DE GRUYTER Demonstratio Mathematica 2024; 57: 20230161

Research Article

Tahair Rasham, Arjumand Mustafa, Aiman Mukheimer, Muhammad Nazam, and Wasfi Shatanawi*

Novel results for two families of multivalued dominated mappings satisfying generalized nonlinear contractive inequalities and applications

https://doi.org/10.1515/dema-2023-0161
received February 3, 2023; accepted February 23, 2024

Abstract: In this manuscript, we prove new extensions of Nashine, Wardowski, Feng-Liu, and Ćirić-type contractive inequalities using orbitally lower semi-continuous functions in an orbitally complete b-metric space. We accomplish new multivalued common fixed point results for two families of dominated set-valued mappings in an ordered complete orbitally b-metric space. Some new definitions and illustrative examples are given to validate our new results. To show the novelty of our results, applications are given to obtain the solution of nonlinear integral and fractional differential equations. Our results expand the hypothetical consequences of Nashine et al. (*Feng–Liu-type fixed point result in orbital b-metric spaces and application to fractal integral equation*, Nonlinear Anal. Model. Control. 26 (2021), no. 3, 522–533) and Rasham et al. (*Common fixed point results for new Ciric-type rational multivalued-contraction with an application*, J. Fixed Point Theory Appl. 20 (2018), no. 1, Paper No. 45).

Keywords: fixed point, orbitally b-metric space, new extensions of Nashine, Wardowski, Feng-Liu and Ćirić-type contraction, two families of set-valued dominated mappings, integral equations, fractional differential equations

MSC 2020: 47H10, 47H04, 45P05

1 Introduction and preliminaries

The Banach contraction principle for multivalued mappings was given by Nadler [1].

Subsequently, Ćirić [2], Feng and Liu [3], Minak et al. [4], Nicolae [5], and Rasham et al. [6] extended this work. Bakhtin [7] and Czerwik [8–10] studied Banach's theorem in metric and b-metric spaces. Mlaiki et al. [11] discussed Banach's theorem in controlled metric-type space, while Aiadi et al. [12] extended Banach's theorem

*** Corresponding author: Wasfi Shatanawi,** Department of Mathematics and Sciences, Prince Sultan University, Riyadh, Saudi Arabia; Department of Mathematics, Faculty of Science, Hashemite University, Zarqa, Jordan; Department of Medical Research, China Medical University, Taichung 40402, Taiwan, e-mail: wshatanawi@yahoo.com, wshatanawi@psu.edu.sa
Tahair Rasham: Department of Mathematics, University of Poonch Rawalakot, AJK, Pakistan, e-mail: tahairrasham@upr.edu.pk
Arjumand Mustafa: Department of Mathematics, University of Poonch Rawalakot, AJK, Pakistan, e-mail: arjumandmustafa5591@smail.upr.edu.pk
Aiman Mukheimer: Department of Mathematics and Sciences, Prince Sultan University, Riyadh, Saudi Arabia, e-mail: mukheimer@psu.edu.sa
Muhammad Nazam: Department of Mathematics, Allama Iqbal Open University, Islamabad, Pakistan, e-mail: muhammad.nazam@aiou.edu.pk

to controlled J-metric spaces. Haque et al. [13] studied Fredholm-type integral equation in controlled rectangular metric-like spaces. Also, Shatanawi et al. [14] investigated Banach's theorem in tvs-cone metric space by applying Mizoguchi-Takahashi-type theorems. Wardowski [15] presented a seminal extension of Banach's contraction named it F-contraction and established a fixed point results. Consequently, Sgroi and Vetro [16] showed the presence of specific outcomes for multivalued F-contractive mappings and demonstrated the use of their principal hypothesis for integral and functional equations. Nicolae [5] introduced some new fixed point problems obeying Feng-Liu-type contractive conditions that carry out definite roles.

Padcharoen et al. [17] showed periodic fixed point results for advanced-type F-contractive mappings in modular-like-metric-spaces. Aydi et al. [18] discussed F-contraction via admissible mappings in complete metric space and found the solution of the integral equation using their main theorem. Nashine et al. [19] proved some new fixed point results in orbital b-metric spaces and applied their results to prove the existence of a solution to an integral equation under a set of conditions.

Rasham et al. [20] discussed new theorems in modular-like metric spaces linked with fuzzy-dominated F-contractive mappings and showed applications to obtain the solution of fractional and integral equations. Moreover, Rasham et al. [21] achieved some new common fixed points of two families of multivalued dominated mappings obeying advanced rational type $\alpha_* - \psi$ contractive conditions on a closed set in a complete dislocated b-metric space. They used a specific weaker class of strictly increasing function A in place of the class of function F used by Wardowski [15]. Recently, Nashine et al. [19] achieved fixed point results for multivalued mappings fulfilling the Wardowski-Feng-Liu-type contraction on orbitally complete b-metric space; they showed new definitions and examples to clarify their results and proved an application for fractal integral equations. However, Qawaqneh et al. [22] obtained some results for multivalued contractions in b-metric spaces.

In this research work, we prove the existence of fixed point results for a coupled families of dominated set-valued mappings fulfilling Nashine, Wardowski, Feng-Liu, and Ćirić-type contraction for orbitally lower several semi-continuous functions in a complete orbitally b-metric space shown by Nashine et al. [19]. Moreover, some new multi-common fixed point results are proved for two families of dominated set-valued mappings in the setting of ordered complete orbitally b-metric space. Some new definitions and illustrative examples are given to validate our new results. To show the novelty of our results, applications are given to obtain the solution of nonlinear integral equations and fractional differential equations.

Definition 1.1. [7,8] Let Γ be a non-empty set. A function $d_b : \Gamma \times \Gamma \to [0, \infty)$ is said to be a b-metric on Γ with coefficient $b > 1$, if the following axioms hold:

i. If $d_b(x, y) = 0$, iff $x = y$;
ii. $d_b(x, y) = d_b(y, x)$;
iii. $d_b(x, y) \leq b[d_b(x, z) + d_b(z, y)]$ for all $x, y, z \in \Gamma$.

The pair (Γ, d_b) is called a b-metric space (or simply d_b-metric).

Example 1.2. [23] Let $\Gamma = \mathbb{R}^+ \cup \{0\}$. Define $d_b(x, y) = |x - y|^2$ for all $x, y \in \Gamma$, then (Γ, d_b) be a b-metric space with $b = 2$.

Definition 1.3. [24] Let $\mathcal{B}$ be a non-empty subset of Γ and $a \in \Gamma$. Then, $f_0 \in \mathcal{B}$ is supposed to be the best estimation in $\mathcal{B}$ if

$$d_b(a, \mathcal{B}) = d_b(a, f_0), \text{ where } d_b(a, \mathcal{B}) = \inf_{f \in \mathcal{B}} d_b(a, f).$$

Let $\mathfrak{R}(\Gamma)$ represent the set that consists of all compact subsets of Γ.

Definition 1.4. [24] A mapping $\mathcal{H}_{d_b} : \mathfrak{R}(\Gamma) \times \mathfrak{R}(\Gamma) \to \mathbb{R}^+$ defined by

$$\mathcal{H}_{d_b}(J, S) = \max\left\{\sup_{p \in J} d_b(p, S),\ \sup_{q \in S} d_b(J, q)\right\},$$

where $\mathcal{H}_{d_b}$ is called as Hausdorff b-metric on $\mathfrak{R}(\Gamma)$.

Definition 1.5. [15] A F-contraction is a function $T : \mathbb{Z} \to \mathbb{Z}$ fulfilling the following:

there exists $\delta > 0$ so that (for all $\alpha,\ \beta \in \mathbb{Z}$)

$$D(T(\alpha), T(\beta)) > 0 \Rightarrow \delta + F(D(T(\alpha), T(\beta))) \leq F(D(\alpha,\ \beta)).$$

The mapping $F : \mathbb{R}^+ \to \mathbb{R}$ holds the given restrictions:

$(F1)$ F is a function of strictly-increasing mapping;

$(F2)$ $\lim_{j\to+\infty} \rho_j = 0$ iff $\lim_{j\to+\infty} F(\rho_j) = -\infty$, for each non-negative sequence $\{\rho_j\}_{j=1}^{\infty}$;

$(F3)$ for all $\theta \in (0,1)$, $\lim_{j\to\infty} \rho_j^{\theta} F(\rho_j) = 0$;

$(F4)$ there is a $\delta > 0$ so that for each non-negative sequence $\{\rho_j\}$,

$\delta + F(b\rho_j) \leq F(\rho_{j-1})$ for all $j \in \mathbb{N}$, then $\delta + F(b^j \rho_j) \leq (b^{j-1}\rho_{j-1})$ for all $j \in \mathbb{N}$.

Definition 1.6. [19] Let $\kappa \geq 1$ be a positive real number and $\Omega\chi_{\kappa}^{*}$ denote the class of $F : \mathbb{R}^+ \to \mathbb{R}$ that fulfilling the assumptions $(F1)-(F4)$ and

$(F5)$ $F(\inf \beta) = \inf F(\beta)$ for all $\beta \subset (0,\ \infty)$ with $\inf \beta > 0$.

Obviously $\Omega\chi_{\kappa}^{*}$ is not an empty set included with $F(a) = \ln a$ or $F(a) = a + \ln a$.

If for each convergent sequence $\{\Phi_j\}$ in $\mathcal{H}$ with $\lim_{j\to+\infty} \Phi_j = \Phi \in \mathcal{H}$, then a mapping $g : \mathcal{H} \to \mathbb{R}$ is said a lower semi-continuous, whenever satisfying $g(\Phi) \leq \lim_{j\to+\infty} \inf g(\Phi_j)$.

