Research Article

O. H. Ezzat*

Jordan triple (α,β)-higher $*$-derivations on semiprime rings

https://doi.org/10.1515/dema-2022-0213
received November 25, 2021; accepted February 16, 2023

Abstract: In this article, we define the following: Let $\mathbb{N}_0$ be the set of all nonnegative integers and $D = (d_i)_{i\in\mathbb{N}_0}$ a family of additive mappings of a $*$-ring R such that $d_0 = id_R$. D is called a *Jordan* (α, β)*-higher* $*$*-derivation* (resp. a *Jordan triple* (α, β)*-higher* $*$*-derivation*) of R if $d_n(a^2) = \sum_{i+j=n} d_i(\beta^j(a))d_j(\alpha^i(a^{*^i}))$ (resp. $d_n(aba) = \sum_{i+j+k=n} d_i(\beta^{j+k}(a))d_j(\beta^k(\alpha^i(b^{*^i})))d_k(\alpha^{i+j}(a^{*^{i+j}})))$ for all $a, b \in R$ and each $n \in \mathbb{N}_0$. We show that the two notions of Jordan (α, β)-higher $*$-derivation and Jordan triple (α, β)-higher $*$-derivation on a 6-torsion free semiprime $*$-ring are equivalent.

Keywords: semiprime rings, involutions, derivations, Jordan $*$-derivations, higher derivations

MSC 2020: Primary 16W25, Secondary 16W10, 39B05, 16N60, 16U80

1 Introduction

Investigating the relation between the two different notions of derivation and Jordan derivation was initiated by Herstein in a very famous article [1]. This area of research deals with establishing equivalence between certain kinds of maps in different classes of rings satisfying specific conditions. Several research studies have been carried out in this active area of research (see [2–5] for example and references therein). In this article, we walk in this direction and obtain a new result concerning the concepts of Jordan (α, β)-higher $*$-derivation and Jordan triple (α, β)-higher $*$-derivation.

Let R be a ring. For any $a, b \in R$, recall that R is *prime* (resp. *semiprime*) if $aRb = \{0\}$ (resp. $aRa = \{0\}$) implies $a = 0$ or $b = 0$ (resp. $a = 0$). Given an integer $n \geq 2$, R is said to be n-torsion free if for $a \in R$, $na = 0$ implies $a = 0$. An additive mapping $a \to a^*$ satisfying $(ab)^* = b^*a^*$ and $(a^*)^* = a$ for all $a, b \in R$ is called an *involution* and R is called a $*$-ring. We assume throughout this article that R denotes a $*$-ring.

An additive mapping $d : R \to R$ is called a *derivation* if $d(ab) = d(a)b + ad(b)$ holds for all $a, b \in R$, and it is called a *Jordan derivation* if $d(a^2) = d(a)a + ad(a)$ for all $a \in R$. A theorem by Herstein [1] shows that in a 2-torsion free prime ring any Jordan derivation is a derivation. A *Jordan triple derivation* is an additive mapping $d : R \to R$ satisfying $d(aba) = d(a)ba + ad(b)a + abd(a)$ for all $a, b \in R$. Every derivation is clearly a Jordan triple derivation. Indeed, it is easy to see that in a 2-torsion free ring every Jordan derivation is a Jordan triple derivation [6, Lemma 3.5]. Brešar [7] extended this result to 2-torsion free semiprime ring.

An additive mapping $d : R \to R$ is called a $*$*-derivation* if $d(ab) = d(a)b^* + ad(b)$ holds for all $a, b \in R$, and it is called a *Jordan* $*$*-derivation* if $d(a^2) = d(a)a^* + ad(a)$ holds for all $a \in R$. This map was initially introduced by Brešar and Vukman [8]. The importance of this map appears in the theory of representability of quadratic functionals with sesquilinear forms. There is a connection between this problem and the

*** Corresponding author: O. H. Ezzat,** Department of Mathematics, College of Science and Arts at Balgarn, University of Bisha, Sabt Al-Alaya(61985), Saudi Arabia; Department of Mathematics, Faculty of Science, Al-Azhar University, Nasr City (11884), Cairo, Egypt, e-mail: oahmed@ub.edu.sa, ohezzat@azhar.edu.eg
ORCID: https://orcid.org/0000-0002-8605-3693

innerness of Jordan $*$-derivations [9]. As a natural extension to results in the previous paragraph, we might guess that mathematician would try to find equivalence on a 2-torsion free prime $*$-ring between Jordan $*$-derivations and $*$-derivations. Unfortunately, it was shown that noncommutative prime $*$-rings do not admit non-trivial $*$-derivations in [8]. Researchers then turned their attention to study the relation between Jordan triple $*$-derivations and Jordan $*$-derivations. A *Jordan triple $*$-derivation* is an additive mapping $d: R \to R$ satisfies $d(aba) = d(a)b^*a^* + ad(b)a^* + abd(a)$ for all $a, b \in R$. We can see that on a 2-torsion free $*$-ring any Jordan $*$-derivation is a Jordan triple $*$-derivation [8, Lemma 2]. Vukman [10] showed that on a 6-torsion free semiprime $*$-ring any Jordan triple $*$-derivation is a Jordan $*$-derivation.

