Research Article

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Best proximity points in $\mathcal{F}$-metric spaces with applications

https://doi.org/10.1515/dema-2022-0191
received December 6, 2021; accepted November 30, 2022

Abstract: The aim of this article is to introduce α-ψ-proximal contraction in the setting of $\mathcal{F}$-metric space and prove the existence of best proximity points for these contractions. As applications of our main results, we obtain coupled best proximity points on $\mathcal{F}$-metric space equipped with an arbitrary binary relation.

Keywords: best proximity point, α-ψ-proximal contraction, non-self mappings, coupled best proximity point

MSC 2020: 46S40, 47H10, 54H25

1 Introduction

The Banach contraction principle [1] is the first result which was introduced by Stefan Banach in 1922 in which notions of fixed point and metric space play an important role. Let Θ, Φ be non-empty subsets of metric space $(\mathcal{M}, \sigma)$. A point $\omega \in \mathcal{M}$ is called a fixed point of $\mathcal{L} : \Theta \to \Phi$ if $\mathcal{L}\omega = \omega$. Because of significance and simplicity of the concept of the fixed point, this conception has been improved and lengthened in many distinct ways. In 2010, Basha [2] gave the conception of best proximity point and extended the famous Banach's contraction principle. For more particular on this perspective, we refer the readers to [3–15]. On the other hand, the well-known extensions of the notion of metric spaces have been done by Bakhtin [16] which was conventionally given by Czerwik [17] in 1993 by generalizing the Banach contraction principle. Jleli and Samet [18] introduced a novel metric space known as $\mathcal{F}$-metric space to extend the classical metric space and b-metric space. Later on, Al-Mezel et al. [19] introduced the notion of generalized $(\alpha\beta$-$\psi)$-contractions in $\mathcal{F}$-metric spaces with the help of α-admissibility of the mapping and obtained fixed-point results in this generalized space.

In this article, we introduce α-ψ-proximal contraction in the background of $\mathcal{F}$-metric space and prove the existence of best proximity points for these contractions.

2 Preliminaries

We state this section with definition of b-metric space in this manner.

Definition 2.1. (See [17]). Let $\mathcal{M} \neq \varnothing$ and $s \geq 1$. A function $\sigma_b : \mathcal{M} \times \mathcal{M} \to [0, \infty)$ is called a b-metric if the following assertions hold:
(b1) $\sigma_b(\omega, \varpi) = 0$ if and only if $\omega = \varpi$;

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(b2) $\sigma_b(\omega, \varpi) = \sigma_b(\varpi, \omega)$,
(b3) $\sigma_b(\omega, \nu) \le s[\sigma_b(\omega, \varpi) + \sigma_b(\varpi, \nu)]$,
$\forall \omega, \varpi, \nu \in \mathcal{M}$.

The pair $(\mathcal{M}, \sigma_b)$ is called a b-metric space.

Jleli and Samet [18] established a fascinating extension of a metric space as follows.
Let $\mathcal{F}$ be the set of functions $f : (0, +\infty) \to \mathbb{R}$ satisfying the following conditions:
($\mathcal{F}$1) $0 < s < \iota \Rightarrow f(s) \le f(\iota)$,
($\mathcal{F}$2) for all $\{\iota_n\} \subseteq \mathbb{R}^+$, $\lim_{n\to\infty}\iota_n = 0 \iff \lim_{n\to\infty} f(\iota_n) = -\infty$.

Definition 2.2. [18] Let $\mathcal{M} \neq \emptyset$ and let $\sigma_{\mathcal{F}} : \mathcal{M} \times \mathcal{M} \to [0, +\infty)$. Assume that $\exists (f, \alpha) \in \mathcal{F} \times [0, +\infty)$ such that
(D1) $(\omega, \varpi) \in \mathcal{M} \times \mathcal{M}$, $\sigma_{\mathcal{F}}(\omega, \varpi) = 0$ if and only if $\omega = \varpi$.
(D2) $\sigma_{\mathcal{F}}(\omega, \varpi) = \sigma_{\mathcal{F}}(\varpi, \omega)$, $\forall (\omega, \varpi) \in \mathcal{M} \times \mathcal{M}$.
(D3) For every $(\omega, \varpi) \in \mathcal{M} \times \mathcal{M}$, for every $N \in \mathbb{N}$, $N \ge 2$, and for every $(u_i)_{i=1}^N \subset \mathcal{M}$, with $(u_1, u_N) = (\omega, \varpi)$, we have

$$\sigma_{\mathcal{F}}(\omega, \varpi) > 0 \Rightarrow f(\sigma_{\mathcal{F}}(\omega, \varpi)) \le f\left(\sum_{i=1}^{N-1} \sigma_{\mathcal{F}}(u_i, u_{i+1})\right) + \alpha.$$

Then $(\mathcal{M}, \sigma_{\mathcal{F}})$ is called an $\mathcal{F}$-metric space.

Example 2.1. (See [18]) The function $\sigma_{\mathcal{F}} : \mathbb{R} \times \mathbb{R} \to [0, +\infty)$

$$\sigma_{\mathcal{F}}(\omega, \varpi) = \begin{cases} (\omega - \varpi)^2 & \text{if } (\omega, \varpi) \in [0, 3] \times [0, 3] \\ |\omega - \varpi| & \text{if } (\omega, \varpi) \notin [0, 3] \times [0, 3], \end{cases}$$

with $f(\iota) = \ln(\iota)$ and $\alpha = \ln(3)$, is an $\mathcal{F}$-metric.

Remark 2.1. It is clear from the definition that any metric space is an $\mathcal{F}$-metric space, but the inverse is not true in general.

Definition 2.3. (See [18]) Let $(\mathcal{M}, \sigma_{\mathcal{F}})$ be an $\mathcal{F}$-metric space.
(i) Let $\{\kappa_n\} \subseteq \mathcal{M}$. The sequence $\{\omega_n\}$ is said to be $\mathcal{F}$-convergent to $\omega \in \mathcal{M}$ if $\{\omega_n\}$ is convergent to ω regarding $\mathcal{F}$-metric $\sigma_{\mathcal{F}}$.
(ii) The sequence $\{\omega_n\}$ is called $\mathcal{F}$-Cauchy, iff

$$\lim_{n,m\to\infty} \sigma_{\mathcal{F}}(\omega_n, \omega_m) = 0.$$

(iii) If every $\mathcal{F}$-Cauchy sequence in $\mathcal{M}$ is $\mathcal{F}$-converges to a point in $\mathcal{M}$, then $(\mathcal{M}, \sigma_{\mathcal{F}})$ is $\mathcal{F}$-complete.

Theorem 2.1. [18] *Let $(\mathcal{M}, d_{\mathcal{F}})$ be an $\mathcal{F}$-metric space and $\mathcal{L} : \mathcal{M} \to \mathcal{M}$ and assume that the conditions given below are satisfied:*
(i) $(\mathcal{M}, \sigma_{\mathcal{F}})$ *is* $\mathcal{F}$-complete,
(ii) $\exists k \in (0, 1)$, *such that*

$$\sigma_{\mathcal{F}}(\mathcal{L}(\omega), \mathcal{L}(\varpi)) \le k\sigma_{\mathcal{F}}(\omega, \varpi).$$

Then $\mathcal{L}$ has a unique fixed point $\omega^* \in \mathcal{M}$. Moreover, for any $\omega_0 \in \mathcal{M}$, the sequence $\{\omega_n\} \subset \mathcal{M}$ defined by

$$\omega_{n+1} = \mathcal{L}(\omega_n), \quad n \in \mathbb{N},$$

is $\mathcal{F}$-convergent to ω^*.

For more characteristics in these ways, we mention the researchers to [18–32].

Motivated with Basha [2], we define the notion of best proximity point in the context of $\mathcal{F}$-metric space in this way.

Definition 2.4. Let $(\mathcal{M}, \sigma_{\mathcal{F}})$ is $\mathcal{F}$-metric space and $\Theta, \Phi \in N(\mathcal{M})$. An element $\omega^* \in \Theta$ is professed to be a best proximity point of $\mathcal{L} : \Theta \to \Phi$ if this assertion hold

$$\sigma_{\mathcal{F}}(\omega^*, \mathcal{L}\omega^*) \leq \sigma_{\mathcal{F}}(\Theta, \Phi).$$

Consistent with Eldred and Veeramani [3], we give the $\mathcal{F}$-distance between the pair (Θ, Φ) of two nonempty sets which satisfy the property P.

