\leq (-1)^{n+1} (-b_h a_2(x)(w''''(x))^n - b_f \beta(x)w^n(y(x))) \gamma(x_0, x) \\ &- a_2(x)(-w''''(x))^n \gamma(x_0, x) \\ &= (-a_2(x)(-w''''(x))^n(1 - b_h) \\ &+ b_f \beta(x)(-w^n(y(x)))) \gamma(x_0, x) \\ &= (-1)^n (-a_2(x)(w''''(x))^n(1 - b_h) \\ &+ b_f \beta(x)(w^n(y(x)))) \gamma(x_0, x) \\ &\leq -b_f \beta(x)w^n(y(x)) \gamma(x_0, x) < 0, \end{aligned}$$

we deduce that $-a_1(x)(-w''''(x))^n \gamma(x_0, x)$ is decreasing. Thus, for $s \geq x \geq x_1$,

$$(a_1(s)\gamma(x_0, s))^{1/n} w''''(s) \leq (a_1(x)\gamma(x_0, x))^{1/n} w''''(x). \tag{34}$$

Dividing both sides of (34) by $(a_1(s)\gamma(x_0, s))^{1/n}$ and integrating from x to b , we get

$$w''(b) \leq w''(x) + (a_1(s)\gamma(x_0, s))^{1/n} w'''(x) \int_x^b \frac{ds}{(a_1(s)\gamma(x_0, s))^{1/n}}.$$

Letting $u \rightarrow \infty$, we arrive that

$$0 \leq w''(x) + (a_1(x)\gamma(x_0, x))^{1/n} w'''(x) \delta(x),$$

which yields

$$-\frac{w'''(x)}{w''(x)} \delta(x) (a_1(x)\gamma(x_0, x))^{1/n} \leq 1.$$

Hence,

$$\frac{a_1(x)(w'''(x))^n}{(w''(x))^n} \geq \frac{-1}{\delta^n(x)\gamma(x_0, x)}.$$

From (32), we find

$$z_5(x) \geq \frac{-1}{\delta^n(x)\gamma(x_0, x)}. \tag{35}$$

and

$$z_5'(x) = \frac{(-a_1(x)(-w'''(x))^n)'}{(w''(x))^n} - n \frac{-a_1(x)(-w'''(x))^{n+1}}{(w''(x))^{n+1}}.$$

From (4) and (32), we obtain

$$\begin{aligned} z_5'(x) &= -b_h \frac{a_2(x)}{a_1(x)} z_5(x) - b_f \beta(x) \frac{w^n(\gamma(x))}{(w''(x))^n} - n \frac{z_5^{\frac{n+1}{n}}(x)}{a_1^{\frac{1}{n}}(x)} \\ &= -b_h \frac{a_2(x)}{a_1(x)} z_5(x) - b_f \beta(x) \frac{w^n(\gamma(x))}{(w''(\gamma(x)))^n} \frac{(w''(\gamma(x)))^n}{(w''(x))^n} \\ &\quad - n \frac{z_5^{\frac{n+1}{n}}(x)}{a_1^{\frac{1}{n}}(x)}. \end{aligned} \tag{36}$$

By Lemma 2.1, we find

$$w(x) \geq \frac{\mu}{2} x^2 w''(x). \tag{37}$$

Thus, from (35) and (37), we see

$$z_5'(x) \leq \frac{b_h a_2(x)}{a_1(x) \delta^n(x) \gamma(x_0, x)} - b_f \beta(x) \left(\frac{\mu}{2} \gamma^2(x)\right)^n - n \frac{z_5^{\frac{n+1}{n}}(x)}{a_1^{\frac{1}{n}}(x)}.$$