Ali and Fošner [11, Theorem 2.1] defined a concept that extended Jordan $*$-derivations. Let α, β be two endomorphisms of R. An additive mapping $d: R \to R$ is said to be a *Jordan $(\alpha, \beta)^*$-derivation* (resp. Jordan triple *$(\alpha, \beta)^*$-derivation*) if $d(a^2) = d(a)\alpha(a^*) + \beta(a)d(a)$ (resp. $d(aba) = d(a)\alpha(b^*a^*) + \beta(a)d(b)\alpha(a^*) + \beta(ab)d(a)$) holds for all $a, b \in R$. In a 2-torsion free ring, every Jordan $(\alpha, \beta)^*$-derivation is a Jordan triple $(\alpha, \beta)^*$-derivation. For the converse statement this is not the case in general [11, Example 2.4]. It was proved in [11] that Jordan $(\alpha, \beta)^*$-derivation and Jordan triple $(\alpha, \beta)^*$-derivation, such that β is an automorphism, are equivalent on a 6-torsion free semiprime $*$-ring.

Let $D = (d_i)_{i\in\mathbb{N}_0}$ be a family of additive mappings of a ring R such that $d_0 = id_R$ and $\mathbb{N}_0$ is the set of all nonnegative integers. Then D is called a *higher derivation* (resp. a *Jordan higher derivation*) of R if for each $n \in \mathbb{N}_0$, $d_n(ab) = \sum_{i+j=n} d_i(a)d_j(b)$ (resp. $d_n(a^2) = \sum_{i+j=n} d_i(a)d_j(a)$) holds for all $a, b \in R$. Obviously, every higher derivation is a Jordan higher derivation. Ferrero and Haetinger [12] extended on 2-torsion free semiprime rings the famous theorem due to Herstein [1] for higher derivations. A family $D = (d_i)_{i\in\mathbb{N}_0}$ of additive mappings of a ring R, where $d_0 = id_R$, is called a *Jordan triple higher derivation* if $d_n(aba) = \sum_{i+j+k=n} d_i(a)d_j(b)d_k(a)$ holds for all $a, b \in R$. Ferrero and Haetinger [12] proved that in a 2-torsion free ring every Jordan higher derivation is a Jordan triple higher derivation. They also showed that in a 2-torsion-free semiprime ring every Jordan triple higher derivation is a higher derivation.

Ezzat [13], inspired by the notions of $*$-derivation and higher derivation, introduced the notions of *higher $*$-derivations*, *Jordan higher $*$-derivations* and *Jordan triple higher $*$-derivations*. He proved that the notions of Jordan triple higher $*$-derivations and Jordan higher $*$-derivations are coincident on 6-torsion free semiprime $*$-rings.

In this article motivated by results in [11,13], we give the definitions of the notions of *(α, β)-higher $*$-derivations*, *Jordan (α, β)-higher $*$-derivations* and *Jordan triple (α, β)-higher $*$-derivations*. Our main purpose is to show that every Jordan triple (α, β)-higher $*$-derivation on a 6-torsion free semiprime $*$-ring is a Jordan (α, β)-higher $*$-derivation. Moreover, in order to show the complete equivalence, we prove that in a 2-torsion free $*$-ring every Jordan (α, β)-higher $*$-derivation is a Jordan triple (α, β)-higher $*$-derivation. Therefore, we can deduce that these two concepts are coincident in 6-torsion free semiprime $*$-ring. This result is a proper extension to the main theorems mentioned in [10,11,13].

2 Preliminaries

We begin by the following definition:

Definition. Let $\mathbb{N}_0$ be the set of all nonnegative integers, α, β two endomorphisms of R, and let $D = (d_i)_{i\in\mathbb{N}_0}$ be a family of additive mappings of R such that $d_0 = id_R$. Assume also that $*$ is an involution defined on R such that for all $a \in R$, $a^{*^0} = a$, and $a^{*^2} = (a^*)^* = a$. Hence, for nonnegative even powers s, $a^{*^s} = a$, and for nonnegative odd powers t, $a^{*^t} = a^*$. D is called:

(a) an *(α, β)-higher $*$-derivation* of R if for each positive integer n,

$$d_n(ab) = \sum_{i+j=n} d_i(\beta^j(a))d_j\Big(\alpha^i\Big(b^{*^i}\Big)\Big) \quad \text{for all } a, b \in R;$$

(b) a *Jordan* (α,β)*-higher* $*$*-derivation* of R if for each positive integer n,

$$d_n(a^2)=\sum_{i+j=n} d_i(\beta^j(a))d_j\big(\alpha^i\big(a^{*^i}\big)\big)\quad \text{for all } a\in R;$$

(c) a *Jordan triple* (α,β)*-higher* $*$*-derivation* of R if for each positive integer n,

$$d_n(aba)=\sum_{i+j+k=n} d_i(\beta^{j+k}(a))d_j\big(\beta^k\big(\alpha^i\big(b^{*^i}\big)\big)\big)d_k\big(\alpha^{i+j}\big(a^{*^{i+j}}\big)\big)\quad \text{for all } a,b\in R.$$

In this definition, we note that if α,β are the identity maps, then we obtain the notion of higher $*$-derivations. Also the first member of this family is an $(\alpha,\beta)^*$-derivation. Therefore, the interesting thing about this new concept is that it covers the notions of higher $*$-derivations, $(\alpha,\beta)^*$-derivations, and $*$-derivations.