Definition 2.5. Let $(\mathcal{M}, \sigma_{\mathcal{F}})$ is $\mathcal{F}$-metric space and $\Theta, \Phi \in N(\mathcal{M})$, then $\sigma_{\mathcal{F}}(\Theta, \Phi)$ is $\mathcal{F}$-distance between two nonempty sets Θ and Φ. Now define Θ_0 and Φ_0 by

$$\Theta_0 = \{\omega \in \Theta : \exists u \in \Phi \text{ such that } \sigma_{\mathcal{F}}(\omega, u) = \sigma_{\mathcal{F}}(\Theta, \Phi)\}$$
$$\Phi_0 = \{u \in \Phi : \exists \omega \in \Theta \text{ such that } \sigma_{\mathcal{F}}(\omega, u) = \sigma_{\mathcal{F}}(\Theta, \Phi)\}.$$

Then (Θ, Φ) is called to have the property P if $\Theta_0 \neq \emptyset$ and

$$\omega, \varpi \in \Theta_0, u, v \in \Phi_0, \quad \sigma_{\mathcal{F}}(\omega, u) = \sigma_{\mathcal{F}}(\varpi, v) = \sigma_{\mathcal{F}}(\Theta, \Phi) \Rightarrow \sigma_{\mathcal{F}}(\omega, \varpi) = \sigma_{\mathcal{F}}(u, v).$$

Definition 2.6. Let $(\mathcal{M}, \sigma_{\mathcal{F}})$ is $\mathcal{F}$-metric space and $\Theta, \Phi \in N(\mathcal{M})$. A mapping $\mathcal{L} : \Theta \to \Phi$ is called α-proximal admissible (α-prox admis) if $\exists$ a function $\alpha : \Theta \times \Theta \to [0, \infty)$ such that

$$\left.\begin{array}{l} \alpha(\omega, \varpi) \geq 1 \\ \sigma_{\mathcal{F}}(u, \mathcal{L}\omega) = \sigma_{\mathcal{F}}(\Theta, \Phi) \\ \sigma_{\mathcal{F}}(v, \mathcal{L}\varpi) = \sigma_{\mathcal{F}}(\Theta, \Phi) \end{array}\right\} \quad \Rightarrow \quad \alpha(u, v) \geq 1,$$

where $\omega, \varpi, u, v \in \Theta$.

3 Best proximity point results in $\mathcal{F}$-metric spaces

We represent by Ψ the collection of non-decreasing functions $\psi : [0, \infty) \to [0, \infty)$ such that $\sum_{n=1}^{\infty} \psi^n(\imath) < \infty$ for each $\imath > 0$. If $\psi \in \Psi$, then $\psi(\imath) < \imath$ for all $\imath > 0$. We also denote $N(\mathcal{M})$ and $Cl(\mathcal{M})$ as set of non-empty subsets of $\mathcal{M}$ and closed subsets of $\mathcal{M}$, respectively.

Definition 3.1. Let $(\mathcal{M}, \sigma_{\mathcal{F}})$ is $\mathcal{F}$-metric space and $\Theta, \Phi \in N(\mathcal{M})$. A mapping $\mathcal{L} : \Theta \to \Phi$ is said to be an α-ψ-proximal contraction if there exists $\psi \in \Psi$and $\alpha : \Theta \times \Theta \to [0, \infty)$ such that

$$\alpha(\omega, \varpi)\sigma_{\mathcal{F}}(\mathcal{L}\omega, \mathcal{L}\varpi) \leq \psi(\sigma_{\mathcal{F}}(\omega, \varpi)) \tag{1}$$

$\forall \omega, \varpi \in \Theta$.

Theorem 3.1. *Let $(\mathcal{M}, \sigma_{\mathcal{F}})$ is complete $\mathcal{F}$-metric space and $\Theta, \Phi \in Cl(\mathcal{M})$ such that $\Theta_0 \neq \emptyset$. Let $\alpha : \Theta \times \Theta \to [0, \infty)$ and $\psi \in \Psi$. Assume that $\mathcal{L} : \Theta \to \Phi$ be an α-ψ-proximal contraction and α-prox admis satisfying these assertions:*

(i) $\mathcal{L}(\Theta_0) \subseteq \Phi_0$ and (Θ, Φ) fulfils the property P;

(ii) $\exists \omega_0, \omega_1 \in \Theta_0$ such that

$$\sigma_{\mathcal{F}}(\omega_1, \mathcal{L}\omega_0) = \sigma_{\mathcal{F}}(\Theta, \Phi), \quad \text{and} \quad \alpha(\omega_0, \omega_1) \geq 1.$$

(iii) $\mathcal{L}$ is continuous.

Then $\exists \omega^* \in \Theta$ such that $\sigma_{\mathcal{F}}(\omega^*, \mathcal{L}\omega^*) \leq \sigma_{\mathcal{F}}(\Theta, \Phi)$.

Proof. By the hypothesis (ii), $\exists \omega_0, \omega_1 \in \Theta_0$ such that

$$\sigma_{\mathcal{F}}(\omega_1, \mathcal{L}\omega_0) = \sigma_{\mathcal{F}}(\Theta, \Phi), \quad \alpha(\omega_0, \omega_1) \geq 1. \tag{2}$$

Since $\mathcal{L}(\Theta_0) \subseteq \Phi_0$, $\exists \omega_2 \in \Theta_0$ such that

$$\sigma_{\mathcal{F}}(\omega_2, \mathcal{L}\omega_1) = \sigma_{\mathcal{F}}(\Theta, \Phi).$$

Now, we have $\alpha(\omega_0, \omega_1) \geq 1$, $\sigma_{\mathcal{F}}(\omega_1, \mathcal{L}\omega_0) = \sigma_{\mathcal{F}}(\Theta, \Phi)$, and $\sigma_{\mathcal{F}}(\omega_2, \mathcal{L}\omega_1) = \sigma_{\mathcal{F}}(\Theta, \Phi)$. As the mapping $\mathcal{L}$ is α-prox admis, we obtain $\alpha(\omega_1, \omega_2) \geq 1$. Hence,

$$\sigma_{\mathcal{F}}(\omega_2, \mathcal{L}\omega_1) = \sigma_{\mathcal{F}}(\Theta, \Phi), \quad \alpha(\omega_1, \omega_2) \geq 1. \tag{3}$$

Again since $\mathcal{L}(\Theta_0) \subseteq \Phi_0$, $\exists \omega_3 \in \Theta_0$ such that

$$\sigma_{\mathcal{F}}(\omega_3, \mathcal{L}\omega_2) = \sigma_{\mathcal{F}}(\Theta, \Phi).$$

Now, we have $\alpha(\omega_1, \omega_2) \geq 1$, $\sigma_{\mathcal{F}}(\omega_2, \mathcal{L}\omega_1) = \sigma_{\mathcal{F}}(\Theta, \Phi)$ and $\sigma_{\mathcal{F}}(\omega_3, \mathcal{L}\omega_2) = \sigma_{\mathcal{F}}(\Theta, \Phi)$. As the mapping $\mathcal{L}$ is α-prox admis, we obtain $\alpha(\omega_2, \omega_3) \geq 1$. Hence,

$$\sigma_{\mathcal{F}}(\omega_3, \mathcal{L}\omega_2) = \sigma_{\mathcal{F}}(\Theta, \Phi), \quad \alpha(\omega_2, \omega_3) \geq 1. \tag{4}$$

By pursuing in this way, by induction, we can generate $\{\omega_n\} \subset \Theta_0$ such that

$$\sigma_{\mathcal{F}}(\omega_{n+1}, \mathcal{L}\omega_n) = \sigma_{\mathcal{F}}(\Theta, \Phi), \quad \alpha(\omega_n, \omega_{n+1}) \geq 1, \tag{5}$$

$\forall n \in \mathbb{N} \cup \{0\}$. Assume that $\omega_k = \omega_{k+1}$ for some k. From (5), we have

$$\sigma_{\mathcal{F}}(\omega_k, \mathcal{L}\omega_k) = \sigma_{\mathcal{F}}(\omega_{k+1}, \mathcal{L}\omega_k) = \sigma_{\mathcal{F}}(\Theta, \Phi),$$

i.e., ω_k is a best proximity point of $\mathcal{L}$. Hence, we assume that $\sigma_{\mathcal{F}}(\omega_{n-1}, \omega_n) > 0$ for all $n \in \mathbb{N} \cup \{0\}$. As (Θ, Φ) satisfies the property P, we summarize from (5) that

$$\sigma_{\mathcal{F}}(\omega_n, \omega_{n+1}) = \sigma_{\mathcal{F}}(\mathcal{L}\omega_{n-1}, \mathcal{L}\omega_n),$$

$\forall n \in \mathbb{N} \cup \{0\}$. So by (1), we have

$$\sigma_{\mathcal{F}}(\omega_n, \omega_{n+1}) \leq \alpha(\omega_n, \omega_{n+1})\sigma_{\mathcal{F}}(\omega_n, \omega_{n+1}) = \alpha(\omega_n, \omega_{n+1})\sigma_{\mathcal{F}}(\mathcal{L}\omega_{n-1}, \mathcal{L}\omega_n) \leq \psi(\sigma_{\mathcal{F}}(\omega_{n-1}, \omega_n)) \tag{6}$$