The proof is complete. □

Theorem 2.14. *Let the functions $\zeta, \vartheta \in C^1([x_0, \infty), (0, \infty))$ such that*

$$\limsup_{x \rightarrow \infty} \int_{x_0}^x \left(b_f \zeta(s) \beta(s) - \left(\frac{2}{Ms^2} \right)^n \frac{a_1(s) \zeta(s) (\chi(s))^{n+1}}{(n+1)^{n+1}} \right) ds = \infty. \tag{38}$$

$$\frac{\vartheta(x)}{\delta(x)(a_1(x)\gamma(x_0, x))^{1/n}} + \vartheta'(x) \leq 0 \tag{39}$$

and, for $M > 0$ and $\mu \in (0, 1)$,

$$\limsup_{x \rightarrow \infty} \int_{x_0}^x \left(b_f \beta(s) \left(\frac{\mu \gamma^2(s)}{2} \frac{\vartheta(\gamma(s))}{\vartheta(s)} \delta(s) \right)^n \gamma(x_0, s) - \tilde{\sigma}(s) \right) ds = \infty, \tag{40}$$

then every solution of (4) is oscillatory.

Proof. Proceeding as in the proof of Theorem 2.6.

For case (N_1) . By Lemma 2.12, we find (30) holds. From Lemma 2.4, we set

$$F = \chi(x), \quad E = nMt^2 / (2(a_1(x)\zeta(x))^{1/n}) \text{ and } u = z_4,$$

we have

$$z_4'(x) \leq -b_f \zeta(x) \beta(x) + \left(\frac{2}{Mt^2} \right)^n \frac{a_1(x) \zeta(x) (\chi(x))^{n+1}}{(n+1)^{n+1}}. \tag{41}$$

Integrating from x_1 to x , we obtain

$$\int_{x_1}^x \left(b_f \zeta(s) \beta(s) - \left(\frac{2}{Ms^2} \right)^n \frac{a_1(s) \zeta(s) (\chi(s))^{n+1}}{(n+1)^{n+1}} \right) ds \leq z_4(x_1),$$

which contradicts (38).

For case (N_2) . From the proof of Lemma 2.13, we see

$$\frac{w'''(x)}{w''(x)} \geq \frac{-1}{\delta(x)(a_1(x)\gamma(x_0, x))^{1/n}}.$$

From (39), we obtain

$$\begin{aligned} \left(\frac{w''(x)}{\vartheta(x)} \right)' &= \frac{w'''(x)\vartheta(x) - w''(x)\vartheta'(x)}{\vartheta^2(x)} \\ &\geq \frac{w''(x)}{\vartheta^2(x)} \left(\frac{\vartheta(x)}{\delta(x)(a_1(x)\gamma(x_0, x))^{1/n}} + \vartheta'(x) \right) \geq 0, \end{aligned}$$

and this means that $w''(x)/\vartheta(x)$ is nondecreasing. So, it follows from $\gamma(x) \geq x$ that

$$\frac{w''(\gamma(x))}{w''(x)} \geq \frac{\vartheta(\gamma(x))}{\vartheta(x)}.$$

Thus, by (36) and (37), we get

$$z_5'(x) \leq \frac{b_h a_2(x)}{a_1(x) \delta^n(x) \gamma(x_0, x)} - b_f \beta(x) \left(\frac{\mu}{2} \gamma^2(x)\right)^n \left(\frac{\vartheta(\gamma(x))}{\vartheta(x)}\right)^n - n \frac{z_5^{\frac{n+1}{n}}(x)}{a_1^{\frac{1}{n}}(x)}. \tag{42}$$

Multiplying (42) by $\delta^n(x) \gamma(x_0, x)$ and integrating from x_1 to x , we get

$$\begin{aligned} & \delta^n(x) \gamma(x_0, x) z_5(x) - \delta^n(x_1) \gamma(x_0, x_1) z_5(x_1) - \int_{x_1}^x \frac{a_2(s)}{a_1(s)} ds \\ & + n \int_{x_1}^x a_1^{-\frac{1}{n}}(s) \delta^{n-1}(s) \gamma(x_0, s) \sigma(s) z_5(s) ds \\ & + \int_{x_1}^x b_f \beta(s) \left(\frac{\mu}{2} \gamma^2(s)\right)^n \left(\frac{\vartheta(\gamma(s))}{\vartheta(s)}\right)^n \delta^n(s) \gamma(x_0, s) ds \\ & + n \int_{x_1}^x \frac{z_5^{\frac{n+1}{n}}(s)}{a_1^{\frac{1}{n}}(s)} \delta^n(s) \gamma(x_0, s) ds \\ & \leq 0. \end{aligned}$$