We shall need the next basic lemma in the proofs of the main results.

Lemma 2.1. ([7], Lemma 1.1) *Let R be a 2-torsion free semiprime ring. If $a,b\in R$ are such that $arb=0$ for all $r\in R$, then $bra=ab=ba=0$.*

3 Main results

Throughout this section, we will use the following notation:

Notation. Let $D=(d_i)_{i\in\mathbb{N}_0}$ be a Jordan triple higher $*$-derivation of R. For every fixed $n\in\mathbb{N}_0$ and each $a,b\in R$, we denote by $A_n(a)$ and $B_n(a,b)$ the elements of R defined by:

$$A_n(a)=d_n(a^2)-\sum_{i+j=n} d_i(\beta^j(a))d_j\big(\alpha^i\big(a^{*^i}\big)\big),$$
$$B_n(a,b)=d_n(ab+ba)-\sum_{i+j=n} d_i(\beta^j(a))d_j\big(\alpha^i\big(b^{*^i}\big)\big)-\sum_{i+j=n} d_i(\beta^j(b))d_j\big(\alpha^i\big(a^{*^i}\big)\big).$$

We now give the following lemma.

Lemma 3.1. *Let R be a 2-torsion free semiprime $*$-ring and let $D=(d_i)_{i\in\mathbb{N}_0}$ a Jordan triple (α,β)-higher $*$-derivation of R such that β is an automorphism of R and $\alpha\beta=\beta\alpha$. If $A_m(a)=0$ for all $a\in R$ and for each $m\le n$, then $\beta^n(a^2)A_n(a)=A_n(a)\beta^n(a^2)=0$ for all $a\in R$ and for each $n\in\mathbb{N}_0$.*

Proof. We can compute the value of $L=d_n(a^2ba^2)$ in the following two ways:

First by substitution of aba for b in the definition of Jordan triple (α,β)-higher $*$-derivation. So we look at L as $d_n(a(aba)a)$ and by using the definition twice, we obtain

$$\begin{aligned}
L&=\sum_{i+j+k=n} d_i(\beta^{j+k}(a))d_j\big(\beta^k\big(\alpha^i\big((aba)^{*^i}\big)\big)\big)d_k\big(\alpha^{i+j}\big(a^{*^{i+j}}\big)\big)\\
&=\sum_{i+j+k=n} d_i(\beta^{j+k}(a))\Bigg(\sum_{p+q+r=j} d_p\big(\beta^{q+r+k}\big(\alpha^i\big(a^{*^i}\big)\big)\big)d_q\big(\beta^{r+k}\big(\alpha^{p+i}\big(b^{*^{p+i}}\big)\big)\big)d_r\big(\beta^k\big(\alpha^{p+q+i}\big(a^{*^{p+q+i}}\big)\big)\big)\Bigg)d_k\big(\alpha^{i+j}\big(a^{*^{i+j}}\big)\big)\\
&=\sum_{i+p+q+r+k=n} d_i(\beta^{p+q+r+k}(a))d_p\big(\beta^{q+r+k}\big(\alpha^i\big(a^{*^i}\big)\big)\big)d_q\big(\beta^{r+k}\big(\alpha^{p+i}\big(b^{*^{p+i}}\big)\big)\big)d_r\big(\beta^k\big(\alpha^{p+q+i}\big(a^{*^{p+q+i}}\big)\big)\big)\\
&\quad\times d_k\big(\alpha^{i+p+q+r}\big(a^{*^{i+p+q+r}}\big)\big)\\
&=\sum_{i+p=n} d_i(\beta^p(a))d_p\big(\alpha^i\big(a^{*^i}\big)\big)\alpha^n\big(b^{*^n}a^{2^{*^n}}\big)+\beta^n(a^2b)\sum_{r+k=n} d_r(\beta^r(a))d_k\big(\alpha^r\big(a^{*^r}\big)\big)\\
&\quad+\sum_{\substack{i+p+q+r+k=n\\ i+p\neq n,\, r+k\neq n}} d_i(\beta^{p+q+r+k}(a))d_p\big(\beta^{q+r+k}\big(\alpha^i\big(a^{*^i}\big)\big)\big)d_q\big(\beta^{r+k}\big(\alpha^{p+i}\big(b^{*^{p+i}}\big)\big)\big)d_r\big(\beta^k\big(\alpha^{p+q+i}\big(a^{*^{p+q+i}}\big)\big)\big)\\
&\quad\times d_k\big(\alpha^{i+p+q+r}\big(a^{*^{i+p+q+r}}\big)\big).
\end{aligned}$$