$\forall n \geq 0$. By using the monotonicity of ψ and (6), we obtain

$$\sigma_{\mathcal{F}}(\omega_n, \omega_{n+1}) \leq \psi^n(\sigma_{\mathcal{F}}(\omega_0, \omega_1))$$

$\forall n \in \mathbb{N} \cup \{0\}$. Let $\varepsilon > 0$ be fixed and $(f, \alpha) \in \mathcal{F} \times [0, +\infty)$ be such that (D_3) is satisfied. By $(\mathcal{F}_2)$, $\exists \delta > 0$ such that

$$0 < \imath < \delta \Rightarrow f(\imath) < f(\varepsilon) - \alpha. \tag{7}$$

Let $n(\varepsilon) \in \mathbb{N}$ such that $\sum_{n \geq n(\delta)} \psi^n(\sigma_{\mathcal{F}}(\omega_0, \omega_1)) < \delta$. Hence, by (6), (7), and $(\mathcal{F}_1)$, we have

$$f\left(\sum_{i=n}^{m-1} \psi^i(\sigma_{\mathcal{F}}(\omega_0, \omega_1))\right) \leq f\left(\sum_{n \geq n(\delta)} \psi^n(\sigma_{\mathcal{F}}(\omega_0, \omega_1))\right) < f(\varepsilon) - \alpha \tag{8}$$

for $m > n > n(\varepsilon)$. By using (D_3) and (8), we obtain $\sigma_{\mathcal{F}}(\omega_n, \omega_m) > 0$, $m > n > n(\varepsilon)$ implies

$$f(\sigma_{\mathcal{F}}(\omega_m, \omega_n)) \leq f\left(\sum_{i=n}^{m-1} \sigma_{\mathcal{F}}(\omega_i, \omega_{i+1})\right) + \alpha \leq f\left(\sum_{i=n}^{m-1} \psi^i(\sigma_{\mathcal{F}}(\omega_0, \omega_1))\right) + \alpha < f(\varepsilon),$$

which implies by $(\mathcal{F}_1)$ that $\sigma_{\mathcal{F}}(\omega_m, \omega_n) < \varepsilon$, $m > n > n(\varepsilon)$. This proves that $\{\omega_n\}$ is $\mathcal{F}$-Cauchy. Since $(\mathcal{M}, \sigma_{\mathcal{F}})$ is $\mathcal{F}$-complete and Θ is closed, $\exists \omega^* \in \Theta$ such that $\{\omega_n\}$ is $\mathcal{F}$-convergent to ω^*, i.e.,

$$\lim_{n \to \infty} \sigma_{\mathcal{F}}(\omega_n, \omega^*) = 0, \tag{9}$$

i.e., $\mathcal{L}\omega_n \to \mathcal{L}\omega^*$ as $n \to \infty$. By using the continuity of $\sigma_{\mathcal{F}}$, we obtain

$$\sigma_{\mathcal{F}}(\Theta, \Phi) = \sigma_{\mathcal{F}}(\omega_{n+1}, \mathcal{L}\omega_n) \to \sigma_{\mathcal{F}}(\omega^*, \mathcal{L}\omega^*)$$

as $n \to \infty$. Therefore, $\sigma_{\mathcal{F}}(\omega^*, \mathcal{L}\omega^*) = \sigma_{\mathcal{F}}(\Theta, \Phi)$. □

(κ) If $\{\omega_n\} \subseteq \Theta$ such that $\alpha(\omega_n, \omega_{n+1}) \geq 1$, for all n and $\omega_n \to \omega \in \Theta$ as $n \to \infty$, then $\exists$ $\{\omega_{n(k)}\}$ of $\{\omega_n\}$ such that $\alpha(\omega_{n(k)}, \omega) \geq 1$, for all k.

Theorem 3.2. *Let $(\mathcal{M}, \sigma_{\mathcal{F}})$ be a complete $\mathcal{F}$-metric space and Θ, $\Phi \in Cl(\mathcal{M})$ such that $\Theta_0 \neq \varnothing$. Let $\alpha : \Theta \times \Theta \to [0, \infty)$ and $\psi \in \Psi$. Assume that $\mathcal{L} : \Theta \to \Phi$ be an α-ψ-proximal contraction and α-proximal admissible satisfies these assertions:*
(i) $\mathcal{L}(\Theta_0) \subseteq \Phi_0$ and (Θ, Φ) satisfies the property P;
(ii) $\exists \omega_0, \omega_1 \in \Theta_0$ such that

$$\sigma_{\mathcal{F}}(\omega_1, \mathcal{L}\omega_0) = \sigma_{\mathcal{F}}(\Theta, \Phi), \quad \text{and} \quad \alpha(\omega_0, \omega_1) \geq 1.$$

(iii) (J) holds.

Then $\exists \omega^* \in \mathcal{M}$ such that $\sigma_{\mathcal{F}}(\omega^*, \mathcal{L}\omega^*) \leq \sigma_{\mathcal{F}}(\Theta, \Phi)$.

Proof. Backing the result of Theorem 3.1, $\exists$ $\{\omega_n\} \subset \Theta$ such that (1) holds and $\{\omega_n\}$ is $\mathcal{F}$-convergent to ω^*, i.e.,

$$\lim_{n\to\infty} \sigma_{\mathcal{F}}(\omega_n, \omega^*) = 0.$$

By the property (κ), $\exists$ $\{\omega_{n(k)}\}$ of $\{\omega_n\}$ such that $\alpha(\omega_{n(k)}, \omega^*) \geq 1$, for all k. We declare that $\mathcal{L}\omega_{n(k)} \to \mathcal{L}\omega^*$ as $k \to \infty$. So by (1), we obtain

$$\sigma_{\mathcal{F}}(\mathcal{L}\omega_{n(k)}, \mathcal{L}\omega^*) \leq \alpha(\omega_{n(k)}, \omega^*)\sigma_{\mathcal{F}}(\mathcal{L}\omega_{n(k)}, \mathcal{L}\omega^*) \leq \psi(\sigma_{\mathcal{F}}(\omega_{n(k)}, \omega^*))$$

$\forall k$. By taking $k \to \infty$ and using the continuity of $\sigma_{\mathcal{F}}$, we have

$$\sigma_F(\Theta, \Phi) = \sigma_{\mathcal{F}}(\omega_{n(k)+1}, \mathcal{L}\omega_{n(k)}) \to \sigma_{\mathcal{F}}(\omega^*, \mathcal{L}\omega^*)$$

as $n \to \infty$. Therefore,

$$\sigma_{\mathcal{F}}(\omega^*, \mathcal{L}\omega^*) = \sigma_F(\Theta, \Phi)$$

thus proved. □

Definition 3.2. Let $\mathcal{L} : \Theta \to \Phi$ and $\alpha : \Theta \times \Theta \to [0, \infty)$. The mapping $\mathcal{L}$ is said to be (α, σ_F)-regular if for all $(\omega, \varpi) \in \alpha^{-1}[0, 1)$, $\exists$ $\varrho \in \Theta_0$ such that

$$\alpha(\omega, \varpi) \geq 1 \quad \text{and} \quad \alpha(\varpi, \varrho) \geq 1.$$

Theorem 3.3. *Besides to the supposition of Theorem 3.1 (respectively Theorem 3.2), assume that $\mathcal{L}$ is (α, σ)-regular. Then, $\exists$ $\omega^* \in \Theta$ such that $\sigma_{\mathcal{F}}(\omega^*, \mathcal{L}\omega^*) \leq \sigma_{\mathcal{F}}(\Theta, \Phi)$, which is unique.*

Proof. It is clear from the Theorem 3.1 that the set of best proximity points of $\mathcal{L}$ is non-empty. Assume that $\exists$ $\varpi^* \in \Theta_0$ of $\mathcal{L}$, i.e.,

$$\sigma_{\mathcal{F}}(\mathcal{L}\omega^*, \omega^*) = \sigma_{\mathcal{F}}(\mathcal{L}\varpi^*, \varpi^*) = \sigma_{\mathcal{F}}(\Theta, \Phi). \tag{10}$$

By using the property P and (10), we obtain that

$$\sigma_{\mathcal{F}}(\mathcal{L}\omega^*, \mathcal{L}\varpi^*) = \sigma_{\mathcal{F}}(\omega^*, \varpi^*). \tag{11}$$

We discuss two cases.