Definition 1.7. [25] Let A be a non-empty set, $E \subseteq A$ and $\Theta : A \times A \to [0, +\infty)$. A mapping $L : A \to \mathfrak{R}(\mathrm{A})$ satisfying

$$\Theta_*(La, Lb) = \inf\{\Theta(t, z) : t\epsilon La, z\epsilon Lb\} \geq 1, \text{ whenever } \Theta(t, z) \geq 1, \text{ for all } t, z \in \mathrm{A}$$

is called an Θ_*- admissible.

A mapping $L : A \to \mathfrak{R}(\mathrm{A})$ satisfying $\Theta_*(a, La) = \inf\{\Theta(a, h) : h\,\epsilon La\} \geq 1$ is called a Θ_*- dominated on E.

Example 1.8. [25] The function $\Theta_* : A \times A \to [0,\ \infty)$ is given by

$$\Theta_*(p, q) = \begin{cases} 1, \text{if} & p > q \\ \frac{1}{4}, \text{if} & p \leq q, \end{cases}$$

and mappings $\mathrm{G}, R : A \to \mathfrak{R}(A)$ defined by

$\mathrm{G}s = [-4 + s, -3 + s]$ and $Rm = [-2 + m,\ -1 + m]$ respectively. Then, both G and R are not $\Theta_* -$ admissible, but they are $\Theta_* -$ dominated.

2 Main results

Let $(\Gamma,\ d_b)$ be a complete b-metric space and $\mathrm{e}_0 \in \Gamma$, and let $\{P_\mu : \mu \in \text{Ѣ}\}$ and $\{R_\vartheta : \vartheta \in \wp\}$ be two families of set-valued mappings from Γ to $\mathfrak{R}(\Gamma)$. Let $c \in \text{Ѣ}$ and $\mathrm{e}_1 \in P_c(\mathrm{e}_0)$, then $d_b(\mathrm{e}_0,\ P_c(\mathrm{e}_0)) = d_b(\mathrm{e}_0, \mathrm{e}_1)$. Let $\mathrm{e}_2 \in R_g(\mathrm{e}_1)$ be such that $d_b(\mathrm{e}_1, R_g(\mathrm{e}_1)) = d_b(\mathrm{e}_1, \mathrm{e}_2)$, where $g \in \wp$. Proceeding this method, we achieve a sequence $\{R_\vartheta P_\mu(\mathrm{e}_j)\}$ in Γ, where $\mathrm{e}_{2j+1} \in P_z(\mathrm{e}_{2j})$, $\mathrm{e}_{2j+2} \in R_l(\mathrm{e}_{2j+1})$, where $z \in \text{Ѣ}$, $l \in \wp$ and $j \in \mathbb{N} \cup \{0\}$. Also, $d_b(\mathrm{e}_{2j}, P_z(\mathrm{e}_{2j})) = d_b(\mathrm{e}_{2j}, \mathrm{e}_{2j+1})$, $d_b(\mathrm{e}_{2j+1}, R_l(\mathrm{e}_{2j+1})) = d_b(\mathrm{e}_{2j+1}, \mathrm{e}_{2j+2})$, then $\{R_\vartheta P_\mu(\mathrm{e}_j)\}$ is the sequence in Γ generated by an initial guess point e_0. If $\{P_\mu : \mu \in \text{Ѣ}\} = \{R_\vartheta : \vartheta \in \wp\}$, then we write $\{\Gamma P_\mu(\mathrm{e}_j)\}$ alternative of $\{R_\vartheta P_\mu(\mathrm{e}_j)\}$.

Definition 2.1. Let $\{P_\mu : \mu \in \text{Ѣ}\}$ and $\{R_\vartheta : \vartheta \in \wp\}$ be two families of set-valued mappings from Γ to $\mathfrak{R}(\Gamma)$, $\mathfrak{R} \in \Omega\chi_{\kappa}^{*}$ and $\mathscr{L} : (0, \infty) \to (0, \infty)$. For each $\mathrm{e}, \varpi \in \Gamma$ with $\max\{d_b(\mathrm{e}, P_\mu(\mathrm{e})), d_b(\varpi, R_\vartheta(\varpi))\} > 0$, define a set $E_{\mathscr{L}}^{\varpi} \subseteq \Gamma$ as:

$$E_{\mathscr{L}}^{\varpi} = \left\{ \begin{array}{c} \varpi \in P_{\mu}(\mathrm{e}), z \in R_{\vartheta}(\varpi): F(d_b(\varpi, z)) \le F\left(b\left[\max\left\{d_b(\mathrm{e}, R_{\vartheta}(\mathrm{e})), d_b(\varpi, R(\varpi)), \frac{d_b(\mathrm{e}, P_{\mu}(\mathrm{e})).d_b(\varpi, R_{\vartheta}(\varpi))}{1+d_b(\mathrm{e},\varpi)}\right\}\right]\right) \\ +\mathscr{L}(d_b(\varpi, z)) \end{array} \right\}.$$

Let $P_\mu,\ R_\vartheta : \Gamma \to \Gamma$, and for any $\varpi_0 \in \Gamma$, $O(\varpi_0) = \{\varpi_0,\ P_\mu(\varpi_0),\ R_\vartheta(\varpi_1), \ldots\ldots\}$ denote the orbit of ϖ_0. A function $g : \Gamma \to R$ is said to be a (P_μ, R_ϑ)-orbitally lower semi-continuous if $g(\varpi) < \lim_{n\to+\infty} \inf g(\varpi_0)$ for all sequences $\{P_\mu R_\vartheta(\varpi_n)\} \subset O(\varpi_0)$ with $\lim_{n\to+\infty}\{P_\mu R_\vartheta(\varpi_n)\} = \varpi \in X$.

Definition 2.2. Let $P_\mu, R_\vartheta : \Gamma \to \mathfrak{R}(\Gamma)$ be a pair of families of set-valued mappings on $(\Gamma,\ d_b)$. And let orbit for a couple $(P_\mu,\ R_\vartheta)$ be a point $\varpi_0 \in \Gamma$ represented by $O(\varpi_0)$ and defined a sequence $\{\varpi_n :\ \varpi_n \in P_\mu R_\vartheta(\varpi_{n-1})\}$.

Definition 2.3. Let $P_\mu,\ R_\vartheta : \Gamma \to \mathfrak{R}(\Gamma)$ be two families of set-valued mappings on $(\Gamma,\ d_b)$. If Cauchy sequence $\{\varpi_n :\ \varpi_n \in P_\mu R_\vartheta(\varpi_{n-1})\}$ converges in b-metric Γ, then Γ is called (P_μ, R_ϑ)-orbitally complete.

It is remarked that orbitally complete b-metric space may be not complete. Now, we start our main results.

Theorem 2.4. *Let* $(\Gamma,\ d_b)$ *be an orbitally complete b-metric space.* Let $\varpi_0 \in \Gamma$, $\Theta : \Gamma \times \Gamma \to [0, \infty)$ *and* $\{P_\mu : \mu \in Ƕ\}$, $\{R_\vartheta : \vartheta \in ℘\}$ *from* Γ to $\mathfrak{R}(\Gamma)$ *be two families of* Θ_**-dominated multivalued mappings and* $F \in \Omega\chi_\kappa^*$. *Suppose that the following conditions hold*:

i. the mappings $z \mapsto \max\{d_b(\mathrm{e}, P_\mu(\mathrm{e})), d_b(\varpi, R_\vartheta(\varpi))\}$ are orbitally lower semi-continuous, there exist functions $\tau, \mathscr{L} : (0, \infty) \to (0, \infty)$ so that for each $t \ge 0$

$$\tau(t) > \mathscr{L}(t), \liminf_{s\to t^+} \tau(t) > \liminf_{s\to t^+} \mathscr{L}(t);$$

ii. for all $\mathrm{e}, \varpi \in \{P_\mu R_\vartheta(\mathrm{e}_j)\}$ with $\alpha(\mathrm{e}, \varpi) \ge 1$ or $\alpha(\varpi, \mathrm{e}) \ge 1$ and $\max\{d_b(\mathrm{e}, P_\mu(\mathrm{e})), d_b(\varpi, R_\vartheta(\varpi))\} > 0$, there exists $\varpi \in E_{\mathscr{L}}^{\mathrm{e}}$ satisfying

$$\tau(d_b(\mathrm{e}, \varpi)) + F\left(b\left[\max\left\{d_b(\mathrm{e}, P_\mu(\mathrm{e})), d_b(\varpi, R_\vartheta(\varpi)), \frac{d_b(\mathrm{e}, P_\mu(\mathrm{e})), d_b(\varpi, R_\vartheta(\varpi))}{1 + d_b(\mathrm{e}, \varpi)}\right\}\right]\right) \le F(d_b(\mathrm{e}, \varpi)). \quad (2.1)$$

If (2.1) exists, then P_μ and R_ϑ admit a multi-common fixed point q in Γ.