Now the second way to compute L is the substitution of a^2 for a in the definition of Jordan triple (α,β)-higher $*$-derivation and using our assumption that $A_m(a)=0$ for $m<n$ to obtain

$$
\begin{aligned}
L &= \sum_{i+j+k=n} d_i\Big(\beta^{j+k}(a^2)d_j\Big(\beta^k\Big(\alpha^i(b)^{*^i}\Big)\Big)\Big)d_k\Big(\alpha^{i+j}\Big(a^{2*^{i+j}}\Big)\Big)\\
&= d_n(a^2)\alpha^n(b^{*^n}a^{2^{*^n}}) + \beta^n(a^2b)d_n(a^2) + \sum_{\substack{i+j+k=n\\ i\neq n,k\neq n}} d_i(\beta^{j+k}(a^2))d_j\Big(\beta^k\Big(\alpha^i(b)^{*^i}\Big)\Big)d_k\Big(\alpha^{i+j}\Big(a^{2*^{i+j}}\Big)\Big)\\
&= d_n(a^2)\alpha^n(b^{*^n}a^{2^{*^n}}) + \beta^n(a^2b)d_n(a^2) + \sum_{\substack{i+j+k=n\\ i\neq n,k\neq n}}\left(\sum_{u+v=i} d_u(\beta^{v+j+k}(a))d_v\Big(\beta^{j+k}\Big(\alpha^u\Big(a^{*^u}\Big)\Big)\Big)\right)d_j\Big(\beta^k\Big(\alpha^i(b)^{*^i}\Big)\Big)\\
&\quad\times\left(\sum_{s+t=k} d_s\Big(\beta^t\Big(\alpha^{i+j}\Big(a^{*^{i+j}}\Big)\Big)\Big)d_t\Big(\alpha^{s+i+j}\Big(a^{*^{i+j+s}}\Big)\Big)\right)\\
&= d_n(a^2)\alpha^n(b^{*^n}a^{2^{*^n}}) + \beta^n(a^2b)d_n(a^2) + \sum_{\substack{u+v+j+s+t=n\\ u+v\neq n,s+t\neq n}} d_u(\beta^{v+j+k}(a))d_v\Big(\beta^{j+k}\Big(\alpha^u\Big(a^{*^u}\Big)\Big)\Big)d_j\Big(\beta^k\Big(\alpha^{u+v}(b)^{*^{u+v}}\Big)\Big)\\
&\quad\times d_s\Big(\beta^t\Big(\alpha^{u+v+j}\Big(a^{*^{u+v+j}}\Big)\Big)\Big)d_t\Big(\alpha^{s+u+v+j}\Big(a^{*^{u+v+j+s}}\Big)\Big).
\end{aligned}
$$

Now, subtracting the two values so obtained for L and using our notation we find that

$$A_n(a)\alpha^n\Big(b^{*^n}a^{2^{*^n}}\Big) + \beta^n(a^2b)A_n(a) = 0. \tag{3.1}$$

In case n is even (3.1) reduces to

$$A_n(a)\alpha^n(ba^2) + \beta^n(a^2b)A_n(a) = 0 \quad \text{for all } a,b\in R. \tag{3.2}$$

Replacing b by ra^2b, $r\in R$, gives

$$A_n(a)\alpha^n(ra^2)\alpha^n(ba^2) + \beta^n(a^2r)\beta^n(a^2b)A_n(a) = 0 \quad \text{for all } a,b,r\in R.$$

Using (3.2) for the value of $A_n(a)\alpha^n(ra^2)$ yields that

$$-\beta^n(a^2r)A_n(a)\alpha^n(ba^2) + \beta^n(a^2r)\beta^n(a^2b)A_n(a) = 0 \quad \text{for all } a,b,r\in R.$$

Again, using (3.2) for the value of $A_n(a)\alpha^n(ba^2)$ yields, in view of R is 2-torsion free that

$$\beta^n(a^2r)\beta^n(a^2b)A_n(a) = 0 \quad \text{for all } a,b,r\in R.$$

Now put $r=b\beta^{-n}(A_n(a))r$ in the last expression, we reach to

$$\beta^n(a^2b)A_n(a)\beta^n(r)\beta^n(a^2b)A_n(a) = 0 \quad \text{for all } a,b,r\in R.$$

Since β onto, this implies that

$$\beta^n(a^2b)A_n(a)R\beta^n(a^2b)A_n(a) = \{0\} \quad \text{for all } a,b\in R.$$

Hence, by the semiprimeness of R, we obtain $\beta^n(a^2b)A_n(a)=0$ for all $a,b\in R$. Again since β is onto we have $\beta^n(a^2)RA_n(a)=\{0\}$ for all $a\in R$, and by Lemma 2.1, we reach to $\beta^n(a^2)A_n(a)=A_n(a)\beta^n(a^2)=0$ for all $a\in R$.