Case 1. If $\alpha(\omega^*, \varpi^*) \geq 1$, by using (10), we obtain that

$$\sigma_{\mathcal{F}}(\omega^*, \varpi^*) = \sigma_{\mathcal{F}}(\mathcal{L}\omega^*, \mathcal{L}\varpi^*) \leq \alpha(\omega^*, \varpi^*)\sigma_{\mathcal{F}}(\mathcal{L}\omega^*, \mathcal{L}\varpi^*) \leq \psi(\sigma_{\mathcal{F}}(\omega^*, \varpi^*)).$$

Since $\psi(\iota) < \iota$, for all $\iota > 0$, the aforementioned inequality satisfies only if $\sigma_{\mathcal{F}}(\omega^*, \varpi^*) = 0$, i.e., $\omega^* = \varpi^*$.

Case 2. If $\alpha(\omega^*, \varpi^*) < 1$.

By supposition, $\exists\ \varrho_0 \in \Theta_0$ such that $\alpha(\omega^*, \varrho_0) \geq 1$ and $\alpha(\varpi^*, \varrho_0) \geq 1$. Since $\mathcal{L}(\Theta_0) \subseteq \Phi_0$, there exists $\varrho_1 \in \Theta_0$ such that

$$\sigma_{\mathcal{F}}(\varrho_1, \mathcal{L}\varrho_0) = \sigma_{\mathcal{F}}(\Theta, \Phi).$$

Now, we have

$$\alpha(\omega^*, \varrho_0) \geq 1$$

$$\sigma_{\mathcal{F}}(\omega^*, \mathcal{L}\omega^*) = \sigma_{\mathcal{F}}(\Theta, \Phi),$$

$$\sigma_{\mathcal{F}}(\varrho_1, \mathcal{L}\varrho_0) = \sigma_{\mathcal{F}}(\Theta, \Phi).$$

As $\mathcal{L}$ is α-prox admis, so we have $\alpha(\omega^*, \varrho_1) \geq 1$. Hence,

$$\sigma_{\mathcal{F}}(\varrho_1, \mathcal{L}\varrho_0) = \sigma_{\mathcal{F}}(\Theta, \Phi) \quad \text{and} \quad \alpha(\omega^*, \varrho_1) \geq 1.$$

By pursuing in this way, we can generate a sequence $\{\varrho_n\}$ in Θ_0 such that

$$\sigma_{\mathcal{F}}(\varrho_{n+1}, \mathcal{L}\varrho_n) = \sigma_{\mathcal{F}}(\Theta, \Phi) \quad \text{and} \quad \alpha(\omega^*, \varrho_n) \geq 1 \tag{12}$$

$\forall n \geq 0$. By property P and (12), we obtain that

$$\sigma_{\mathcal{F}}(\varrho_{n+1}, \omega^*) = \sigma_{\mathcal{F}}(\mathcal{L}\varrho_n, \mathcal{L}\omega^*) \tag{13}$$

$\forall n \in \mathbb{N} \cup \{0\}$. Since $\mathcal{L}$ is an α–ψ-proximal contraction, we have

$$\sigma_{\mathcal{F}}(\varrho_{n+1}, \omega^*) = \sigma_{\mathcal{F}}(\mathcal{L}\varrho_n, \mathcal{L}\omega^*) \leq \alpha(\varrho_n, \omega^*)\sigma_{\mathcal{F}}(\mathcal{L}\varrho_n, \mathcal{L}\omega^*) \leq \psi(\sigma_{\mathcal{F}}(\varrho_n, \omega^*))$$

$\forall n \geq 0$. By induction, we can obtain

$$\sigma_{\mathcal{F}}(\varrho_n, \omega^*) \leq \psi^n(\sigma_{\mathcal{F}}(\varrho_0, \omega^*)) \tag{14}$$

$\forall n \geq 0$. Assume that $\varrho_0 = \omega^*$. Then by (13), we obtain

$$\sigma_{\mathcal{F}}(\varrho_1, \omega^*) = \sigma_{\mathcal{F}}(\mathcal{L}\varrho_0, \mathcal{L}\omega^*) = \sigma_{\mathcal{F}}(\mathcal{L}\omega^*, \mathcal{L}\omega^*) = 0,$$

that is, $\varrho_1 = \omega^*$. By pursuing in this way and inductively, we have $\varrho_n = \omega^*$, for all $n \geq 0$. Assume $\sigma_{\mathcal{F}}(\varrho_0, \omega^*) > 0$. By taking limit as $n \to \infty$ in (14), we establish that $\varrho_n \to \omega^*$ whenever $n \to \infty$. Thus, in all the discussed cases, we obtain $\varrho_n \to \omega^*$ as $n \to \infty$. Likewise, we can show that $\varrho_n \to \varpi^*$ as $n \to \infty$. By uniqueness of the limit, we obtain that $\omega^* = \varpi^*$. □

4 Applications

Theorem 4.1. *Let $(\mathcal{M}, \sigma_{\mathcal{F}})$ is complete $\mathcal{F}$-metric space, $\Theta, \Phi \in Cl(\mathcal{M})$ such that $\Theta_0 \neq \varnothing$. Let $\psi \in \Psi$. Suppose that $\mathcal{L} : \Theta \to \Phi$ satisfying*

(i) $\mathcal{L}(\Theta_0) \subseteq \Phi_0$ and (Θ, Φ) satisfying the property P;

(ii) $\sigma_{\mathcal{F}}(\mathcal{L}\omega, \mathcal{L}\varpi) \leq \psi(\sigma_{\mathcal{F}}(\omega, \varpi))$, for all $\omega, \varpi \in \Theta$.

Then $\exists \omega^* \in \mathcal{M}$ such that $\sigma_{\mathcal{F}}(\omega^*, \mathcal{L}\omega^*) \leq \sigma_{\mathcal{F}}(\Theta, \Phi)$.

Proof. Define $\alpha : \Theta \times \Theta \to [0, \infty)$ by

$$\alpha(\omega, \varpi) = 1$$

$\forall \omega, \varpi \in \Theta$. Evidently $\mathcal{L}$ is α-prox admis by the definition of α, and also it is an α–ψ-proximal contraction. Otherwise, for any $\omega \in \Theta_0$, since $\mathcal{L}(\Theta_0) \subseteq \Phi_0$, $\exists \varpi \in \Theta_0$ such that $\sigma(\mathcal{L}\omega, \varpi) = \sigma(\Theta, \Phi)$. Furthermore, from the hypothesis (ii), we obtain

$$\sigma_{\mathcal{F}}(\mathcal{L}\omega, \mathcal{L}\varpi) \leq \psi(\sigma_{\mathcal{F}}(\omega, \varpi)) < \sigma_{\mathcal{F}}(\omega, \varpi).$$

From the aforementioned inequality, we have $\mathcal{L}$ is a continuous. Thus, all the assumptions of Theorem 3.1 are fulfilled and $\exists$ is the best proximity point of $\mathcal{L}$ directly from Theorem 3.1. Furthermore, from the definition of α and from Theorem 3, we obtain that this best proximity point is unique. □

If we take $\psi(\imath) = k\imath$, where $0 < k < 1$ in Theorem 4.1, we establish this result.

Theorem 4.2. *Let $(\mathcal{M}, \sigma_{\mathcal{F}})$ is complete $\mathcal{F}$-metric space, $\Theta, \Phi \in Cl(\mathcal{M})$ such that $\Theta_0 \neq \emptyset$. Let $\psi \in \Psi$. Assume that $\mathcal{L} : \Theta \to \Phi$ satisfying these assertions:*
(i) $\mathcal{L}(\Theta_0) \subseteq \Phi_0$ and (Θ, Φ) satisfies the property P;
(ii) $\exists\, k \in (0, 1)$ such that $\sigma_{\mathcal{F}}(\mathcal{L}\omega, \mathcal{L}\varpi) \leq k\sigma_{\mathcal{F}}(\omega, \varpi)$, for all $\omega, \varpi \in \Theta$.

Then $\exists \omega^* \in \mathcal{M}$ such that $\sigma_{\mathcal{F}}(\omega^*, \mathcal{L}\omega^*) \leq \sigma_{\mathcal{F}}(\Theta, \Phi)$.

4.1 Results on $\mathcal{F}$-metric space endowed with binary relation

Let $(\mathcal{M}, \sigma_{\mathcal{F}})$ be an $\mathcal{F}$-metric space and R be any binary relation on $\mathcal{M}$ and let

$$S = R \cup R^{-1}.$$

Evidently,

$$\omega, \varpi \in \mathcal{M}, \quad \omega S \varpi \quad \text{if and only if} \quad \omega R \varpi \quad \text{or} \quad \varpi R \omega.$$

Definition 4.1. A mapping $\mathcal{L} : \Theta \to \Phi$ is called a proximal comparative mapping if

$$\left.\begin{array}{c} \omega_1 S \omega_2 \\ \sigma_F(u_1, \mathcal{L}u_1) = \sigma_F(\Theta, \Phi) \\ \sigma_F(u_2, \mathcal{L}u_2) = \sigma_F(\Theta, \Phi) \end{array}\right\} \Rightarrow u_1 S u_2$$

$\forall \omega_1, \omega_2, u_1, u_2 \in \Theta$.