Proof. Assume P_μ and R_ϑ both have no fixed point, then for all $\mathrm{e}, \varpi \in \Gamma$, we have $\max\{d_b(\mathrm{e}, P_\mu(\mathrm{e})), d_b(\varpi, R_\vartheta(\varpi))\} > 0$. Since $P_\mu(\mathrm{e}),\ R_\vartheta(\varpi) \in \mathfrak{R}(\Gamma)$ for every $\mathrm{e}, \varpi \in \Gamma$ and $F \in \Omega\chi_\kappa^*$, it is very easy to show $E_{\mathscr{L}}^{\mathrm{e}}$ is not an empty set for every $\mathrm{e}, \varpi \in \Gamma$ (proof is similar as given in first line of [18]). As $\{P_\mu : \mu \in Ƕ\}$, $\{R_\vartheta : \vartheta \in ℘\}$ are two families of Θ_*- dominated multivalued mappings on $\{R_\vartheta P_\mu(\mathrm{e}_j)\}$, by applying definition, we have $\Theta_*(\mathrm{e}_{2j}, P_\mu(\mathrm{e}_{2j})) \ge 1$ and $\Theta_*(\mathrm{e}_{2i+1}, R_\vartheta(\mathrm{e}_{2j+1})) \ge 1$ for all $j \in \mathbb{N}$. As $\Theta_*(\mathrm{e}_{2j}, P_\mu(\mathrm{e}_{2j})) \ge 1$, this implies that $\inf\{\Theta(\mathrm{e}_{2j}, b) : b \in P_\mu(\mathrm{e}_{2j})\} \ge 1$, and therefore, $\Theta(\mathrm{e}_{2j}, \mathrm{e}_{2j+1}) \ge 1$. If $\mathrm{e}_0 \in \Gamma$ is any initial point then $\mathrm{e}_{2j}, \mathrm{e}_{2j+1} \in E_{\mathscr{L}}^{\mathrm{e}_0}$, and using (2.1), we have

$$\tau(d_b(\mathrm{e}_{2j}, \mathrm{e}_{2j+1})) + F\left(b\left[\max\left\{d_b(\mathrm{e}_{2j}, P_\mu(\mathrm{e}_{2j})), d_b(\mathrm{e}_{2j+1}, R_\vartheta(\mathrm{e}_{2j+1})), \frac{d_b(\mathrm{e}_{2j}, P_\mu(\mathrm{e}_{2j})).\ d_b(\mathrm{e}_{2j+1}, R_\vartheta(\mathrm{e}_{2j+1}))}{1 + d_b(\mathrm{e}_{2j}, \mathrm{e}_{2j+1})}\right\}\right]\right)$$
$$\le F(d_b(\mathrm{e}_{2j}, \mathrm{e}_{2j+1}))$$

$$\tau(d_b(\mathrm{e}_{2j}, \mathrm{e}_{2j+1})) + F\left(b\left[\max\left\{d_b(\mathrm{e}_{2j}, \mathrm{e}_{2j+1}), d_b(\mathrm{e}_{2j+1}, \mathrm{e}_{2j+2}), \frac{d_b(\mathrm{e}_{2j}, \mathrm{e}_{2j+1}).\ d_b(\mathrm{e}_{2j+1}, \mathrm{e}_{2j+2})}{1 + d_b(\mathrm{e}_{2j}, \mathrm{e}_{2j+1})}\right\}\right]\right) \le F(d_b(\mathrm{e}_{2j}, \mathrm{e}_{2j+1})),$$

$$\tau(d_b(\mathrm{e}_{2j}, \mathrm{e}_{2j+1})) + F(b[\max\{d_b(\mathrm{e}_{2j}, \mathrm{e}_{2j+1}), d_b(\mathrm{e}_{2j+1}, \mathrm{e}_{2j+2}), d_b(\mathrm{e}_{2j+1}, \mathrm{e}_{2j+2})\}]) \le F(d_b(\mathrm{e}_{2j}, \mathrm{e}_{2j+1})).$$

This implies that

$$\tau(d_b(e_{2j}, e_{2j+1})) + F(b[\max\{d_b(e_{2j}, e_{2j+1}), d_b(e_{2j+1}, e_{2j+2})\}]) \leq F(d_b(e_{2j}, e_{2j+1})). \tag{2.2}$$

If, $\max\{d_b(e_{2j}, e_{2j+1}), d_b(e_{2j+1}, e_{2j+2})\} = d_b(e_{2j}, e_{2j+1})$, then from (2.2), we have

$$\tau(d_b(e_{2j}, e_{2j+1})) + F(b[d_b(e_{2j}, e_{2j+1})]) \leq F(d_b(e_{2j}, e_{2j+1})),$$

which is a contradiction. According to $(F1)$, the function F is strictly increasing. So, we obtain

$$\max\{d_b(e_{2j}, e_{2j+1}), d_b(e_{2j+1}, e_{2j+2})\} = d_b(e_{2j+1}, e_{2j+2}),$$

for all $j \in \mathbb{N} \cap \{0\}$, we deduce that

$$\tau(d_b(e_{2j}, e_{2j+1})) + F(b[d_b(e_{2j+1}, e_{2j+2})]) \leq F(d_b(e_{2j}, e_{2j+1})), \tag{2.3}$$

for each $j \in \mathbb{N} \cap \{0\}$, also from inequality (2.3) and using $(F4)$ we obtain

$$\tau(d_b(e_{2j}, e_{2j+1})) + F(b^{2j+1}[d_b(e_{2j+1}, e_{2j+2})]) \leq F(b^{2j} d_b(e_{2j}, e_{2j+1})). \tag{2.4}$$

Since, $e_{j+1} \in E_{\mathscr{L}}^{\varpi}$, then by the definition of $E_{\mathscr{L}}^{\varpi}$, we have

$$F(d_b(e_{2j}, e_{2j+1})) \leq F(d_b(e_{2j}, R_\vartheta e_{2j})) + \mathscr{L} d_b(e_{2j}, e_{2j+1}),$$

which implies that

$$F(b^{2j} d_b(e_{2j}, e_{2j+1})) \leq F(b^{2j} d_b(e_{2j}, R e_{2j})) + \mathscr{L} d_b(e_{2j}, e_{2j+1}). \tag{2.5}$$

From (2.4) and (2.5), we have

$$F(b^{2j+1}[d_b(e_{2j+1}, e_{2j+2})]) \leq F(b^{2j} d_b(e_{2j}, e_{2j+1})) + \mathscr{L} d_b(e_{2j}, e_{2j+1}) - \tau(d_b(e_{2j}, e_{2j+1})). \tag{2.6}$$

Similarly, for each $j \in \mathbb{N} \cap \{0\}$, we have

$$F(b^{2j}[d_b(e_{2j}, e_{2j+1})]) \leq F(b^{2j-1} d_b(e_{2j-1}, e_{2j})) + \mathscr{L} d_b(e_{2j-1}, e_{2j}) - \tau(d_b(e_{2j-1}, e_{2j})). \tag{2.7}$$

Substituting (2.7) into (2.5), we have

$$\begin{aligned} F(b^{2j+1}[d_b(e_{2j+1}, e_{2j+2})]) \leq {} & F(b^{2j-1} d_b(e_{2j-1}, e_{2j})) + \mathscr{L} d_b(e_{2j}, e_{2j+1}) + \mathscr{L} d_b(e_{2j-1}, e_{2j}) - \tau(d_b(e_{2j-1}, e_{2j})) \\ & - \tau(d_b(e_{2j}, e_{2j+1})). \end{aligned} \tag{2.8}$$

Now, put $2j + 1 = a$ and also $d_b(e_{2j}, e_{2j+1}) = \sigma_a$ for each $a \in \mathbb{N} \cap \{0\}$, $\sigma_a > 0$ and also from the inequality (2.8), it is observed that sequence $\{\sigma_a\}$ is non-increasing. Furthermore, if there lies a $\varepsilon > 0$ so that $\lim_{a \to +\infty} \sigma_a = \varepsilon$. As $\varepsilon > 0$, let $\Upsilon(t) = \liminf_{t \to s^+} \tau(t) - \liminf_{t \to s^+} \eta(t) \geq 0$. Then, using (2.6), we obtain

$$\begin{aligned} F(b^a \sigma_a) &\leq F(b^{a-1} \sigma_a) - \Upsilon(\sigma_a), \\ &\leq F(b^{a-1} \sigma_{a-1}) - \Upsilon(\sigma_a) - \Upsilon(\sigma_{a-1}), \\ &\vdots \\ &\leq F(\sigma_0) - \Upsilon(\sigma_a) - \Upsilon(\sigma_{a-1}) - \ldots - \Upsilon(\partial_0). \end{aligned} \tag{2.9}$$

Take l_a, which is the largest number from the $\{0, 1, 2, 3, \ldots, a-1\}$ so that

$$\Upsilon(\sigma_{l_a}) = \min\{\Upsilon(\sigma_0), \Upsilon(\sigma_1), \Upsilon(\sigma_2), \ldots, \Upsilon(\sigma_a)\}.$$

For each $a \in \mathbb{N}$, so $\{\sigma_a\}$ is the sequence of non-decreasing. From inequality (2.9), we obtain

$$F(b^a \sigma_a) \leq F(\sigma_0) - a\Upsilon(\sigma_{l_a}).$$

Similarly, from (2.6), we can obtain

$$F(b^{a+1} d_b(e_{a+1}, R_\vartheta(e_{a+1}))) \leq F(b^a d_b(e_0, e_1)) - a\Upsilon(\sigma_{l_a}). \tag{2.10}$$

Now, we consider two different cases about the sequence $\{\Upsilon(\sigma_{l_a})\}$.

Case I. For all, $a \in \mathbb{N}$, there must exist $e > a$ so that $\Upsilon(\sigma_{l_e}) > \Upsilon(\sigma_{l_a})$. Then, we obtain a subsequence $\{\Upsilon(\sigma_{l_{a_k}})\}$ of $\{\sigma_{l_a}\}$ with $\Upsilon(\sigma_{l_{a_k}}) > \Upsilon(\sigma_{l_{a_{k+1}}})$ for all k. Since $\sigma_{l_{a_k}} \to \delta^+$, we deduce that

$$\liminf_{t\to\delta^+}\Upsilon(\sigma_{l_{g_k}}) > 0.$$

Hence,

$$F(b^{a_k}\sigma_{a_k}) \leq F(\sigma_0) - a^k\Upsilon(\sigma_{l_{a_k}}),$$

for every k. Consequently, $\lim_{k\to\infty} F(b^{a_k}\sigma_{a_k}) = -\infty$, and by $(F2)$ $\lim_{k\to+\infty} b^{a_k}\sigma_{a_k} = 0$, which is not true that $\lim_{k\to+\infty}\sigma_{a_k} > 0$, for any $b > 1$.