Using Lemma 2.4, we set

$$E = \delta^n(s) \gamma(x_0, s) / a_1^{\frac{1}{n}}(s), \quad F = \int_{x_1}^x a_1^{-\frac{1}{n}}(s) \delta^{n-1}(s) \gamma(x_0, s) \sigma(s),$$

$$u = z_5(x).$$

Thus, we get

$$\begin{aligned} & \delta^n(x) \gamma(x_0, x) z_5(x) - \delta^n(x_1) \gamma(x_0, x_1) z_5(x_1) - \int_{x_1}^x \frac{a_2(s)}{a_1(s)} ds \\ & + \int_{x_1}^x b_f \beta(s) \left(\frac{\mu}{2} \gamma^2(s)\right)^n \left(\frac{\vartheta(\gamma(s))}{\vartheta(s)}\right)^n \delta^n(s) \gamma(x_0, s) ds \\ & + \int_{x_1}^x \frac{n^{(n+1)} \zeta(s) \sigma^{n+1}(s) \gamma(x_0, s)}{\delta(s) a_1^{\frac{1}{n}}(x)} ds \\ & \leq 0. \end{aligned}$$

Hence, by (35), we obtain

$$\begin{aligned} & \int_{x_1}^x \left(b_f \beta(s) \left(\frac{\mu \gamma^2(s)}{2} \frac{\vartheta(\gamma(s))}{\vartheta(s)}\right)^n \delta(s) \right) \gamma(x_0, s) - \tilde{\sigma}(s) ds \\ & \leq \delta^n(x) \gamma(x_0, x) z_5(x_1) + 1, \end{aligned}$$

which contradicts (40).

Theorem 2.14 is proved.

Example 2.15. Let the equation:

$$(x^3(w'''(x)))^3)' + \frac{\beta_0}{x^7} w^3(\gamma x) = 0, \tag{43}$$

where $x \geq 1, \gamma \in (0, 1]$ and $\beta_0 > 0$. Let $n = k = 3, \gamma_j(x) = \gamma x, a_1(x) = x^3$, and $\beta(x) = \beta_0/x^7$. So, we obtain

$$\bar{R}(x) = \lambda \left(\frac{\beta_0}{6}\right)^{1/3} \gamma \frac{1}{2x^2}.$$

By Corollary 2.7, we find (43) is oscillatory if

$$\beta_0 > \frac{6^3}{e \left(\ln \frac{1}{\gamma}\right) \gamma^6},$$

$$\beta_0 > \left(\frac{3^4}{2}\right) \frac{1}{\gamma^9},$$

and

$$\beta_0 > 6 \left(\frac{1}{4\gamma}\right)^3.$$

So, equation (43) is oscillatory if

$$\beta_0 > \max \left\{ \left(\frac{3^4}{2}\right) \frac{1}{\gamma^9}, 6 \left(\frac{1}{4\gamma}\right)^3 \right\} = \left(\frac{3^4}{2}\right) \frac{1}{\gamma^9}. \tag{44}$$

Example 2.16. Let the equation

$$(x^2(w'''(x)))' + \frac{x}{2} w'''(x) + \frac{\beta_0}{x^2} w(2x) = 0, \tag{45}$$

where $\beta_0 > 0$ is a constant. Let $n = 1, x_0 = 1, a_1(x) = x^2, a_2(x) = x/2, \beta(x) = x$, and $\gamma(x) = 2x$, we now set $\zeta(x) = x, b_h = b_f = 1$, then

$$\gamma(x_0, x) = \exp \left(\int_{x_0}^x \frac{a_2(u)}{a_1(u)} du \right) = x^{1/2},$$

$$\delta(x) = \int_x^\infty \frac{ds}{(a_1(s) \gamma(x_0, s))^{\frac{1}{n}}} = \frac{2x^{-3/2}}{3}, \quad \vartheta(x) = \frac{2x^{-3/2}}{3},$$

$$\sigma(x) = \frac{1}{\gamma^{\frac{1}{n}}(x_0, x)} - \frac{\delta(x) a_2(x) a_1^{1-n/n}(x)}{n} = \frac{2x^{-1/2}}{3},$$

$$\chi(x) = \frac{-1}{2x},$$

and

$$\tilde{\sigma}(x) = \frac{a_2(x)}{a_1(x)} + \frac{n^{(n+1)} \zeta(x) \sigma^{n+1}(x) \gamma(x_0, x)}{\delta(x) a_1^{\frac{1}{n}}(x)} = \frac{2x^{-1/3}}{3}.$$