Theorem 4.3. *Let $(\mathcal{M}, \sigma_{\mathcal{F}})$ is complete $\mathcal{F}$- metric space, $\Theta, \Phi \in Cl(\mathcal{M})$ such that $\Theta_0 \neq \emptyset$. Let R be any binary relation on $\mathcal{M}$. Assume that $\mathcal{L} : \Theta \to \Phi$ is continuous satisfying*
(i) $\mathcal{L}(\Theta_0) \subseteq \Phi_0$ and (Θ, Φ) satisfies the P-Property;
(ii) $\mathcal{L}$ is a proximal comparative mapping,
(iii) $\exists \omega_0, \omega_1 \in \Theta_0$ such that

$$\sigma_{\mathcal{F}}(\omega_1, \mathcal{L}\omega_0) = \sigma_{\mathcal{F}}(\Theta, \Phi), \quad \text{and} \quad \omega_0 S \omega_1,$$

(iv) $\exists \psi \in \Psi$ such that

$$\omega, \varpi \in \Theta, \quad \omega S \varpi \quad \text{implies } \sigma_{\mathcal{F}}(\mathcal{L}\omega, \mathcal{L}\varpi) \leq \psi(\sigma_{\mathcal{F}}(\omega, \varpi)), \tag{15}$$

Then $\exists \omega^* \in \mathcal{M}$ such that $\sigma_{\mathcal{F}}(\omega^*, \mathcal{L}\omega^*) \leq \sigma_{\mathcal{F}}(\Theta, \Phi)$.

Proof. Define $\alpha : \Theta \times \Theta \to [0, \infty)$ by:

$$\alpha(\omega, \varpi) = \begin{cases} 1 & \text{if } \omega S \varpi \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that

$$\begin{cases}\alpha(\omega_1, \omega_2) \geq 1 \\ \sigma_{\mathcal{F}}(u_1, \mathcal{L}\omega_1) = \sigma_{\mathcal{F}}(\Theta, \Phi) \\ \sigma_{\mathcal{F}}(u_2, \mathcal{L}\omega_2) = \sigma_{\mathcal{F}}(\Theta, \Phi)\end{cases}$$

for some $\omega_1, \omega_2, u_1, u_2 \in \Theta$. By the definition of α, we obtain that

$$\begin{cases}\omega_1 S \omega_2, \\ \sigma_{\mathcal{F}}(u_1, \mathcal{L}\omega_1) = \sigma_{\mathcal{F}}(\Theta, \Phi) \\ \sigma_{\mathcal{F}}(u_2, \mathcal{L}\omega_2) = \sigma_{\mathcal{F}}(\Theta, \Phi).\end{cases}$$

Then by supposition (ii), we obtain that $u_1 S u_2$. Now by the definition of α, we have $\alpha(u_1, u_2) \geq 1$. Thus, we established that $\mathcal{L}$ is α-prox admis. Supposition (iii) yields

$$\sigma_{\mathcal{F}}(\omega_1, \mathcal{L}\omega_0) = \sigma_{\mathcal{F}}(\Theta, \Phi)$$

and $\alpha(\omega_0, \omega_1) \geq 1$. Finally, condition (iv) implies that

$$\alpha(\omega, \varpi)\sigma_{\mathcal{F}}(\mathcal{L}\omega, \mathcal{L}\varpi) \leq \psi(\sigma_{\mathcal{F}}(\omega, \varpi))$$

i.e., $\mathcal{L}$ is an α–ψ-proximal contraction. Thus, all the assumption of Theorem 3.1 hold, and the required result comes directly from this result. □

If we want to omit the continuity of $\mathcal{L}$, then we use this assumption:

($\mathscr{H}$) if $\{\omega_n\}$ in $\mathcal{M}$ and $\omega \in \mathcal{M}$ are such that $\omega_n S \omega_{n+1}$, for all $n \geq 0$ and $\lim_{n\to\infty}\sigma_{\mathcal{F}}(\omega_n, \omega) = 0$, then $\exists$ $\{\omega_{n(k)}\}$ of $\{\omega_n\}$ such that $\omega_{n(k)} S \omega$ for all k.

Theorem 4.4. *Let $\Theta, \Phi \in Cl(\mathcal{M})$, where $(\mathcal{M}, \sigma_{\mathcal{F}})$ is complete $\mathcal{F}$-metric space such that $\Theta_0 \neq \emptyset$. Let $\mathcal{R}$ be a binary relation over $\mathcal{M}$. Suppose that $\mathcal{L} : \Theta \to \Phi$ satisfying*

(i) *$\mathcal{L}(\Theta_0) \subseteq \Phi_0$ and (Θ, Φ) fulfils the property P;*

(*ii*) *$\mathcal{L}$ is a proximal comparative mapping,*

(*iii*) *$\exists \omega_0, \omega_1 \in \Theta_0$ such that*

$$\sigma_{\mathcal{F}}(\omega_1, \mathcal{L}\omega_0) = \sigma_{\mathcal{F}}(\Theta, \Phi), \quad \text{and} \quad \omega_0 S \omega_1,$$

(*iv*) *$\exists \psi \in \Psi$ such that*

$$\omega, \varpi \in \Theta, \omega S \varpi \quad \text{implies} \quad \sigma_{\mathcal{F}}(\mathcal{L}\omega, \mathcal{L}\varpi) \leq \psi(\sigma_{\mathcal{F}}(\omega, \varpi)),$$

(*v*) *$(\mathcal{H})$ holds.*

Then $\exists \omega^* \in \mathcal{M}$ such that $\sigma_{\mathcal{F}}(\omega^*, \mathcal{L}\omega^*) \leq \sigma_{\mathcal{F}}(\Theta, \Phi)$.

Proof. If we consider $\alpha : \Theta \times \Theta \to [0, \infty)$ given by

$$\alpha(\omega, \varpi) = \begin{cases}1 & \text{if } \omega S \varpi \\ 0 & \text{otherwise.}\end{cases}$$

and by observing that assertion (H) yields condition (J), then from Theorem 3.2 we obtain the conclusion. □

Theorem 4.5. *Addition to the assumptions of Theorem 4.3 (respectively Theorem 4.4),assume that these conditions fulfils: for all $(\omega, \varpi) \in \Theta \times \Theta$ with $(\omega, \varpi) \notin S$, $\exists \varrho \in \Theta_0$ such that $\omega S \varrho$ and $\varpi S \varrho$. Then there exists $\omega^* \in \mathcal{M}$ such that $\sigma_{\mathcal{F}}(\omega^*, \mathcal{L}\omega^*) \leq \sigma_{\mathcal{F}}(\Theta, \Phi)$ which is unique.*

4.2 Coupled best proximity points results

Definition 4.2. A point $(\omega^*, \varpi^*) \in \Theta \times \Theta$ is professed to be a coupled best proximity point of J if

$$\sigma_{\mathcal{F}}(\omega^*, J(\omega^*, \varpi^*)) = \sigma_{\mathcal{F}}(\varpi^*, J(\varpi^*, \omega^*)) = \sigma_{\mathcal{F}}(\Theta, \Phi).$$

We establish the following notions.

$$\mathfrak{M} = \mathcal{M} \times \mathcal{M}, \quad \aleph_1 = \Theta \times \Theta, \quad \aleph_2 = \Phi \times \Phi.$$

Define $\mathcal{L} : \aleph_1 \to \aleph_2$ by

$$\mathcal{L}(\omega, \varpi) = (J(\omega, \varpi), J(\varpi, \omega)),$$

$\forall (\omega, \varpi) \in \aleph_1$. We supply the product set $\mathfrak{M}$ with $\sigma'_{\mathcal{F}}$ given by:

$$\sigma'_{\mathcal{F}}((\omega, \varpi), (u, v)) = \frac{\sigma_{\mathcal{F}}(\omega, u) + \sigma_{\mathcal{F}}(\varpi, v)}{2}.$$

Evidently, if $(\mathcal{M}, \sigma_{\mathcal{F}})$ is $\mathcal{F}$-complete, then $(\mathfrak{M}, \sigma'_{\mathcal{F}})$ is F-complete.

Definition 4.3. A mapping $J : \Theta \times \Theta \to \Phi$ is professed to be bi-proximal comparative (bi-prox comp) mapping if

$$\left.\begin{array}{c} \omega_1 S \omega_2, \varpi_1 S \varpi_2 \\ \sigma_F(u_1, J(\omega_1, \varpi_1)) = \sigma_F(\Theta, \Phi) \\ \sigma_F(u_2, J(\omega_2, \varpi_2)) = \sigma_F(\Theta, \Phi) \end{array}\right\} \Rightarrow u_1 S u_2$$

$\forall \omega_1, \omega_2, \varpi_1, \varpi_2, u_1, u_2 \in \Theta$.