In case n is odd (3.1) reduces to

$$A_n(a)\alpha^n(b^*a^{2*}) + \beta^n(a^2b)A_n(a) = 0 \quad \text{for all } a,b\in R. \tag{3.3}$$

Putting $b=ra^2b$ gives for all $a,b,r\in R$ that

$$A_n(a)\alpha^n(b^*a^{2*})\alpha^n(r^*a^{2*}) + \beta^n(a^2r)\beta^n(a^2b)A_n(a) = 0.$$

Substituting the value of $A_n(a)\alpha^n(b^*a^{2*})$ from (3.3) in the last relation gives for all $a,b,r\in R$ that

$$-\beta^n(a^2b)A_n(a)\alpha^n(r^*a^{2*}) + \beta^n(a^2r)\beta^n(a^2b)A_n(a) = 0.$$

Again by using (3.3) for the value of $A_n(a)\alpha^n(r^*a^{2*})$, we obtain for all $a, b, r \in R$

$$\beta^n(a^2)(\beta^n(ba^2r) + \beta^n(ra^2b))A_n(a) = 0. \tag{3.4}$$

Taking $r = b$ in (3.4) leads, in view of R is 2-torsion free, to

$$\beta^n(a^2b)\beta^n(a^2b)A_n(a) = 0 \quad \text{for all } a, b \in R. \tag{3.5}$$

Now putting $r = b\beta^{-n}(A_n(a))r$ in (3.4) gives for all $a, b, r \in R$

$$\beta^n(a^2b)\beta^n(a^2b)A_n(a)\beta^n(r)A_n(a) + \beta^n(a^2b)A_n(a)\beta^n(ra^2b)A_n(a) = 0.$$

But using (3.5), the first summand of the last equation is zero. Hence, we obtain $\beta^n(a^2b)A_n(a)\beta^n(ra^2b)A_n(a) = 0$ for all $a, b, r \in R$. Surjectiveness of β leads to $\beta^n(a^2b)A_n(a)R\beta^n(a^2b)A_n(a) = \{0\}$ for all $a, b \in R$ and since R is semiprime we obtain $\beta^n(a^2)\beta^n(b)A_n(a) = 0$ for all $a, b \in R$. Again by using the surjectiveness of β, we find $\beta^n(a^2)RA_n(a) = \{0\}$ for all $a \in R$. Thus, since R is semiprime we obtain by Lemma 2.1 that $A_n(a)\beta^n(a^2) = \beta^n(a^2)A_n(a) = 0$ for all $a \in R$.

So for either two cases the required result holds and our lemma is proved. □

We will use the following remark without any explicit mention in the steps of the next theorem proof.

Remark 3.1. Straightforward computations show that $A_n(a + b) = A_n(a) + A_n(b) + B_n(a, b)$, $A_n(a) = A_n(-a)$ and $B_n(-a, b) = -B_n(a, b)$ for each pair $a, b \in R$ and for all nonnegative integer n.

Now, we are ready to prove our main results

Theorem 3.2. *Let R be a 6-torsion free semiprime $*$-ring and β an automorphism of R. Then every Jordan triple (α, β)-higher $*$-derivation $D = (d_i)_{i\in\mathbb{N}_0}$ of R, with $\alpha\beta = \beta\alpha$, is a Jordan (α, β)-higher $*$-derivation of R.*

Proof. We will use induction on n in our proof. We see trivially that $A_0(a) = 0$ for all $a \in R$. In case $n = 1$, we obtain from [11, Theorem 2.1] that $A_1(a) = 0$ for all $a \in R$. So we suppose that $A_m(a) = 0$ for all $a \in R$ and $m < n$. Now we can use Lemma 3.1 to obtain

$$A_n(a)\beta^n(a^2) = 0 \qquad \text{for all } \ a \in R \tag{3.6}$$

and

$$\beta^n(a^2)A_n(a) = 0 \quad \text{for all } a \in R. \tag{3.7}$$

The replacement of $a + b$ for a in (3.6) gives

$$\begin{aligned} &A_n(a)\beta^n(b^2) + A_n(b)\beta^n(a^2) + B_n(a, b)\beta^n(a^2 + b^2) + (A_n(a) + A_n(b) + B_n(a, b))\beta^n(ab + ba) \\ &\quad = 0 \ \text{ for all } a, b \in R. \end{aligned} \tag{3.8}$$

By replacing a by $-a$ in (3.8) we obtain

$$\begin{aligned} &A_n(a)\beta^n(b^2) + A_n(b)\beta^n(a^2) - B_n(a, b)\beta^n(a^2 + b^2) - (A_n(a) + A_n(b) - B_n(a, b))\beta^n(ab + ba) \\ &\quad = 0 \ \text{ for all } a, b \in R. \end{aligned} \tag{3.9}$$

Adding (3.8) and (3.9) we obtain, since R is 2-torsion free that

$$B_n(a, b)\beta^n(a^2 + b^2) + (A_n(a) + A_n(b))\beta^n(ab + ba) = 0 \quad \text{for all } a, b \in R. \tag{3.10}$$

Substituting $2a$ for a in (3.10) gives in view of the fact that R is 2-torsion free that

$$4B_n(a, b)\beta^n(a^2) + B_n(a, b)\beta^n(b^2) + 4A_n(a)\beta^n(ab + ba) + A_n(b)(ab + ba) = 0 \quad \text{for all } a, b \in R. \tag{3.11}$$

Comparing (3.10) and (3.11) we have, since R is 3-torsion free, that

$$B_n(a, b)\beta^n(a^2) + A_n(a)\beta^n(ab + ba) = 0 \quad \text{for all } a, b \in R. \tag{3.12}$$