Thus, we get

$$\square \quad \limsup_{x \rightarrow \infty} \int_{x_0}^x \left(b_f \zeta(s) \beta(s) - \left(\frac{2}{Ms^2}\right)^n a_1(s) \zeta(s) \chi(s)^{n+1} \right) ds = \infty$$

and

$$\frac{\vartheta(x)}{\delta(x)(a_1(x)\gamma(x_0, x))^{1/n}} + \vartheta'(x) = 0.$$

Also,

$$\limsup_{x \rightarrow \infty} \int_{x_0}^x \left(b_f \beta(s) \left(\frac{\mu \gamma^2(s)}{2} \frac{\vartheta(\gamma(s))}{\vartheta(s)} \delta(s) \right)^n \gamma(x_0, s) - \bar{\sigma}(s) \right) ds = \infty.$$

By Theorem 2.14, we find (45) is oscillatory.

3 Conclusion

In this article, by using the Riccati method and the comparison technique, some new oscillation conditions for fourth-order differential equations are established with delay and advanced terms. Conditions complement and extend some of the results published on this topic. Two examples are discussed to illustrate the efficiency of the main results. Furthermore, nonlinear equations contribute in many applications of mathematical physics, biological and medical physics, engineering, complex networks, aviation, sociophysics and econophysics. The future work is to further study the oscillatory properties of neutral differential equations with p -Laplacian:

$$(a_1(x)(w'''(x))^{p-1})' + \sum_{j=1}^r \beta_j(x) w^{p-1}(\gamma_j(x)) = 0$$

and

$$(a_1(x)(w'''(x))^{p-1})' + a_2(x)h(w'''(x)) + \beta(x)f(w(\gamma(x))) = 0.$$

Under the assumptions:

$$\int_{x_0}^{\infty} \frac{1}{a_1^{1/p-1}(s)} ds < \infty$$

and

$$\int_{x_0}^{\infty} \left[\frac{1}{a_1(s)} \exp \left(- \int_{x_0}^s \frac{a_2(u)}{a_1(u)} du \right) \right]^{1/p-1} ds = \infty,$$

where $p > 1$.

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References

- [1] Hale JK. Theory of functional differential equations. New York, NY, USA: Springer; 1977.
- [2] Agarwal RP, Bohner M, Li T, Zhang C. A new approach in the study of oscillatory behavior of even-order neutral delay differential equations. Appl Math Comput. 2013;225:787–94.
- [3] Agarwal R, Grace S, O'Regan D. Oscillation theory for difference and functional differential equations. Dordrecht, The Netherlands: Kluwer Academic Publisher; 2000.
- [4] Saker S. Oscillation theory of delay differential and difference equations: second and third orders. LAP Lambert Academic Publishing; 2010.
- [5] Baculikova B. Oscillation of second-order nonlinear noncanonical differential equations with deviating argument. Appl Math Lett. 2019;91:68–75.
- [6] Dzirna J, Jadlovská I. A note on oscillation of second-order delay differential equations. Appl Math Lett. 2017;69:126–32.
- [7] Bohner M, Grace SR, Jadlovská I. Sharp oscillation criteria for second-order neutral delay differential equations. Math Meth Appl Sci. 2020;43:10041–53.
- [8] Xing G, Li T, Zhang C. Oscillation of higher-order quasi linear neutral differential equations. Adv Differ Equ. 2011;2011:1–10.
- [9] Moaaz O, Awrejcewicz J, Bazighifan O. A new approach in the study of oscillation criteria of even-order neutral differential equations. Mathematics. 2020;8:179.
- [10] Agarwal RP, Bazighifan O, Ragusa MA. Nonlinear neutral delay differential equations of fourth-order: oscillation of solutions. Entropy. 2021;23:129.
- [11] Chatzarakis GE, Li T. Oscillation criteria for delay and advanced differential equations with nonmonotone arguments. Complexity. 2018;2018:8237634.
- [12] Bazighifan O, Ghanim F, Awrejcewicz J, Al-Ghafri KS, Al-Kandari M. New criteria for oscillation of half-linear differential equations with p -laplacian-like operators. Mathematics. 2021;9:2584.
- [13] Jarad F, Abdeljawad T, Hammouch Z. On a class of ordinary differential equations in the frame of Atangana-Baleanu fractional derivative. Chaos Soliton Fractal. 2018;117:16–20.
- [14] Atangana A, Hammouch Z. Fractional calculus with power law: The cradle of our ancestors. Europ Phys J Plus. 2019;134(9):429.
- [15] Ghanim F, Al-Janaby HF, Bazighifan O. Some new extensions on fractional differential and integral properties for Mittag-