Theorem 4.6. *Let $\Theta, \Phi \in Cl(\mathcal{M})$, where $(\mathcal{M}, \sigma_{\mathcal{F}})$ is complete $\mathcal{F}$-metric space such that $\Theta_0 \neq \varnothing$ and R be a binary relation on $\mathcal{M}$. Assume that the mapping $J : \Theta \times \Theta \to \Phi$ is continuous satisfying*

(*i*) $J(\Theta_0 \times \Theta_0) \subseteq \Phi_0$ *and* (Θ, Φ) *fulfils the property P,*

(*ii*) J *is a bi-prox comp mapping,*

(*iii*) $\exists\ \omega_0, \varpi_0, \omega_1, \varpi_1 \in \Theta_0$ *such that*

$$\sigma_{\mathcal{F}}(\omega_1, J(\omega_0, \varpi_0)) = \sigma_{\mathcal{F}}(\varpi_1, J(\varpi_0, \omega_0)) = \sigma_{\mathcal{F}}(\Theta, \Phi), \quad \text{and} \quad \omega_0 S \omega_1, \varpi_0 S \varpi_1,$$

(*iv*) $\exists$ a $\psi \in \Psi$ *in such that*

$$\omega, \varpi, u, v \in \Theta, \omega S u, \varpi S v \;\Rightarrow\; \sigma_{\mathcal{F}}(J(\omega, \varpi), J(u, v)) \leq \psi\left(\frac{\sigma_{\mathcal{F}}(\omega, u) + \sigma_{\mathcal{F}}(\varpi, v)}{2}\right). \tag{16}$$

Then J possess a coupled best proximity point.

Proof. Define a binary relation R_2 on $\mathcal{M}$ by $(\omega, \varpi), (u, v) \in \mathfrak{M}$, $(\omega, \varpi) R_2 (u, v)$ if and only if $\omega S u, \varpi S v$. If we represent by S_2 the symmetric relation devoted to R_2, evidently, we obtain $S_2 = R_2$. We assert that $\mathcal{L} : \Theta \times \Theta \to \Phi$ has a best proximity point $(\omega^*, \varpi^*) \in \Theta_0 \times \Theta_0$, such that

$$\sigma'_{\mathcal{F}}((\omega^*, \varpi^*), \mathcal{L}(\omega^*, \varpi^*)) = \sigma'_{\mathcal{F}}(\aleph_1, \aleph_2).$$

Represent by:

$$\mathcal{A}_0 = \{(\ell_1, \ell_2) \in \aleph_1, \sigma'_{\mathcal{F}}((\ell_1, \ell_2), (\hbar_1, \hbar_2)) = \sigma'_{\mathcal{F}}(\aleph_1, \aleph_2) \quad \text{for some } (\hbar_1, \hbar_2) \in \mathcal{B}_0\}$$
$$\mathcal{B}_0 = \{(\hbar_1, \hbar_2) \in \aleph_2, \sigma'_{\mathcal{F}}((\ell_1, \ell_2), (\hbar_1, \hbar_2)) = \sigma'_{\mathcal{F}}(\aleph_1, \aleph_2) \quad \text{for some } (\ell_1, \ell_2) \in \mathcal{A}_0\}.$$

We can observe that

$$\sigma'_{\mathcal{F}}(\aleph_1, \aleph_2) = \sigma_{\mathcal{F}}(\Theta, \Phi).$$

In fact, we have

$$
\begin{aligned}
\sigma'_{\mathcal{F}}(\aleph_1, \aleph_2) &= \inf\{\sigma'_{\mathcal{F}}((\ell_1, \ell_2), (\hbar_1, \hbar_2)) : (\ell_1, \ell_2) \in \aleph_1, (\hbar_1, \hbar_2) \in \aleph_2\} \\
&= \frac{1}{2} \inf\{\sigma_{\mathcal{F}}((\ell_1, \hbar_1) + (\ell_2, \hbar_2)) : (\ell_1, \hbar_1) \in \aleph_1 \times \aleph_2, (\ell_2, \hbar_2) \in \aleph_1 \times \aleph_2\} \\
&= \frac{1}{2}(\inf\{\sigma_{\mathcal{F}}((\ell_1, \hbar_1) : (\ell_1, \hbar_1)) \in \Theta \times \Phi\} + \inf\{\sigma_{\mathcal{F}}((\ell_2, \hbar_2)) : (\ell_2, \hbar_2) \in \Theta \times \Phi\}) \\
&= \frac{1}{2}(\sigma_{\mathcal{F}}(\Theta, \Phi) + \sigma_{\mathcal{F}}(\Theta, \Phi)) \\
&= \sigma_{\mathcal{F}}(\Theta, \Phi).
\end{aligned}
$$

Now, let $(\ell_1, \ell_2) \in \mathcal{A}_0$. Then there exists $(\hbar_1, \hbar_2) \in \aleph_2$ such that

$$\sigma'_{\mathcal{F}}((\ell_1, \ell_2), (\hbar_1, \hbar_2)) = \sigma'_{\mathcal{F}}(\aleph_1, \aleph_2),$$

that is,

$$\sigma_{\mathcal{F}}((\ell_1, \hbar_1) + (\ell_2, \hbar_2)) = 2\sigma_{\mathcal{F}}(\Theta, \Phi).$$

Thus, we have

$$
\begin{cases}
\sigma_{\mathcal{F}}((\ell_1, \hbar_1) + (\ell_2, \hbar_2)) = 2\sigma_{\mathcal{F}}(\Theta, \Phi) \\
\sigma_{\mathcal{F}}(\ell_1, \hbar_1) \geq \sigma_{\mathcal{F}}(\Theta, \Phi) \\
\sigma_{\mathcal{F}}(\ell_2, \hbar_2) \geq \sigma_{\mathcal{F}}(\Theta, \Phi),
\end{cases}
$$

which implies that

$$\sigma_{\mathcal{F}}(\ell_1, \hbar_1) = \sigma_{\mathcal{F}}(\ell_2, \hbar_2) = \sigma_{\mathcal{F}}(\Theta, \Phi).$$

This implies that $(\ell_1, \ell_2) \in \Theta_0 \times \Theta_0$. Similarly, if $(\ell_1, \ell_2) \in \Theta_0 \times \Theta_0$, we have $(\ell_1, \ell_2) \in \mathcal{A}_0$. Thus, we proved that $\Theta_0 \times \Theta_0 = \mathcal{A}_0$. Likewise, we can prove that $\Phi_0 \times \Phi_0 = \mathcal{B}_0$. Since $\Theta_0 \neq \emptyset$, then $\mathcal{A}_0 \neq \emptyset$. Otherwise, from (i), we have

$$\mathcal{L}(\aleph_1) = \{(J(\omega, \varpi), J(\varpi, \omega)) : (\omega, \varpi) \in \Theta_0 \times \Theta_0\} \subset J(\Theta_0 \times \Theta_0) \times J(\Theta_0 \times \Theta_0) \subseteq \mathcal{B}_0.$$

Suppose now that for some $(\ell_1, \ell_2), (\omega_1, \omega_2) \in \aleph_1, (\hbar_1, \hbar_2), (\varpi_1, \varpi_2) \in \aleph_2$, we have

$$
\begin{aligned}
\sigma'_{\mathcal{F}}((\ell_1, \ell_2), (\hbar_1, \hbar_2)) = \sigma'_{\mathcal{F}}(\aleph_1, \aleph_2) \\
\sigma'_{\mathcal{F}}((\omega_1, \omega_2), (\varpi_1, \varpi_2)) = \sigma'_{\mathcal{F}}(\aleph_1, \aleph_2).
\end{aligned}
$$

This implies that

$$
\begin{aligned}
\sigma_{\mathcal{F}}((\ell_1, \hbar_1)) = \sigma_{\mathcal{F}}((\ell_2, \hbar_2)) = \sigma_{\mathcal{F}}(\Theta, \Phi) \\
\sigma_{\mathcal{F}}((\omega_1, \varpi_1)) = \sigma_{\mathcal{F}}((\omega_2, \varpi_2)) = \sigma_{\mathcal{F}}(\Theta, \Phi).
\end{aligned}
$$

Since (Θ, Φ) fulfils the property P, we obtain that

$$\sigma_{\mathcal{F}}((\ell_1, \omega_1)) = \sigma_{\mathcal{F}}((\varpi_1, \hbar_1))$$

and

$$\sigma_{\mathcal{F}}((\ell_2, \omega_2)) = \sigma_{\mathcal{F}}((\varpi_2, \hbar_2)),$$

which implies that

$$\sigma'_{\mathcal{F}}((\ell_1, \ell_2), (\omega_1, \omega_2)) = \sigma'_{\mathcal{F}}((\hbar_1, \hbar_2), (\varpi_1, \varpi_2)).$$