Case II. As $a_0 \in \mathbb{N}$ such that $\Upsilon(\sigma_{l_{a_0}}) > \Upsilon(\Upsilon_{l_e})$ for all $e > a_0$. Then,

$$F(\sigma_e) \leq F(\sigma_0) - e\Upsilon(\sigma_{l_e}), \text{ for all } e > a_0.$$

Hence, $\lim_{e\to+\infty} F(\sigma_e) = -\infty$, and by $(F2)$, $\lim_{e\to+\infty}\sigma_e = 0$, which is wrong due to the $\lim_{e\to+\infty}\sigma_e > 0$. Hence, $\lim_{e\to+\infty}\sigma_e = 0$. From $(\Re 3)$, there is $k \in (0,1)$ so that $\lim_{a\to+\infty}(b^a\sigma_a)^k F(b^a\sigma_a) = -\infty$, and from (2.10), the upcoming conditions satisfies for all $a \in \mathbb{N}$:

$$(b^a\sigma_a)^k F(b^a\sigma_a) - (b^a\sigma_a)^k F(\sigma_0) \leq (b^a\sigma_a)^k(F(\sigma_0) - a\Upsilon(\sigma_{l_a})) - (b^a\sigma_a)^k F(\sigma_0) = -a(b^a\sigma_a)^k\Upsilon(\sigma_{l_a}) \leq 0. \tag{2.11}$$

Taking $a \to +\infty$ in (2.11), we obtain

$$\lim_{a\to+\infty} F(b^a\sigma_a)^k\,\Upsilon(\sigma_{l_a}) = 0.$$

Since $\zeta = \lim_{a\to+\infty}\Upsilon > 0$ there exists $a_0 \in \mathbb{N}$, such that $\Upsilon(\sigma_{l_a}) > \frac{\zeta}{2}$ for all $a \neq a_0$. Thus,

$$F(b^a\sigma_a)^k\frac{\zeta}{2} < F(b^a\sigma_a)^k\Upsilon(\sigma_{l_a}), \tag{2.12}$$

for each $a > a_0$. Letting $a \to +\infty$ in (2.12), we have

$$0 \leq \lim_{a\to+\infty} F(b^a\sigma_a)^k\frac{\zeta}{2} < \lim_{a\to+\infty} F(b^a\sigma_a)^k\Upsilon(\sigma_{l_a}) = 0.$$

That is,

$$\lim_{a\to+\infty} F(b^a\sigma_a)^k = 0, \tag{2.13}$$

from (2.13), there exists $a_1 \in \mathbb{N}$ such that $F(b^a\sigma_a)^k \leq 1$ for all $a > a_1$,

$$(b^a\sigma_a)^k \leq \frac{1}{a} \Longrightarrow \sigma_a \leq \frac{1}{b^a a^{1/k}}, \text{ for all } a > a_1.$$

The convergence of the series $\sum_{a=1}^{\infty} b^a\sigma_a$ implies that $\{\mathrm{e}_a\}$ is a Cauchy sequence in Γ, and it is obvious that Γ is orbitally complete b-metric space, there exists $q \in O(\varpi_0)$ so that $\mathrm{e}_a \to q$ when $a \to +\infty$. By (2.10) and $(F2)$, we find $\lim_{a\to+\infty} d_b(\mathrm{e}_a, R_\vartheta(\mathrm{e}_a)) = 0$. Since $q \to d_b(\mathrm{e}, R_\vartheta(\mathrm{e}))$ is orbitally lower semi-continuous and $\Theta(\mathrm{e}_a, R_\vartheta(\mathrm{e}_a)) > 1$, we have

$$\begin{aligned} 0 \leq d_b(q, R_\vartheta(q)) &\leq \lim_{a\to+\infty}\inf d_b(\mathrm{e}_a, R(\mathrm{e}_a)) \leq \lim_{a\to+\infty}\inf d_b(\mathrm{e}_a, \mathrm{e}_{a+1}),\\ &\leq \lim_{a\to+\infty} b[d_b(\mathrm{e}_a, \mathrm{e}) + d_b(\mathrm{e}, \mathrm{e}_{a+1})] = 0. \end{aligned}$$

Hence,

$$q \in R_\vartheta(q).$$

As the series $\sum_{a=1}^{\infty} b^a\sigma_a$ convergent, and $\{\xi_a\}$ be a Cauchy sequence in Γ, and it is clear that Γ is an orbitally complete $b-$metric space, there must exist $q \in O(\varpi_0)$ so that $\xi_a \to q$ as $a \to +\infty$. Applying (2.10) and $(F2)$, we obtain $\lim_{a\to+\infty} d_b(\xi_a, P_\mu(\xi_a)) = 0$. Since $q \to d_b(\xi, P_\mu(\xi))$ is orbitally lower semi-continuous and also $\Theta(\xi_a, P_\mu(\xi_a)) > 1$, we have

$$0 \le d_b(q, P_\mu(q)) \le \lim_{a\to+\infty} \inf d_b(\xi_a, P_\mu(\xi_a)) \le \lim_{a\to+\infty} \inf d_b(\xi_a, \xi_{a+1}),$$
$$\le \lim_{a\to+\infty} b[d_b(\xi_a, y) + d_b(\xi, \xi_{a+1})] = 0.$$

Therefore, $q \in P_\mu(q)$. Hence, $\{P_\mu : \mu \in Ƕ\}$ and $\{R_\vartheta : \vartheta \in \wp\}$ both admit a multi-common fixed point q in Γ.

Example 2.5. Let $\Gamma = [0, 1]$ and define a function $d_b : \Gamma \times \Gamma \to \mathbb{R}$ by $d_b(u, v) = |u - v|^2$ for all $u, v \in \Gamma$. Then, the function d_b becomes b-metric on Γ with $b = 2$. Also, we define two families of multi-maps $P_\mu, R_\vartheta : \Gamma \to \mathfrak{R}(\Gamma)$ by

$$P_\mu(u) = \left[\frac{u}{2m}, \frac{3u}{2m}\right], \text{ if } u \in \Gamma, \text{ where } m = 1, 2, 3, \ldots$$

and

$$R_\vartheta(q) = \left[\frac{u}{5n}, \frac{4u}{5n}\right], \text{ if } u \in \Gamma, \text{ where } n = 1, 2, 3, \ldots$$

Suppose $u_0 = 1$, then $d_b(u_0, P_\mu(u_0)) = d_b(1, P_\mu(1)) = d_b\left(1, \frac{1}{2}\right)$. So, $u_1 = \frac{1}{2}$. Now, $d_b(u_1, R_\vartheta(u_1)) = d_b\left(\frac{1}{2}, R_\vartheta\left(\frac{1}{2}\right)\right) = d_b\left(\frac{1}{2}, \frac{1}{10}\right)$. So, $u_1 = \frac{1}{10}$. Now, $d_b(u_2, P_\mu(u_2)) = d_b\left(\frac{1}{10}, P_\mu\left(\frac{1}{10}\right)\right) d_b\left(\frac{1}{10}, \frac{1}{20}\right)$. So, $u_1 = \frac{1}{20}$. Continuing, this method we obtain a sequence of the form $\{R_\vartheta P_\mu(u_n)\} = \left\{1, \frac{1}{2}, \frac{1}{10}, \frac{1}{20}, \ldots\right\}$. Consider the function $\Theta : \Gamma \times \Gamma \to \mathbb{R}$ defined as

$$\Theta(u, v) = \begin{cases} 1, \text{if} & u > v \\ \frac{1}{2}, & \text{otherwise.} \end{cases}$$

Now, for all $u, v \in \{R_\vartheta P_\mu(u_n)\}$ with either $\Theta(u, v) \ge 1$ or $\Theta(v, u) \ge 1$. Define $F : \mathbb{R}^+ \to \mathbb{R}$ by $F(u) = \ln(u)$ for each $u \in \mathbb{R}^+$ and $\tau(t) = 1/8$, $\eta(t) = 1/10$ for all $t \in (0, \infty)$. Then, obviously, $F \in \Omega\chi^*_\kappa$ and $\tau(t) > \mathscr{L}(t)$, $\lim_{s\to t^+}\inf\tau(t) > \lim_{s\to t^+}\inf\mathscr{L}(t)$.