Multiply (3.12) by $A_n(a)\beta^n(a)$ from the right and using (3.7) we obtain

$$A_n(a)\beta^n(ab)A_n(a)\beta^n(a) + A_n(a)\beta^n(b)\beta^n(a)A_n(a)\beta^n(a) = 0 \quad \text{for all } a, b \in R. \tag{3.13}$$

Substituting b by ba in (3.13) and multiplying by $\beta^n(a)$ from the left we obtain using that β is onto $(\beta^n(a)A_n(a)\beta^n(a))R(\beta^n(a)A_n(a)\beta^n(a)) = \{0\}$ for all $a, b \in R$. But since R is semiprime $\beta^n(a)A_n(a)\beta^n(a) = 0$ for all $a \in R$. So (3.13) reduces to $A_n(a)\beta^n(a)\beta^n(b)A_n(a)\beta^n(a) = 0$, for all $a, b \in R$. Since β is onto, we have $A_n(a)\beta^n(a)RA_n(a)\beta^n(a) = \{0\}$, for all $a \in R$. Again, since R is semiprime, we have

$$A_n(a)\beta^n(a) = 0 \quad \text{for all } a \in R. \tag{3.14}$$

In view of (3.14), (3.12) reduces to $B_n(a, b)\beta^n(a^2) + A_n(a)\beta^n(ba) = 0$ for all $a, b \in R$. Multiplying this relation by $\beta^n(a)$ from left and by $A_n(a)$ from right we obtain for all $a, b \in R$, $\beta^n(a)A_n(a)\beta^n(b)\beta^n(a)A_n(a) = 0$. Since β is onto we obtain for all $a \in R$, $\beta^n(a)A_n(a)R\beta^n(a)A_n(a) = \{0\}$ and by the semiprimeness of R we have

$$\beta^n(a)A_n(a) = 0 \quad \text{for all } a \in R. \tag{3.15}$$

Linearizing (3.14) we have

$$A_n(a)\beta^n(b) + A_n(b)\beta^n(a) + B_n(a, b)\beta^n(a + b) = 0 \text{ for all } a, b \in R. \tag{3.16}$$

Taking $a = -a$ in (3.16) we obtain

$$A_n(a)\beta^n(b) - A_n(b)\beta^n(a) + B_n(a, b)\beta^n(a - b) = 0 \text{ for all } a, b \in R. \tag{3.17}$$

Adding (3.16) and (3.17) we obtain, since R is 2-torsion free, that

$$A_n(a)\beta^n(b) + B_n(a, b)\beta^n(a) = 0 \text{ for all } a, b \in R. \tag{3.18}$$

Right multiplication (3.18) by $A_n(a)$ and using (3.15) gives for all $a, b \in R$, $A_n(a)\beta^n(b)A_n(a) = 0$. Since β is onto we obtain $A_n(a)RA_n(a) = 0$ for all $a \in R$. By the semiprimeness of R, we obtain $A_n(a) = 0$ for all $a \in R$. □

To obtain the equivalence between Jordan triple (α, β)-higher $*$-derivations and Jordan (α, β)-higher $*$-derivations we turn to the other direction in the following theorem.

Theorem 3.3. *Let R be a 2-torsion free $*$-ring. Then every Jordan (α, β)-higher $*$-derivation $D = (d_i)_{i\in\mathbb{N}_0}$ of R is a Jordan triple (α, β)-higher $*$-derivation of R.*

Proof. By the definition, we have for all $a, b \in R$ and for each positive integer n

$$d_n(a^2) = \sum_{i+j=n} d_i(\beta^j(a))d_j\big(\alpha^i\big(a^{*^i}\big)\big). \tag{3.19}$$

Put $w = a + b$ and using (3.19)

$$\begin{aligned} d_n(w^2) &= \sum_{i+j=n} d_i(\beta^j(a + b))d_j\big(\alpha^i\big((a + b)^{*^i}\big)\big) \\ &= \sum_{i+j=n} (d_i\beta^j(a))d_j\big(\alpha^i\big(a^{*^i}\big)\big) + (d_i\beta^j(a))d_j\big(\alpha^i\big(b^{*^i}\big)\big) + d_i(\beta^j(b))d_j\big(\alpha^i\big(a^{*^i}\big)\big) + d_i(\beta^j(b))d_j\big(\alpha^i\big(b^{*^i}\big)\big), \end{aligned}$$

and

$$\begin{aligned} d_n(w^2) &= d_n(a^2 + ab + ba + b^2) \\ &= d_n(a^2) + d_n(b^2) + d_n(ab + ba) \\ &= \sum_{l+m=n} d_l(\beta^m(a))d_m\big(\alpha^l\big(a^{*^l}\big)\big) + \sum_{r+s=n} d_r(\beta^s(b))d_s\big(\alpha^r\big(b^{*^r}\big)\big) + d_n(ab + ba). \end{aligned}$$

Subtracting the last two expressions of $d_n(w^2)$ gives for all $a, b \in R$ and each positive integer n

$$d_n(ab+ba)=\sum_{i+j=n}\left(d_i(\beta^j(a))d_j\left(\alpha^i\left(b^{*^i}\right)\right)+d_i(\beta^j(b))d_j\left(\alpha^i\left(a^{*^i}\right)\right)\right). \tag{3.20}$$