Thus, we showed that $(\aleph_1, \aleph_2)$ fulfils the property P. Assume that for some

$$(\ell_1, \ell_2), (\omega_1, \omega_2), (u_1, u_2), (v_1, v_2) \in \aleph_1,$$

we have

$$(\ell_1, \ell_2)\mathcal{S}_2(\omega_1, \omega_2)$$

$$\sigma'_{\mathcal{F}}((u_1, u_2), \mathcal{L}(\ell_1, \ell_2)) = \sigma'_{\mathcal{F}}(\aleph_1, \aleph_2)$$

$$\sigma'_{\mathcal{F}}((v_1, v_2), \mathcal{L}(\omega_1, \omega_2)) = \sigma'_{\mathcal{F}}(\aleph_1, \aleph_2).$$

This implies that

$$\ell_1\mathcal{S}\omega_1,\ \ell_2\mathcal{S}\omega_2$$
$$\sigma_{\mathcal{F}}(u_1, J(\ell_1, \ell_2)) = \sigma_{\mathcal{F}}(\Theta, \Phi)$$
$$\sigma_{\mathcal{F}}(v_1, J(\omega_1, \omega_2)) = \sigma_{\mathcal{F}}(\Theta, \Phi),$$

and

$$\ell_2\mathcal{S}\omega_2,\ \ell_1\mathcal{S}\omega_1$$
$$\sigma_{\mathcal{F}}(u_2, J(\ell_2, \ell_1)) = \sigma_{\mathcal{F}}(\Theta, \Phi)$$
$$\sigma_{\mathcal{F}}(v_2, J(\omega_2, \omega_1)) = \sigma_{\mathcal{F}}(\Theta, \Phi).$$

Since $\mathcal{J}$ is a bi-prox comp, so we have

$$u_1\mathcal{S}v_1,\ u_2\mathcal{S}v_2,$$

that is,

$$(u_1, u_2)\mathcal{S}_2(v_1, v_2).$$

which shows $\mathcal{L}$ is a prox comp. Now, from condition (iii), we obtain

$$\sigma_{\mathcal{F}}(\omega_1, J(\omega_0, \varpi_0)) + \sigma_{\mathcal{F}}(\varpi_1, J(\varpi_0, \omega_0)) = 2\sigma_{\mathcal{F}}(\Theta, \Phi)$$

and

$$(\omega_0, \varpi_0)\mathcal{S}_2(\omega_1, \varpi_1),$$

which implies that

$$\sigma_{\mathcal{F}}^{/}((\omega_1, \varpi_1), \mathcal{L}(\omega_0, \varpi_0)) = \sigma_{\mathcal{F}}^{/}(\aleph_1, \aleph_2)$$

and

$$(\omega_0, \varpi_0)\mathcal{S}_2(\omega_1, \varpi_1).$$

Moreover, if $(\omega, \varpi), (u, v) \in \aleph_1$ are such that

$$(\omega_0, \varpi_0)\mathcal{S}_2(u, v),$$

i.e., $\omega\mathcal{S}u$ and $\varpi\mathcal{S}v$, from condition (iv), we have

$$\sigma_{\mathcal{F}}(J(\omega, \varpi), J(u, v)) \le \psi\left(\frac{\sigma_{\mathcal{F}}(\omega, u) + \sigma_{\mathcal{F}}(\varpi, v)}{2}\right) \tag{17}$$

and

$$\sigma_{\mathcal{F}}(J(\varpi, \omega), J(v, u)) \le \psi\left(\frac{\sigma_{\mathcal{F}}(\omega, u) + \sigma_{\mathcal{F}}(\varpi, v)}{2}\right). \tag{18}$$

Adding (17) to (18), we obtain that

$$\sigma_{\mathcal{F}}^{/}((J(\omega, \varpi), J(\varpi, \omega)), (J(u, v), J(v, u))) \le \psi(\sigma_{\mathcal{F}}^{/}((\omega, \varpi), (u, v))),$$

i.e.,

$$\sigma_{\mathcal{F}}^{/}(\mathcal{L}(\omega, \varpi), \mathcal{L}(u, v)) \le \psi(\sigma_{\mathcal{F}}^{/}((\omega, \varpi), (u, v))).$$

Now, all the assumptions of Theorem 4.3 hold, and thus, we conclude then that J possess a best proximity point $(\omega^*, \varpi^*) \in \mathcal{A}_0$, that is, $(\omega^*, \varpi^*) \in \Theta_0 \times \Theta_0$, which satisfies

$$\sigma_{\mathcal{F}}^{/}((\omega^*, \varpi^*), \mathcal{L}(\omega^*, \varpi^*)) = \sigma_{\mathcal{F}}^{/}(\aleph_1, \aleph_2).$$

As we already proved that

$$\sigma_{\mathcal{F}}^{/}(\aleph_1, \aleph_2) = \sigma_{\mathcal{F}}(\Theta, \Phi),$$

so the aforementioned equality implies immediately that

$$\sigma_{\mathcal{F}}(\omega, J(\omega^*, \varpi^*)) = \sigma_{\mathcal{F}}^{/}(\varpi^*, J(\varpi^*, \omega^*)) = \sigma_{\mathcal{F}}(\Theta, \Phi). \qquad \square$$

Theorem 4.7. *Let $(\mathcal{M}, \sigma_{\mathcal{F}})$ is complete $\mathcal{F}$- metric space, $\Theta, \Phi \in Cl(\mathcal{M})$ such that $\Theta_0 \neq \emptyset$. Let R is binary relation over $\mathcal{M}$. Assume that the mapping $\mathcal{J} : \Theta \times \Theta \to \Phi$ is continuous satisfying*

(i) *$\mathcal{J}(\Theta_0 \times \Theta_0) \subseteq \Phi_0$ and (Θ, Φ) fulfils the property P;*
(ii) *$\mathcal{J}$ is a bi-proximal comparative mapping,*
(iii) *there exist $\omega_0, \varpi_0, \omega_1, \varpi_1 \in \Theta_0$ such that*

$$\sigma_{\mathcal{F}}(\omega_1, \mathcal{J}(\omega_0, \varpi_0)) = \sigma_{\mathcal{F}}(\varpi_1, \mathcal{J}(\varpi_0, \omega_0)) = \sigma_{\mathcal{F}}(\Theta, \Phi), \quad \text{and} \quad \omega_0 S \omega_1, \quad \varpi_0 S \varpi_1,$$

(iv) *$\exists a\ \psi \in \Psi$ in such that*

$$\omega, \varpi, u, v \in \Theta, \omega S u, \varpi S v \ \Rightarrow\ \sigma_{\mathcal{F}}(J(\omega, \varpi), J(u, v)) \leq \psi\left(\frac{\sigma_{\mathcal{F}}(\omega, u) + \sigma_{\mathcal{F}}(\varpi, v)}{2}\right),$$

(v) *(H) holds.*

Then $\mathcal{J}$ has a coupled best proximity point.

Theorem 4.8. *Besides the assumptions of* Theorem 4.6 (*respectively* Theorem 4.7), *assume that this condition holds*: $\forall(\omega, \varpi) \in \Theta \times \Theta, \exists \varrho \in \Theta_0$ *such that $\omega S \varrho$ and $\varpi S \varrho$. Then $\mathcal{L}$ has a unique best proximity point (ω^*, ϖ^*) $\in \Theta \times \Theta$. Moreover, we have $\omega^* = \varpi^*$.*

Proof. Suppose (ω, ϖ), $(u, v) \in \Theta \times \Theta$. By the assumptions, there exists $z_1 \in \Theta_0$ such that $\omega \mathcal{S} \varrho_1$ and $u \mathcal{S} \varrho_1$. Likewise, there exists $\varrho_2 \in \Theta_0$ such that $\varpi \mathcal{S} \varrho_2$ and $v \mathcal{S} \varrho_2$. Hence, we obtain $(\omega, \varpi) \mathcal{S}_2 (\varrho_1, \varrho_2)$ and $(u, v) \mathcal{S}_2 (\varrho_1, \varrho_2)$, where $(\varrho_1, \varrho_2) \in \Theta_0 \times \Theta_0$. Now, by Theorem 4.5, we establish that $\mathcal{L}$ has a unique best proximity point, i.e., a unique coupled best proximity point of $\mathcal{L}$. $\square$

By setting $\Theta = \Phi$ in Theorem 3.1, we establish this result.