Therefore, $u \to \max\{d_b(u, P_\mu(u)), d_b(v, R_\vartheta(v))\}$ is orbitally lower semi-continuous. For $u \in \Gamma$, we have $u = \left(\frac{4v}{5}\right)^2 \in E^{e}_{\mathscr{L}}$, and for this using the inequality, we have

$$\tau(d_b(u, v)) + F\left(b\left[\max\left\{d_b(u, P_\mu(u)), d_b(v, R_\vartheta(v)), \frac{d_b(u, P_\mu(u)).\ d_b(v, R_\vartheta(v))}{1 + d_b(u, v)}\right\}\right]\right),$$
$$= \frac{1}{8} + \ln\left[2\max\left\{\left(u - \frac{u}{2m}\right)^2, \left(v - \frac{v}{5n}\right)^2, \frac{\left(u - \frac{u}{2m}\right)^2 . \left(v - \frac{v}{5n}\right)^2}{1 + |u - v|^2}\right\}\right],$$
$$= \frac{1}{8} + \ln\left[2\max\left\{d_b\left(u, \left[\frac{u}{2m}, \frac{3u}{2m}\right]\right), d_b\left(u, \left[\frac{v}{5n}, \frac{4v}{5n}\right]\right), \frac{d_b\left(u, \left[\frac{u}{2m}, \frac{3u}{2m}\right]\right).\ d_b\left(u, \left[\frac{v}{5n}, \frac{4v}{5n}\right]\right)}{1 + d_b(u, v)}\right\}\right]$$
$$\le \ln(|u - v|^2),$$

which implies that

$$\tau(d_b(u,v)) + F\left(b\left[\max\left\{d_b(u,P_\mu(u)), d_b(v,R_\vartheta(v)), \frac{d_b(u,P_\mu(u)).\ d_b(v,R_\vartheta(v))}{1+d_b(u,v)}\right\}\right]\right) \le F(d_b(u,v)).$$

Note that $\frac{1}{2}, 0 \in \Gamma$, $\Theta\left(\frac{1}{2}, 0\right) \ge 1$; then, we have

$$\tau\left(d_b\left(\frac{1}{2},0\right)\right) + F\left(2\left[\max\left\{d_b\left(\frac{1}{2},P_1\left(\frac{1}{2}\right)\right), d_b(0,R_2(0)), \frac{d_b\left(\frac{1}{2},P_1\left(\frac{1}{2}\right)\right).\ d_b(0,R_2(0))}{1+d_b\left(\frac{1}{2},0\right)}\right\}\right]\right) \le F\left(d_b\left(\frac{1}{2},0\right)\right).$$

So, whole assumptions of Theorem 2.4 are satisfied for $F(e) = \ln(e)$, where $e > 0$.

Let Q be a non-empty set, and $u \preccurlyeq Q$ means there is $v \in Q$ such that $u \preccurlyeq v$. The function $P_\mu : \Gamma \to \mathfrak{R}(\Gamma)$ is multi $\preccurlyeq$-dominated on Q, if $u \preccurlyeq P_\mu u$ for any $u \in \Gamma$.

We are proving next result for $\preccurlyeq$-dominated set-valued mappings on $\{R_\vartheta P_\mu(\hbar_j)\}$ in an ordered orbitally complete b-metric space.

Theorem 2.6. *Let* $(\Gamma, \preccurlyeq, d_b)$ *be an ordered complete orbitally b-metric space.* Let $\hbar_0 \in \Gamma$, $\Theta : \Gamma \times \Gamma \to [0,\infty)$ *and* $\{P_\mu : \mu \in \text{Ƕ}\}$, $\{R_\vartheta : \vartheta \in \wp\}$ *be two families of* Θ_**-dominated multivalued mappings from* Γ *to* $\mathfrak{R}(\Gamma)$ *and* $F \in \Omega\chi_\kappa^*$. *Suppose that following assumptions hold*:

i. the mappings $z \mapsto \max\{d_b(\mathrm{e}, P_\mu(\mathrm{e})), d_b(\varpi, R_\vartheta(\varpi))\}$ are orbitally lower semi-continuous, and $\tau, \mathscr{L} : (0,\infty) \to (0,\infty)$ are the continuous functions so that for all $t \ge 0$

$$\tau(t) > \mathscr{L}(t), \liminf_{s\to t^+}\tau(t) > \liminf_{s\to t^+}\mathscr{L}(t);$$

ii. for all $\mathrm{e}, \varpi \in \{P_\mu R_\vartheta(\hbar_j)\}$ with either $\mathrm{e} \preccurlyeq \varpi$ or $\varpi \preccurlyeq \mathrm{e}$ and $\max\{d_b(\mathrm{e}, P_\mu(\mathrm{e})), d_b(\varpi, R_\vartheta(\varpi))\} > 0$, also $\{P_\mu R_\vartheta(\hbar_j)\} \to \hbar^*$, there exists $\varpi \in E_{\mathscr{L}}^{\varpi}$, $\preccurlyeq$ satisfying

$$\tau(d_b(\mathrm{e},\varpi)) + F\left(b\left[\max\left\{d_b(\mathrm{e},P_\mu(\mathrm{e})), d_b(\varpi,R_\vartheta(\varpi)), \frac{d_b(\mathrm{e},P_\mu(\mathrm{e})), d_b(\varpi,R_\vartheta(\varpi))}{1+d_b(\mathrm{e},\varpi)}\right\}\right]\right) \le F(d_b(\mathrm{e},\varpi)). \quad (2.14)$$

Also, if (2.14) holds for $\hbar^*$, $\hbar^* \preccurlyeq \hbar_j$ or $\hbar_j \preccurlyeq \hbar^*$, where $j = \{0,1,2,3,\ldots.\}$, then $\hbar^*$ is the multi-common fixed point of both $\{P_\mu : \mu \in \text{Ƕ}\}$ and $\{R_\vartheta : \vartheta \in \wp\}$ in Γ.

Proof. Let $\Theta : \Gamma \times \Gamma \to [0,+\infty)$ be a function defined by $\Theta(\hbar, q) = 1$ for each $\hbar \in \Gamma$ with $\hbar \preccurlyeq q$, families of Θ_*-dominated multivalued mappings on Γ, so $\hbar \preccurlyeq P_\mu(\mathrm{e})$ and $\hbar \preccurlyeq R_\vartheta(\varpi)$ for every $\hbar \in \Gamma$. This implies that $\hbar \preccurlyeq u$ for every $u \in P_\mu(\mathrm{e})$ and $\hbar \preccurlyeq u$ for each $\hbar \in R_\vartheta(\varpi)$. So, $\Theta(\hbar, u) = 1$ for every $u \in P_\mu(\mathrm{e})$ and $\Theta(\hbar, u) = 1$ for every $\hbar \in R_\vartheta(\varpi)$. This implies that $\inf\{\Theta(\hbar, q) : q \in P_\mu(\mathrm{e})\} = 1$, and $\inf\{\Theta(\hbar, q) : q \in R_\vartheta(\varpi)\} = 1$. □

Hence, $\Theta_*(\hbar, R_\vartheta(\mathrm{e})) = 1$, $\Theta_*(\hbar, R_\vartheta(\varpi)) = 1$ for each $\hbar \in \Gamma$. So, $\{P_\mu : \mu \in \text{Ƕ}\}$, $\{R_\vartheta : \vartheta \in \wp\}$ be two families of Θ_*-dominated multivalued mappings on Γ. Again, if (2.14) holds, it can be written as

$$\tau(d_b(\mathrm{e},\varpi)) + F\left(b\left[\max\left\{d_b(\mathrm{e},P_\mu(\mathrm{e})), d_b(\varpi,R_\vartheta(\varpi)), \frac{d_b(\mathrm{e},P_\mu(\mathrm{e})), d_b(\varpi,R_\vartheta(\varpi))}{1+d_b(\mathrm{e},\varpi)},\right\}\right]\right) \le F(d_b(\mathrm{e},\varpi)),$$

for each $\hbar, q$ in $\{R_\vartheta P_\mu(\hbar_j)\}$, with either $\Theta(\hbar, q) \ge 1$ or $\Theta(q, \hbar) \ge 1$. Then, from Theorem 2.6, $\{R_\vartheta P_\mu(\hbar_j)\}$ is the convergent sequence in Γ as $\{R_\vartheta P_\mu(\hbar_j)\} \to \hbar^* \in \Gamma$. Now, $\hbar_j, \hbar^* \in \Gamma$ and either $\hbar_j \preccurlyeq \hbar^*$, or $\hbar^* \preccurlyeq \hbar_j$ implies that the either $\Theta(\hbar_j, \hbar^*)$ or $\Theta(\hbar^*, \hbar_j) \ge 1$. Hence, all conditions of Theorem 2.4 are hold. Then, both $\{P_\mu : \mu \in \text{Ƕ}\}$ and $\{R_\vartheta : \vartheta \in \wp\}$ have a multi-fixed point $\hbar^*$ in Γ.

3 Application to integral equation

In the setting of different abstract spaces using generalized contractions, a specified number of well-known authors observed sufficient and compulsory conditions for the solution of linear and nonlinear first and second types of both (Fredholm and Volterra)-type integrals in the field of fixed point theory. Rasham et al. [21] showed some new fixed-point results for two families of multifunction, and they used their fundamental outcome to analyze the important circumstances to solved integral equations. For additional huge current outcomes presence with fundamental integral applications, click here (see [23,26–28]).

Theorem 3.1. *Let* (Γ, d_b) *be an orbitally complete b-metric space. Let* $\varpi_0 \in \Gamma, \Theta : \Gamma \times \Gamma \to [0, \infty)$ *and* $P_\mu, R_\vartheta : \Gamma \to \mathfrak{R}(\Gamma)$ *be a pair of self-dominated mappings and* $F \in \Omega\chi_\kappa^*$. *Suppose that the following conditions hold*:

i. The mappings $z \mapsto \max\{d_b(\mathrm{e}, P_\mu(\mathrm{e})), d_b(\varpi, R_\vartheta(\varpi))\}$ are orbitally lower semi-continuous, there exist functions $\tau, \mathscr{L} : (0, \infty) \to (0, \infty)$ so that for each $t \geq 0$;

$$\tau(t) > \mathscr{L}(t), \liminf_{s\to t^+} \tau(t) > \liminf_{s\to t^+} \mathscr{L}(t);$$

ii. for all $\mathrm{e}, \varpi \in \{\mathrm{e}_j\}$ with $\alpha(\mathrm{e}, \varpi) \geq 1$ or $\alpha(\varpi, \mathrm{e}) \geq 1$; and $\max\{d_b(\mathrm{e}, P_\mu(\mathrm{e})), d_b(\varpi, R_\vartheta(\varpi))\} > 0$, there exists $\varpi \in E_{\mathscr{L}}^{\mathrm{e}}$ satisfying

$$\tau(d_b(\mathrm{e}, \varpi)) + F\left(b\left[\max\left\{d_b(\mathrm{e}, P_\mu(\mathrm{e})), d_b(\varpi, R_\vartheta(\varpi)), \frac{d_b(\mathrm{e}, P_\mu(\mathrm{e})), d_b(\varpi, R_\vartheta(\varpi))}{1 + d_b(\mathrm{e}, \varpi)}\right\}\right]\right) \leq F(d_b(\mathrm{e}, \varpi)). \tag{3.1}$$

If (3.1) holds, then P_μ and R_ϑ have a common fixed point q in Γ.