Now put $c=a(ab+ba)+(ab+ba)a$. Using (3.20) we obtain

$$\begin{aligned}
d_n(c)&=\sum_{i+j=n}d_i(\beta^j(a))d_j\left(\alpha^i\left((ab+ba)^{*^i}\right)\right)+\sum_{i+j=n}d_i(\beta^j(ab+ba))d_j\left(\alpha^i\left(a^{*^i}\right)\right)\\
&=\sum_{i+r+s=n}\left(d_i(\beta^{r+s}(a))d_r\left(\beta^s\left(\alpha^i\left(a^{*^i}\right)\right)\right)d_s\left(\alpha^{i+r}\left(b^{*^{i+r}}\right)\right)+d_i(\beta^{r+s}(a))d_r\left(\beta^s\left(\alpha^i\left(b^{*^i}\right)\right)\right)d_s\left(\alpha^{i+r}\left(a^{*^{i+r}}\right)\right)\right)\\
&\quad+\sum_{k+l+j=n}\left(d_k(\beta^{l+j}(a))d_l\left(\beta^j\left(\alpha^k\left(b^{*^k}\right)\right)\right)d_j\left(\alpha^{k+l}\left(a^{*^{k+l}}\right)\right)+d_k(\beta^{l+j}(b))d_l\left(\beta^j\left(\alpha^k\left(a^{*^k}\right)\right)\right)d_j\left(\alpha^{k+l}\left(a^{*^{k+l}}\right)\right)\right)\\
&=\sum_{i+r+s=n}d_i(\beta^{r+s}(a))d_r\left(\beta^s\left(\alpha^i\left(a^{*^i}\right)\right)\right)d_s\left(\alpha^{i+r}\left(b^{*^{i+r}}\right)\right)+2\sum_{i+j+k=n}d_i(\beta^{j+k}(a))d_j\left(\beta^k\left(\alpha^i\left(b^{*^i}\right)\right)\right)d_k\left(\alpha^{i+j}\left(a^{*^{i+j}}\right)\right)\\
&\quad+\sum_{k+l+j=n}d_k(\beta^{l+j}(b))d_l\left(\beta^j\left(\alpha^k\left(a^{*^k}\right)\right)\right)d_j\left(\alpha^{k+l}\left(a^{*^{k+l}}\right)\right).
\end{aligned}$$

Also, we have

$$\begin{aligned}
d_n(c)&=d_n(2aba+(a^2b+ba^2))\\
&=2d_n(aba)+d_n(a^2b+ba^2)\\
&=2d_n(aba)+\sum_{i+r+s=n}d_i(\beta^{r+s}(a))d_r\left(\beta^s\left(\alpha^i\left(a^{*^i}\right)\right)\right)d_s\left(\alpha^{i+r}\left(b^{*^{i+r}}\right)\right)\\
&\quad+\sum_{k+l+j=n}d_k(\beta^{l+j}(b))d_l\left(\beta^j\left(\alpha^k\left(a^{*^k}\right)\right)\right)d_j\left(\alpha^{k+l}\left(a^{*^{k+l}}\right)\right).
\end{aligned}$$

Subtracting the last two expressions of $d_n(c)$ and using the fact that R is 2-torsion free we obtain for all $a,b\in R$ and each positive integer n

$$d_n(aba)=\sum_{i+j+k=n}d_i(\beta^{j+k}(a))d_j\left(\beta^k\left(\alpha^i\left(b^{*^i}\right)\right)\right)d_k\left(\alpha^{i+j}\left(a^{*^{i+j}}\right)\right).$$

This proves the theorem. □

By Theorems 3.2 and 3.3, we can deduce the following:

Theorem 3.4. *If β is an automorphism such that $\alpha\beta=\beta\alpha$, then the notions of a Jordan (α,β)-higher ∗-derivation and a Jordan triple (α,β)-higher ∗-derivation on a 6-torsion free semiprime ∗-ring are equivalent.*

We conclude our results by the following example showing that, without the adopted hypotheses of Theorem 3.4, the equivalence between the two notions of a Jordan (α,β)-higher ∗-derivation and a Jordan triple (α,β)-higher ∗-derivation may not be achieved on arbitrary rings.

Example 3.1. Let S be the set of all 3×3 strictly lower triangular matrices over a commutative ring R,

$$S=\left\{\begin{pmatrix}0&0&0\\a&0&0\\b&c&0\end{pmatrix}\middle|\quad a,b,c\in R\right\}.$$

Define, for all $a,b,c\in R$, the involution $*$ on S by

$$\begin{pmatrix}0&0&0\\a&0&0\\b&c&0\end{pmatrix}^*=\begin{pmatrix}0&0&0\\c&0&0\\b&a&0\end{pmatrix},$$

and the two endomorphisms $\alpha,\beta:S\to S$ by

$$\alpha\begin{pmatrix}0&0&0\\a&0&0\\b&c&0\end{pmatrix}=\begin{pmatrix}0&0&0\\-a&0&0\\b&-c&0\end{pmatrix},\quad \beta\begin{pmatrix}0&0&0\\a&0&0\\b&c&0\end{pmatrix}=\begin{pmatrix}0&0&0\\a&0&0\\-b&-c&0\end{pmatrix}.$$

Indeed β is an automorphism and $\alpha\beta = \beta\alpha$. Finally, suppose that the family $D = (d_i)_{i\in\mathbb{N}_0}$ of mappings, with $d_0 = id_S$, are defined on S for each positive integer n by

$$d_n\begin{pmatrix}0&0&0\\a&0&0\\b&c&0\end{pmatrix}=\begin{pmatrix}0&0&0\\0&0&0\\0&nc&0\end{pmatrix}.$$

Then straightforward calculations show that D is a Jordan triple (α, β)-higher $*$-derivation of S. However, D is not a Jordan (α, β)-higher $*$-derivation of S.