- Leffler confluent hypergeometric function. *Fractal Fract.* 2021;5:143.
- [16] Valliammal N, Ravichandran C, Hammouch Z, Baskonus HM. A new investigation on fractional-ordered neutral differential systems with state-dependent delay. *Int J Nonlinear Sci Numer Simulat.* 2019;20(7–8):803–9.
- [17] Baskonus HM, Hammouch Z, Mekkaoui T, Bulut H. Chaos in the fractional order logistic delay system: circuit realization and synchronization. In *AIP Conference Proceedings*. Vol. 1738. No. 1. AIP Publishing LLC; 2016. p. 290005.
- [18] Jothimani K, Kaliraj K, Hammouch Z, Ravichandran C. New results on controllability in the framework of fractional integrodifferential equations with nondense domain. *Europ Phys J Plus.* 2019;134(9):441.
- [19] Park C, Moaaz O, Bazighifan O. Oscillation results for higher order differential equations. *Axioms.* 2020;9:14.
- [20] Zhang C, Li T, Suna B, Thandapani E. On the oscillation of higher-order half-linear delay differential equations. *Appl Math Lett.* 2011;24:1618–21.
- [21] Baculikova B, Dzurina J, Graef JR. On the oscillation of higher-order delay differential equations. *Math Slovaca.* 2012;187:387–400.
- [22] Moaaz O, Muhib A. New oscillation criteria for nonlinear delay differential equations of fourth-order. *Appl Math Comput.* 2020;377:125192.
- [23] Agarwal RP, Zhang C, Li T. Some remarks on oscillation of second order neutral differential equations. *Appl Math Comput.* 2016;274:178–81.
- [24] Zhang C, Agarwal RP, Bohner M, Li T. New results for oscillatory behavior of even-order half-linear delay differential equations. *Appl Math Lett.* 2013;26:179–83.
- [25] Zhang C, Li T, Saker S. Oscillation of fourth-order delay differential equations. *J Math Sci.* 2014;201:296–308.
- [26] Bazighifan O. On the oscillation of certain fourth-order differential equations with p -Laplacian like operator. *Appl Math Comput.* 2020;386:125475.
- [27] Elabbasy EM, Thandapani E, Moaaz O, Bazighifan O. Oscillation of solutions to fourth-order delay differential equations with middle term. *Open J Math Sci.* 2019;3:191–7.
- [28] El-Deeb AA-M, Bazighifan O, Awrejcewicz J. A variety of dynamic Steffensen-type inequalities on a general time scale. *Symmetry.* 2021;13:1738.
- [29] Chatzarakis GE, Grace SR, Jadlovská I, Li T, Tunç E. Oscillation criteria for third-order Emden-Fowler differential equations with unbounded neutral coefficients. *Complexity.* Vol. 2019; 2019. p. 5691758.
- [30] Bazighifan O, Almutairi A, Almarri B, Marin M. An oscillation criterion of nonlinear differential equations with advanced term. *Symmetry.* 2021;13:843.
- [31] Philos CG. On the existence of nonoscillatory solutions tending to zero at ∞ for differential equations with positive delays. *Arch Math.* 1981;36:168–78.
- [32] Agarwal R, Shieh SL, Yeh CC. Oscillation criteria for second order retarded differential equations. *Math Comput Model.* 1997;26:1–11.