Proof. The proof of Theorem 3.1 is similar to the proof of Theorem 2.4. □

Now, we prove an application by Theorem 3.1 for a system of nonlinear Volterra-type integral equations.

Let $X = (\mathbb{C}[0,1], \mathbb{R}_+)$ be the set of continuous functions on [0,1] endowed with the metric $d_b : X \times X \to \mathbb{R}$ defined by $d_b(f, g) = \sup|f(t) - g(t)|^2$ for all $f, g \in (\mathbb{C}[0,1], \mathbb{R})$ and $t \in [0,1]$. Define $\alpha : X \times X \to \mathbb{R}$ as

$$\alpha(p, q) = \begin{cases} 1, \text{ if } p(t) \leq q(t) \\ \frac{1}{2}, \text{ otherwise} \end{cases}.$$

Take an integral equation:

$$l(t) = \int_0^t \mathcal{K}(t, s, \varkappa(s))\mathrm{d}s, \tag{3.2}$$

where $\mathcal{K} : [0,1] \times [0,1] \times X \to \mathbb{R}$ and l are continuous for all $s, t \in [0,1]$. Our aim is to show the existence of the solution to equation (3.2) by applying Theorem 3.1.

Theorem 3.2. *Let* $X = (\mathbb{C}[0,1], \mathbb{R})$ *and* $P_\mu, R_\vartheta : X \to X$ *be Volterra integral operator defined as follows*:

$$(P_\mu\varkappa)(t) = \int_0^t \mathcal{K}(t, s, \varkappa(s))\mathrm{d}s, \text{ and } (R_\vartheta y)(t) = \int_0^t \mathcal{K}(t, s, y(s))\mathrm{d}s, \text{ where } \varkappa, y \in \mathbb{C}[0, 1].$$

Also $\mathcal{K} : [0,1] \times [0,1] \times X \to \mathbb{R}$ is continuous on $\mathbb{R}$ for all $\varkappa, t \in [0,1]$ and the following hold:

(i) There exists a continuous function $\mu : [0, 1] \to \mathbb{R}_+$ such that $\left(\int_0^t \mu(s)ds\right)^2 \leq \frac{e^{-\tau(t)}}{2}; t > 0$ and satisfying, for all $\varkappa, y \in \mathcal{X}$ such that $\varkappa(t) \leq y(t)$,

$$|\mathcal{K}(t, s, \varkappa(s)) - \mathcal{K}(t, s, y(s))| \leq \mu(s)|\varkappa(s) - y(s)|.$$

(ii) For all $\varkappa,\ y \in X$ such that $\varkappa(t) \leq y(t)$, we have

$$\max\left\{d_b(\varkappa, P_\mu(\varkappa)), d_b(y, R_\vartheta(y)), \frac{d_b(\varkappa, P_\mu(\varkappa)).\ d_b(y, R_\vartheta(y))}{1 + d_b(\varkappa, y)}\right\} \leq |(P_\mu y)(t) - (R_\vartheta y)(t)|^2.$$

Then, (3.2) has a unique solution.

Proof. We give an assurance here that both the multivalued mappings P_μ and R_ϑ hold all the necessary requirements of our main Theorem 3.1, for single-valued mappings. Let $\varkappa \in X = (\mathbb{C}[0,1], \mathbb{R}_+)$ and

$$\begin{aligned}|(P_\mu y)(t) - (R_\vartheta y)(t)|^2 &\leq \left(\int_0^t |\mathcal{K}(t, s, \varkappa(s)) - \mathcal{K}(t, s, y(s))| \mathrm{d}s\right)^2 \\ &\leq \left(\int_0^t \mu(s)|\varkappa(s) - y(s)| \mathrm{d}s\right)^2 \\ &\leq d_b(\varkappa, y)\left(\int_0^t \mu(s) \mathrm{d}s\right)^2, \text{ for all } t, s \in [0, 1].\end{aligned}$$

This implies that

$$|(P_\mu y)(t) - (R_\vartheta y)(t)|^2 \leq \frac{d_b(\varkappa, y)e^{-\tau(t)}}{2}.$$

Thus, by (ii), we have

$$2e^{\tau(t)}\max\left\{d_b(\varkappa, P_\mu(\varkappa)), d_b(y, R_\vartheta(y)), \frac{d_b(\varkappa, P_\mu(\varkappa)).\ d_b(y, R_\vartheta(y))}{1 + d_b(\varkappa, y)}\right\} \leq d_b(\varkappa, y),$$

$$e^{\tau(t)} \times 2\max\left\{d_b(\varkappa, P_\mu(\varkappa)), d_b(y, R_\vartheta(y)), \frac{d_b(\varkappa, P_\mu(\varkappa)).\ d_b(y, R_\vartheta(y))}{1 + d_b(\varkappa, y)}\right\} \leq d_b(\varkappa, y).$$

Taking

$$b = 2,$$

we obtain

$$e^{\tau(t)} \times b\max\left\{d_b(\varkappa, P_\mu(\varkappa)), d_b(y, R_\vartheta(y)), \frac{d_b(\varkappa, P_\mu(\varkappa)).\ d_b(y, R_\vartheta(y))}{1 + d_b(\varkappa, y)}\right\} \leq d_b(\varkappa, y).$$

Taking ln both sides, we obtain

$$\ln e^{\tau(t)} + \ln b\left(\max\left\{d_b(\varkappa, P_\mu(\varkappa)), d_b(y, R_\vartheta(y)), \frac{d_b(\varkappa, P_\mu(\varkappa)).\ d_b(y, R_\vartheta(y))}{1 + d_b(\varkappa, y)}\right\}\right) \leq \ln(d_b(\varkappa, y)).$$

Define a function $F: \mathbb{R}^+ \to \mathbb{R}$ by $F(p) = \ln(p)$ for all $p \in \mathbb{R}^+$ and $\tau(t) = \frac{1}{8}$ and $\eta(t) = \frac{1}{10}$ for each $t \in (0, \infty)$. Then, clearly, $F \in \Omega\chi_\kappa^*$ and $\tau(t) > \eta(t)$, $\lim_{s\to t^+}\inf\tau(t) > \lim_{s\to t^+}\inf\eta(t)$. So, all the conditions of Theorem 3.1 are satisfied. Hence, the integral equation (3.2) has a unique common solution.

4 Application to fractional differential equation

Lacroix (1819) first established the different aspects of fractional differentials. Later, a specific number of well-known writers showed significant fixed point problems in distinct types of distance spaces by applying a variety of different generalized contractions to obtain the solution of fractional differential equations (see [19,25,29]). Some new models related to Caputo–Fabrizio derivative have newly established and well explained (see [30,31]). In the aforementioned section, we prove one new model that exists in b-metric spaces from these types of models.

Let $I = [0,1]$ and $\mathbb{C}(I, \mathbb{R})$ be the space consisting of all continuous functions. The mapping defined

$$2e^{\tau(t)}\max\left\{d_b(\varkappa, P_\mu(\varkappa)), d_b(y, R_\vartheta(y)), \frac{d_b(\varkappa, P_\mu(\varkappa)) \,.\, d_b(y, R_\vartheta(y))}{1 + d_b(\varkappa, y)}\right\} \leq d_b(\varkappa, y).$$

$d_b : \mathbb{C}(I, \mathbb{R}) \times \mathbb{C}(I, \mathbb{R}) \to [0, \infty)$ by $d_b(x, y) = \|x - y\|_\infty^2 = \max_{l\in[0,L]}|x(l) - y(l)|^2$, for each $x, y \in \mathbb{C}(I, \mathbb{R})$.

Then, the setting $(\mathbb{C}(I, \mathbb{R}), d_b)$ becomes a complete b-metric space. Define $\alpha : X \times X \to \mathbb{R}$ as

$$\alpha(p, q) = \begin{cases} 1, & \text{if } p(t) \leq q(t) \\ \frac{1}{2}, & \text{otherwise} \end{cases}.$$

Let $\mathcal{K}_1, \mathcal{K}_2 : I \times \mathbb{R} \to \mathbb{R}$ be two mappings such that $\mathcal{K}_1(l, g(e)), \mathcal{K}_2(l, h(e)) \geq 0$ for all $l \in I$ and $g, h \in \mathbb{C}(I, \mathbb{R})$. We will investigate the following system of fractional differential equations:

$$^{C}D^{v}g(l) = \mathcal{K}_1(l, g(l)); g \in \mathbb{C}(I, \mathbb{R}), \tag{4.1}$$

$$^{C}D^{v}h(l) = \mathcal{K}_2(l, h(l)); h \in \mathbb{C}(I, \mathbb{R}), \tag{4.2}$$

with boundary conditions $g(0) = 0, Ig(1) = g'(0), h(0) = 0, Ih(1) = h'(0)$.