The following corollaries are immediate consequence of Theorem 3.4.

Corollary 3.1. ([11], Theorem 2.1) *If β is an automorphism, then the notions of Jordan $(\alpha, \beta)^*$-derivation and Jordan triple $(\alpha, \beta)^*$-derivation on a 6-torsion free semiprime $*$-ring are equivalent.*

Corollary 3.2. ([13], Theorem 2.3) *The notions of Jordan higher $*$-derivation and Jordan triple higher $*$-derivation on a 6-torsion free semiprime $*$-ring are coincident.*

Corollary 3.3. ([10], Theorem 1) *Let R be a 6-torsion free semiprime $*$-ring. Then every Jordan triple $*$-derivation of R is a Jordan higher $*$-derivation of R.*

4 Discussion and conclusions

In this article, we introduced the notions of *(α, β)-higher $*$-derivations*, *Jordan (α, β)-higher $*$-derivations*, and *Jordan triple (α, β)-higher $*$-derivations*. The definitions of these mappings are motivated by the notion of Jordan $*$-derivation, which is closely connected with the problem of the representability of quadratic functionals by sesquilinear forms. We established that every Jordan triple (α, β)-higher $*$-derivation on a 6-torsion free semiprime $*$-ring is a Jordan (α, β)-higher $*$-derivation. Also, we proved that in a 2-torsion free $*$-ring every Jordan (α, β)-higher $*$-derivation is a Jordan triple (α, β)-higher $*$-derivation. These results make it possible to study some functional equations in prime and semiprime rings with involutions. Moreover, the equivalence between these two new maps can be studied in other different classes of algebraic structures such as triangular algebras.

Acknowledgement: I would like to give my deep thanks to the referees and the handling editor for their valuable comments.

Conflict of interest: The author states no conflict of interest.

References

[1] I. N. Herstein, *Jordan derivations of prime rings*, Proc. Amer. Math. Soc. **8** (1957), no. 6, 1104–1110, DOI: https://doi.org/10.2307/2032688.

[2] A. Fošner and W. Jing, *A note on Jordan derivations of triangular rings*, Aeq. Math. **94** (2020), no. 2, 277–285, DOI: https://doi.org/10.1007/s00010-019-00689-y.

[3] N. Rehman, O. Golbasi, and E. Koc, *Lie ideals and* (α, β)*-derivations of* $*$*-prime rings*, Rend. Circ. Mat. Palermo **62** (2013), no. 2, 245–251, DOI: https://doi.org/10.1007/s12215-013-0119-5.
[4] N. A. Dar and S. Ali, *On the structure of generalized Jordan* $*$*-derivations of prime rings*, Comm. Algebra **49** (2020), no. 4, 1422–1430, DOI: https://doi.org/10.1080/00927872.2020.1837148.
[5] O. H. Ezzat, *Functional equations related to higher derivations in semiprime rings*, Open Math. **19** (2021), no. 1, 1359–1365, DOI: https://doi.org/10.1515/math-2021-0123.
[6] I. N. Herstein, *Topics in Ring Theory*, University of Chicago Press, Chicago, 1969.
[7] M. Brešar, *Jordan mappings of semiprime rings*, J. Algebra. **127** (1989), no. 1, 218–228. DOI: https://doi.org/10.1016/0021-8693(89)90285-8.
[8] M. Brešar and J. Vukman, *On some additive mappings in rings with involution*, Aequationes Math. J. **38** (1989), no. 2, 178–185. DOI: https://doi.org/10.1007/BF01840003.
[9] P. Šemrl, *Quadratic functionals and Jordan* $*$*-derivations*, Studia Math. **97** (1991), no. 3, 157–165.
[10] J. Vukman, *A note on Jordan* $*$*-derivations in semiprime rings with involution*, Int. Math. Forum **1** (2006), no. 13, 617–622, DOI: http://dx.doi.org/10.12988/imf.2006.06053.
[11] S. Ali and A. Fošner, *On Jordan* $(\alpha, \beta)^*$*-derivations in semiprime* $*$*-rings*, Int. J. Algebra **4** (2010), no. 3, 99–108.
[12] M. Ferrero and C. Haetinger, *Higher derivations and a theorem by Herstein*, Quaest. Math. **25** (2002), no. 2, 249–257, DOI: https://doi.org/10.2989/16073600209486012.
[13] O. H. Ezzat, *A note on Jordan triple higher* $*$*-derivations on semiprime rings*, ISRN Algebra **2014** (2014), 1–5, DOI: https://doi.org/10.1155/2014/365424.