Theorem 4.9. *Let $(\mathcal{M}, \sigma_{\mathcal{F}})$ is complete $\mathcal{F}$-metric space and $\Theta \in Cl(\mathcal{M})$. Assume that $\mathcal{L} : \Theta \to \Theta$ satisfies the condition*:

$$\sigma_{\mathcal{F}}(\mathcal{L}\omega, \mathcal{L}\varpi) \leq \psi(\sigma_{\mathcal{F}}(\omega, \varpi))$$

$\forall \omega, \varpi \in \Theta$, *where $\psi \in \Psi$. Then there exists $\omega^* \in \Theta$ such that $\mathcal{L}\omega^* = \omega^*$, which is unique.*

By setting $\psi(\imath) = k\imath$ for some $k \in (0, 1)$ and $\imath > 0$ in Theorem 4.9, we obtain the main result of Jleli and Samet [18].

Theorem 4.10. *Let $(\mathcal{M}, \sigma_{\mathcal{F}})$ is complete $\mathcal{F}$-metric space and $\Theta \in Cl(\mathcal{M})$. Assume that $\mathcal{L} : \Theta \to \Theta$ satisfies the condition*:

$$\sigma_{\mathcal{F}}(\mathcal{L}\omega, \mathcal{L}\varpi) \leq k \sigma_{\mathcal{F}}(\omega, \varpi),$$

$\forall \omega, \varpi \in \Theta$, *where $k \in (0, 1)$. Then $\exists \omega^* \in \Theta$ such that $\mathcal{L}\omega^* = \omega^*$, which is unique.*

Author contributions: The author has accepted responsibility for the entire content of this article and approved its submission.

Conflict of interest: The author states no conflict of interest.

References

[1] S. Banach, *Sur les operations dans les ensembles abstraits et leur applications aux equations integrales*, Fundam. Math. **3** (1922), 133–181.

[2] S. S. Basha, *Extensions of Banachas contraction principle*, Numer. Funct. Anal. Optim. **31** (2010), 569–576.

[3] A. Eldred and P. Veeramani, *Existence and convergence of best proximity points*, J. Math. Anal. Appl. **323** (2006), no. 2, 1001–1006.

[4] B. Samet, C. Vetro, and P. Vetro, *Fixed point theorem for α-ψ-contractive type mappings*, Nonlinear Anal. **75** (2012), 2154–2165.

[5] M. Jleli and B. Samet, *Best proximity points for $(\alpha$-$\psi)$-proximal contractive type mappings and applications*, Bulletin des Sciences Mathematiques **137** (2013), 977–995.

[6] A. Abkar and M. Gabeleh, *Best proximity points for cyclic mappings in ordered metric spaces*, J. Optim. Theory Appl. **150** (2011), 188–193.

[7] A. Abkar and M. Gabeleh, *The existence of best proximity points for multivalued non-self-mappings*, Acad. Cienc. Exactas Fs. Nat. Ser. A Math. RACSAM **107** (2013), 319–325.

[8] G. Gecheva, M. Hristov, D. Nedelcheva, M. Ruseva, and B. Zlatanov, *Applications of coupled fixed points for multivalued maps in the equilibrium in duopoly markets and in aquatic ecosystems*, Axioms **10** (2021), no. 2, 44, DOI: https://doi.org/10.3390/axioms10020044.

[9] M. Hristov, A. Ilchev, and B. Zlatanov, *On some application on coupled and best proximity points theorems*, In:AIP Confer. Proc. **2333** (2021), 080008, DOI: https://doi.org/10.1063/5.0041716.

[10] M. Hristov, A. Ilchev, and B. Zlatanov, *Coupled fixed points for Chatterjea type maps with the mixed monotone property in partially ordered metric spaces*, AIP Confer. Proc. **2172** (2019), 060003, DOI: https://doi.org/10.1063/1.5133531.

[11] B. Zlatanov. *Best proximity points in modular function spaces*, Arabian J. Math. **4** (2015), no. 3, 215–227

[12] A. Ilchev and B. Zlatanov. *Fixed and best proximity points forKannan cyclic contractions in modular function spaces*, J. Fixed Point Theory Appl. **19** (2017), no. 4, 2873–2893.

[13] A. Ilchev and B. Zlatanov, *Coupled fixed points and coupled best proximity points in modular function spaces*, Int. J. Pure Appl. Math. **118** (2018), no. 4, 957–977.

[14] A. Ilchev and B. Zlatanov. *Coupled fixed points and coupled best proximity points for cyclic Kannan type contraction maps in modular function spaces*, Mattex Confer. Proc. **1** (2018), 75–88.

[15] B. Zlatanov, *Coupled best proximity points for cyclic contractive maps and their applications*, Fixed Point Theory **22** (2021), 431–452.

[16] I. A. Bakhtin, *The contraction mapping principle in almost metric spaces*, Funct. Anal. **30** (1989), 26–37.

[17] S. Czerwik, *Contraction mappings in b-metric spaces*, Acta Math. Inform. Univ. Ostra. **1** (1993), 5–11.

[18] M. Jleli and B. Samet, *On a new generalization of Metric Spaces*, J. Fixed Point Theory Appl. **20** (2018), 128.

[19] S. A. Al-Mezel, J. Ahmad, and G. Marino, *Fixed point theorems for generalized $(\alpha\beta$-$\psi)$-Contractions in $\mathcal{F}$-metric spaces with applications*, Mathematics **8** (2020), no. 4, 584, DOI: https://doi.org/10.3390/math8040584.

[20] J. Ahmad, A. S. Al-Rawashdeh, and A. E. Al-Mazrooei, *Fixed point results for $(\alpha, \perp_{\mathcal{F}})$-contractions in orthogonal $\mathcal{F}$-metric spaces with applications*, J. Funct. Spaces **2022** (2022), no. 3, 1–10.

[21] M. Alansari, S. Mohammed, and A. Azam, *Fuzzy fixed point results in $\mathcal{F}$-metric spaces with applications*, J. Function Spaces. **2020** (2020), 5142815, 11 pages.

[22] A. E. Al-Mazrooei and J. Ahmad, *Fixed point theorems for rational contractions in $\mathcal{F}$-metric spaces*, J. Math. Anal. **10** (2019), 79–86.

[23] L. A. Alnaser, D. Lateef, H. A. Fouad, and J. Ahmad, *Relation theoretic contraction results in $\mathcal{F}$-metric spaces*, J. Nonlinear Sci. Appl. **12** (2019), 337–344.

[24] L. A. Alnaser, D. Lateef, H. A. Fouad, and J. Ahmad, *New fixed point theorems with applications to non-linear neutral differential equations*, Symmetry **11** (2019), 602.

[25] O. Alqahtani, E. Karapiiinar, and P. Shahi, *Common fixed point results in function weighted metric spaces*. J. Inequal. Appl. **2019** (2019), 164.

[26] A. Beraž, H. Garai, B. Damjanović, and A. Chanda, *Some interesting results on $\mathcal{F}$-metric spaces*, Filomat **33** (2019), no. 10, 3257–3268.

[27] D. Lateef and D. J. Ahmad, *Dass and Guptaas Fixed point theorem in F-metric spaces*, J. Nonlinear Sci. Appl. **12** (2019), 405–411.

[28] A Hussain, H Al-Sulami, N Hussain, and H. Farooq, *Newly fixed disc results using advanced contractions on $\mathcal{F}$-metric space*, J. Appl. Anal. Comput. **10** (2020), no. 6, 2313–2322.

[29] F. Jahangir, P. Haghmaram, and K. Nourouzi, *A note on F-metric spaces*, J. Fixed Point Theory Appl. **23** (2021), 2.

[30] T. Kanwal, A. Hussain, H. Baghani, and M. de la Sen, *New fixed point theorems in orthogonal $\mathcal{F}$-metric spaces with application to fractional differential equation*, Symmetry **12** (2020), no. 5, 832.

[31] Z. D. Mitrović, H. Aydi, N. Hussain, and A. Mukheimer, *Reich, Jungck, and Berinde common fixed point results on $\mathcal{F}$-metric spaces and an application*, Mathematics. **7** (2019), no. 5, 387.

[32] A. Tomar and M. Joshi, *Relation-theoretic nonlinear contractions in an $\mathcal{F}$-metric space and applications*, Rendiconti del Circolo Matematico di Palermo Series **70** (2021), 835–852.
[33] C. Zhu, J. Chen, J. Chen, C. Chen, and H. Huang, *A new generalization of $\mathcal{F}$-metric spaces and some fixed point theorems and applications*, J. Appl. Anal. Comput. **11** (2021), 2649–2663.
[34] D. Guo and V. Lakshmikantham, *Coupled fixed points of nonlinear operators with application*, Nonlinear Anal. **11** (1987), no. 5, 623–632.
[35] T. Bhaskar and V. Lakshmikantham, *Fixed point theorems in partially ordered metric spaces and applications*, Nonlinear Anal. **65** (2006), no. 7, 1379–1393.