Here, $^{C}D^{v}$ represents the CFD of order v given as

$$^{C}D^{v}g(l) = \frac{1}{\gamma(p - v)}\int_0^l((l - e)^{p-v-1}g(e))\mathrm{d}e,$$

where $p - 1 < v < p$ and $p = [n] + 1$, and $I^{v}g$ is defined by:

$$I^{v}g(l) = \frac{1}{\gamma(v)}\int_0^l((l - e)^{v-1}g(e))\mathrm{d}e, \text{ with } v > 0.$$

Then, equations (4.1) and (4.2) can be modified to

$$g(l) = \frac{1}{\gamma(v)}\int_0^l(l - e)^{v-1}\mathcal{K}_1(e, g(e))\mathrm{d}e + \frac{2l}{\gamma(v)}\int_0^L\int_0^e(e - z)^{v-1}\mathcal{K}_1(z, g(z))\mathrm{d}z\mathrm{d}e,$$

$$h(l) = \frac{1}{\gamma(v)}\int_0^l(l - e)^{v-1}\mathcal{K}_2(e, h(e))\mathrm{d}e + \frac{2l}{\gamma(v)}\int_0^L\int_0^e(e - z)^{v-1}\mathcal{K}_2(z, h(z))\mathrm{d}z\mathrm{d}e.$$

Suppose that

a) there exists $\tau > 0$, such that

$$|\mathcal{K}_1(l, g(e)) - \mathcal{K}_2(l, h(e))| \leq \frac{e^{-\tau}\gamma(v + 1)}{4}|g(e) - h(e)|,$$

for all $e \in I$.

b) There exists $f_0 \in \mathbb{C}(I, \mathbb{R})$, so that for any $l \in I$,

$$g_0(l) \leq \frac{1}{\gamma(v)}\int_0^l(l - e)^{v-1}\mathcal{K}_1(e, u_0(e))\mathrm{d}e + \frac{2l}{\gamma(v)}\int_0^L\int_0^e(e - z)^{v-1}\mathcal{K}_1(z, u_0(z))\mathrm{d}z\mathrm{d}e.$$

$$h_0(l) \leq \frac{1}{\gamma(v)}\int_0^l(l - e)^{v-1}\mathcal{K}_2(e, j_0(e))\mathrm{d}e + \frac{2l}{\gamma(v)}\int_0^L\int_0^e(e - z)^{v-1}\mathcal{K}_2(z, j_0(z))\mathrm{d}z\mathrm{d}e$$

c) Let $X = \{u \in \mathbb{C}(I, \mathbb{R}) : u(l) \geq 0 \text{ for all } l \in I\}$ and define the operator $P_\mu, R_\vartheta : X \to X$by

$$(P_\mu q)(l) = \frac{1}{\gamma(v)}\int_0^l (l-e)^{v-1}\mathcal{K}_1(e, q(e))\mathrm{d}e + \frac{2l}{\gamma(v)}\int_0^L\int_0^e (e-z)^{v-1}\mathcal{K}_1(z, q(z))\mathrm{d}z\mathrm{d}e$$

and

$$(R_\vartheta h)(l) = \frac{1}{\gamma(v)}\int_0^l (l-e)^{v-1}\mathcal{K}_2(e, h(e))\mathrm{d}e + \frac{2l}{\gamma(v)}\int_0^L\int_0^e (e-z)^{v-1}\mathcal{K}_2(z, h(z))\mathrm{d}z\mathrm{d}e$$

satisfying

$$\max\left\{d_b(\varkappa, P_\mu(\varkappa)), d_b(y, R_\vartheta(y)), \frac{d_b(\varkappa, P_\mu(x))\cdot d_b(y, R_\vartheta(y))}{1 + d_b(\varkappa, y)}\right\} \leq |(P_\mu y)(t) - (R_\vartheta y)(t)|^2.$$

Theorem 4.1. *Equations* (4.1) *and* (4.2) *have a common solution in* $\mathbb{C}(I, \mathbb{R})$ *if the conditions* (a)–(c) *are satisfied.*

Proof. Consider

$$|(P_\mu g)(l) - (R_\vartheta u)(l)| = \left| \begin{array}{c} \frac{1}{\gamma(v)}\int_0^l (l-e)^{v-1}\mathcal{K}_1(e, g(e))\mathrm{d}e \\ - \frac{1}{\gamma(v)}\int_0^l (l-e)^{v-1}\mathcal{K}_2(e, u(e))\mathrm{d}e \\ + \frac{2l}{\gamma(v)}\int_0^L\int_0^z (z-w)^{v-1}\mathcal{K}_1(w, g(w))\mathrm{d}w\mathrm{d}z \\ - \frac{2l}{\gamma(v)}\int_0^L\int_0^z (z-w)^{v-1}\mathcal{K}_2(w, u(w))\mathrm{d}w\mathrm{d}z \end{array} \right|,$$

which implies that

$$\begin{aligned}
|(P_\mu g)(l) - (R_\vartheta u)(l)| \leq{}& \left| \int_0^l \left(\frac{1}{\gamma(v)}(l-e)^{v-1}\mathcal{K}_1(e, g(e)) - \frac{1}{\gamma(v)}(l-e)^{v-1}\mathcal{K}_2(e, u(e)) \right)\mathrm{d}e \right| \\
&+ \left| \int_0^L\int_0^z \left(\frac{2}{\gamma(v)}(z-w)^{v-1}\mathcal{K}_1(w, g(w)) - \frac{2}{\gamma(v)}(z-w)^{v-1}\mathcal{K}_2(w, u(w)) \right)\mathrm{d}w\mathrm{d}z \right| \\
\leq{}& \frac{1}{\gamma(v)}\frac{e^{-\tau(t)}\gamma(v+1)}{4}\int_0^l (l-e)^{v-1}(|g(e) - u(e)|)\mathrm{d}e \\
&+ \frac{2}{\gamma(v)}\frac{e^{-\tau(t)}\gamma(v+1)}{4}\int_0^L\int_0^z (z-w)^{v-1}(|g(w) - u(w)|)\mathrm{d}w\mathrm{d}z \\
\leq{}& \frac{1}{\gamma(v)}\frac{e^{-\tau(t)}\gamma(v+1)}{4}\cdot|g(e) - u(e)|\cdot\int_0^l (l-e)^{v-1}\mathrm{d}e + \frac{2}{\gamma(v)}\frac{e^{-\tau(t)}\gamma(v)\cdot\gamma(v+1)}{4(v)\cdot\gamma(v+1)}\cdot|g(e) - u(e)|. \\
&\int_0^L\int_0^z (z-w)^{v-1}\mathrm{d}w\mathrm{d}z \leq \left(\frac{e^{-\tau(t)}\gamma(v)\cdot\gamma(v+1)}{4\gamma(v)\cdot\gamma(v+1)} \right)\cdot|g(e) - u(e)| \\
&+ 2e^{-\tau(t)}B(v+1, 1)\frac{\gamma(v)\cdot\gamma(v+1)}{4\gamma(v)\gamma\cdot(v+1)}\cdot|g(e) - u(e)| \leq \frac{e^{-\tau(t)}}{4}|g(e) - u(e)| + \frac{e^{-\tau(t)}}{2}|g(e) - u(e)| \\
<{}& \frac{e^{-\tau(t)}}{4}|g(e) - u(e)| < \frac{e^{-\tau(t)}}{2}|g(e) - u(e)|.
\end{aligned}$$

This implies that

$$|(P_\mu g)(l) - (R_\vartheta u)(l)| \leq \frac{e^{-\tau(t)}}{2}|g(e) - u(e)|. \tag{4.3}$$

On taking square both sides of inequality (4.3), we deduce that

$$|(P_\mu g)(l) - (R_\vartheta u)(l)|^2 \leq \frac{e^{-2\tau(t)}}{4}|g(e) - u(e)|^2 \leq \frac{e^{-\tau(t)}}{4}|g(e) - u(e)|^2, \tag{4.4}$$

where B is the beta function. By using assumption (c), the inequality (4.4) can be written as

$$4\max\left\{d_b(\varkappa, P_\mu(\varkappa)), d_b(y, R_\vartheta(y)), \frac{d_b(\varkappa, P_\mu(\varkappa)).\ d_b(y, R_\vartheta(y))}{1 + d_b(\varkappa, y)}\right\} \leq e^{-\tau(t)}d_b(\varkappa, y); \varkappa, y \in \mathbb{C}(I, \mathbb{R}). \tag{4.5}$$

Define $F(q(l)) = \ln(q(l))$ for all $q \in \mathbb{C}(I, \mathbb{R}^+)$, and $\tau\ (t) = \frac{1}{8}$ and $\eta(t) = \frac{1}{10}$ for each $t \in (0, \infty)$. Then, clearly, $F \in \Omega\chi_\kappa^*$ and $\tau(t) > \eta(t)$, $\lim_{s\to t^+}\inf\tau(t) > \lim_{s\to t^+}\inf\eta(t)$; then, the inequality (4.5) can be written as

$$\tau(d_b(g, u)) + \ln\left(4\max\left\{d_b(\varkappa, P_\mu(\varkappa)), d_b(y, R_\vartheta(y)), \frac{d_b(\varkappa, P_\mu(\varkappa)).\ d_b(y, R_\vartheta(y))}{1 + d_b(\varkappa, y)}\right\}\right) \leq \ln(d_b(g, u)).$$

All conditions of Theorem 2.4 for a single-valued mapping are verified, and the mappings P_μ and R_ϑ admit a fixed point. Hence, equations (4.1) and (4.2) have a unique common solution.

5 Conclusion

In this manuscript, we establish some newly fixed point findings for two families of dominated multivalued mappings in the settings of orbitally complete b-metric space that fulfill a unique generalized Nashine, Wardowski, Feng-Liu, and Ćirić-type contraction for orbitally lower semi-continuous functions. Moreover, the existence of significant fixed point result for two families of dominated set-valued mappings is presented in an ordered complete orbitally b-metric space. Illustrative examples are given to validate our new outcomes. To show the novelty of our new outcomes, applications are offered for non-linear integral and fractional differential equations. Moreover, in this article, we generalize and improve the results of Rasham et al. [21,25,27], Nashine et al. [19], and many more [5,15,18,23,32–39]. We can further enhance our research work in the future for fuzzy mappings, families of fuzzy mappings L-fuzzy mappings, and bipolar fuzzy mappings.

Acknowledgement: The third and fifth authors would like to thank Prince Sultan University for funding this manuscript through TAS Lab.

Author contributions: T. R and A. M discussed each other and traced out the problem to write this research article. A.M and M. N prepared examples of this article. T. R prepared the first draft. W. S involved in editing – writing and reviewing the whole article. T. R involved in conceptualization. All authors checked and approved final draft of this article.

Conflict of interest: Prof. Wasfi Shatanawi is a member of the Editorial Board of Demonstratio Mathematica but was not involved in the review process of this article.

Ethical approval: The conducted research is not related to either human or animal use.

Data availability statement: The sharing of data is unrelated to this article since no datasets were produced or examined in the course of this study.

References

[1] S. B. Nadler, *Multivalued contraction mappings*, Pac. J. Math. **30** (1969), 475–488.

[2] L. B. Ćirić, *Fixed point for generalized multivalued contractions*, Mat. Vesn. **9** (1972), 265–272.

[3] Y. Feng and S. Liu, *Fixed point theorems for multi-valued contractive mappings and multi-valued Caristi type mappings*, J. Math. Anal. Appl. **317**(1) (2006), 103–112.

[4] G. Minak, M. Olgun, and I. Altun, *A new approach to fixed point theorems for multivalued contractive mappings*, Carpathian J. Math. **31** (2015), no. 2, 241–248.

[5] A. Nicolae, *Fixed point theorems for multi-valued mappings of Feng–Liu type*, Fixed Point Theory **12** (2011), no. 1, 145–154.

[6] T. Rasham, A. Shoaib, N. Hussain, M. Arshad, and S. U. Khan, *Common fixed point results for new Ciric-type rational multivalued -contraction with an application*, J. Fixed Point Theory. Appl. **20** (2018), no. 1, Paper No. 45.

[7] A. Bakhtin, *The contraction mapping principle in almost quasi spaces*, Funkts. Anal. **30** (1989), 26–37.

[8] S. Czerwik, *Contraction mappings in b-metric spaces*, Acta Math. Inform. Univ. Ostrav. **5** (1993), 5–11.

[9] S. Czerwik, *Nonlinear set-valued contraction mappings in b-metric spaces*, Atti Semin. Mat. Fis. Univ. Modena. **46** (1998), no. 2, 263–276.

[10] S. Czerwik, K. Dlutek, and S. L. Sing, *Round-off stability of iteration procedures for set-valued operators in b-metric spaces*, J. Nat. Phys. Sci. **11** (2007), 87–94.

[11] N. Mlaiki, H. Aydi, N. Souayah, and T. Abdeljawad, *Controlled metric type spaces and the related contraction principle*, Maths. **6** (2018), Paper No. 194.

[12] S. S. Aiadi, W. A. M. Othman, K. B. Wang, and N. Mlaiki, *Fixed point theorems in controlled J–metric spaces*, AIMS Maths. **8** (2023), 4753–4763.

[13] S. Haque, F. Azmi, and N. Mlaiki, *Fredholm type integral equation in controlled rectangular metric-like spaces*, Symmetry. **14** (2022), no. 5, Paper No. 991.

[14] W. Shatanawi, V. C. Rajic, S. Radenovic, and A. Al-Rawashdeh, *Mizoguchi-Takahashi-type theorems in tvs-cone metric spaces*, Fixed Point Theory Appl. **2012** (2012), Paper No. 106.

[15] D. Wardowski, *Fixed point theory of a new type of contractive mappings in complete metric spaces*, Fixed Point Theory Appl. **2012** (2012), Paper No. 94.

[16] M. Sgroi, and C. Vetro, *Multi-valued F-contractions and the solution of certain functional and integral equations*, Filomat, **27** (2013), no. 7, 1259–1268.

[17] A. Padcharoen, D. Gopal, P. Chaipunya, and P. Kumam, Fixed point and periodic point results for α-type -contractions in modular metric spaces, Fixed Point Theory Appl. **2016** (2016), Paper No. 39.

[18] H. Aydi, E. Karapinar, and H. Yazidi, *Modified F-contractions via α-admissible mappings and application to integral equations*, Filomat. **31** (2017), no. 5, 1141–1148.

[19] H. K. Nashine, L. K. Dey, R. W. Ibrahimc, and S. Radenović, *Feng–Liu-type Fixed point result in orbital b-metric spaces and application to fractal integral equation*, Nonlinear Anal. Model. Control. **26** (2021), no. 3, 522–533.

[20] T. Rasham, A. Shoaib, C. Park, R. P. Agarwal, and H. Aydi, *On a pair of fuzzy mappings in modular-like metric spaces with applications*, Adv. Difference Equ. **2021** (2021), Paper No. 245.

[21] T. Rasham, A. Shoaib, G. Marino, B. A. S. Alamri, and M Arshad, *Sufficient conditions to solve two systems of integral equations via fixed point results*, J. Ineq. Appl. **2019** (2019), Paper No. 182.

[22] H. Qawaqneh, M. S. M. Noorani, W. Shatanawi, H. Aydi, and H. Alsamir, *Fixed point results for multi-valued contraction in b-metric spaces and an application*, Maths. **7** (2019), no. 2, 132.

[23] H. K. Nashine and Z. Kadelburg, *Cyclic generalized ϕ-contractions in b-metric spaces and an application to integral equations*, Filomat. **28** (2014), no. 10, 2047–2057.

[24] T. Rasham, A. Shoaib, Q. Zaman, and M. S. Shabbir, *Fixed point results for a generalized-contractive mapping on closed ball with application*, Math. Sci. **14** (2020), no. 2, 177–184.

[25] T. Rasham, A. Asif, H. Aydi, and M. D. La Sen, *On pairs of fuzzy dominated mappings and applications*, Adv. Difference Equ. **2021** (2021), Paper No. 417.

[26] B. Alqahtani, H. Aydi, E. Karapınar, and V. Rakočević, *A Solution for Volterra fractional integral equations by hybrid contractions*, Math. **7** (2019), no. 8, Paper No. 694.

[27] T. Rasham, A. Shoaib, N. Hussain, M. Arshad, and S. U. Khan, *Common fixed point results for new Ciric-type rational multivalued F-contraction with an application*, J. Fixed Point Theory. Appl. **20** (2018), no. 1, Paper No. 45.

[28] M. Sgroi and C. Vetro, *Multi-valued -contractions and the solution of certain functional and integral equations*, Filomat **27** (2013), no. 7, 1259–1268.

[29] M. Nazam, C. Park, and M. Arshad, *Fixed point problems for generalized contractions with applications*, Adv. Diffference Equ. **2021** (2021), Paper No. 247.

[30] E. Karapınar, A. Fulga, M. Rashid, L. Shahid, and H. Aydi, *Large contractions on quasi-metric spaces with an application to nonlinear fractional differential equations*, Math. **7** (2019), Paper No. 444.

[31] N. Tuan, H. Mohammadi, and S. Rezapour, *A mathematical model for COVID-19 transmission by using the Caputo fractional derivative*, Chaos Solit. Fractals. **140** (2020), 110–107.

[32] Ö. Acar, G. Durmaz, and G. Minak, Generalized multivalued F-contractions on complete metric spaces, Bull. Iranian Math. Soc. **40** (2014), 1469–1478.

[33] R. P. Agarwal, U. Aksoy, E. Karapınar, and I. M. Erhan, F-contraction mappings on metric-like spaces in connection with integral equations on time scales, Rev. R. Acad. Cienc. Exact Físicas. Nat. Ser. A. Mat. RACSAM **114** (2020), no. 3, Paper No. 147.

[34] J. Ahmad, A. Al-Rawashdeh, and A. Azam, *Some new fixed point theorems for generalized contractions in complete metric spaces*, Fixed Point Theory Appl. **2015** (2015), Paper No. 80.

[35] H. H. Alsulami, E. Karapinar, and H. Piri, *Fixed points of modified F-contractive mappings in complete metric-like spaces*, J. Funct. Spaces. **2015** (2015), Article ID 270971.

[36] E. Karapınar, M. A. Kutbi, H. Piri, and D. O'Regan, *Fixed points of conditionally F-contractions in complete metric-like spaces*, Fixed point Theory and Appl. **2015** (2015), Paper No. 126.

[37] T. Rasham, P. Agarwal, L. S. Abbasi, and S. Jain, *A study of some new multivalued fixed point results in a modular like metric space with graph*, The Journal of Anal. **30** (2022), 833–844.

[38] T. Rasham, G. Marino, and A. Shoaib, *Fixed point for a pair of F-dominated contractive mappings in rectangular b-metric spaces with graph*, Math. **7** (2019), no. 10, Paper No. 884.

[39] S. U. Khan, M. Arshad, T. Rasham, and A. Shoaib, *Some new common fixed points of generalized rational contractive mappings in dislocated metric spaces with application*, Honam Math. J. **39** (2017), no. 2, 161–174.