Research Article

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Non-equilibrium Phase Transitions in 2D Small-World Networks: Competing Dynamics

https://doi.org/10.1515/phys-2019-0001
Received Jul 26, 2018; accepted Nov 08, 2018

Abstract: This article offers a detailed analysis of the Ising model in 2D small-world networks with competing Glauber and Kawasaki dynamics. The non-equilibrium stationary state phase transitions are obtained in these networks. The phase transitions are discussed, and the phase diagrams are obtained via Monte Carlo simulations and finite-size analyzing. We find that as the addition of links increases the phase transition temperature increases and the transition competing probability of tricritical point decreases. For the competition of the two dynamics, ferromagnetic to anti-ferromagnetic phase transitions and the critical endpoints are found in the small-world networks.

Keywords: non-equilibrium stationary state, competing dynamics, small-world networks, phase transitions

PACS: 64.60.De, 64.60.Ht, 89.75.-k

1 Introduction

The dynamical behavior of the Ising model was successfully described with the Glauber [1] and Kawasaki [2] mechanisms. In the past decades, an interesting problem has been attracting much attention, *i.e.*, the competing Glauber-type and Kawasaki-type dynamics, which leads to non-equilibrium steady states [3–6]. This competing mechanism has been applied to the other spin models [7–10] as well, and the emergence of the dynamical tricritical point and self-organization have been reported. The authors found that, for the non-equilibrium models, the universality class of the stationary critical behavior is the same as the equilibrium models [4, 6]. All of these works were built on the regular square lattices.

In 1998, Watts and Strogatz proposed the small-world networks (SWN) [11], which are believed to catch the essence of many network systems in nature and society [12, 13]. SWNs are those intermediate between a regular lattice and a random network. They can be realized by introducing a very small portion of long-range links to a regular lattice. Such networks appear as small-worlds like random graphs, *i.e.*, with a short average node-node distance that grows logarithmically with the network size. Meanwhile, they also usually have large clustering coefficients.

After Watts and Strogatz's work, bursting studies of dynamical processes on SWNs appeared(see [12, 14] for review).The critical phenomena in equilibrium and growing networks including the birth of the giant connected component, percolation, k-core percolation, phenomena near epidemic thresholds, condensation transitions, critical phenomena in spin models placed on networks, synchronization, and self-organized criticality effects in interacting systems on networks were investigated in recent years [14]. Using Monte Carlo simulations, phase transitions of the Ising model built on the SWNs were studied by many authors [15–17], and the mean-field critical exponents were found in 1 to 3 dimension SWNs.

Many authors obtained the exact solutions of the equilibrium phase transitions in a Gaussian system with long range interactions [18, 19]. Zhu and Zhu successfully introduced the SWN effect to the critical dynamics of the spin system. They obtained the analytical results of D-dimensions Gaussian model and 1-dimension Ising model built on the SWNs. In Zhu's work [20], it has been found that the SWN effect may have different types of influence on the Glauber-type dynamics and the Kawasaki-type dynamics. By focusing on the competition of the dynamics, it is easy for us to understand the significance of the SWN effect further and highlight the disparities between the dynamics. We have generalized the competing dynamics of the Gaussian model and 1D Ising model to the SWNs [21, 22]. By considering the mean field nature of the SWN effect, we obtained the critical behaviors and the phase diagrams analytically.

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The 1D Ising model built on SWNs does not show antiferromagnetic phase at any temperature with any competing probability. Moreover, we did not obtain the full phase diagrams of the Gaussian model because of divergence problems. As we assumed that the Hamiltonian of the models have a mean-field nature, we obtained the simple phase diagrams of 1D Ising models in SWNs, in which the critical temperature is constant in non-zero competing probability.

As the mean-field Hamiltonian was used to obtain the analytical results in our previous work, some detail of behaviors of the stationary state transitions were ignored. To obtain more reliable results in this work, we use extensive Monte Carlo simulations to study the non-equilibrium phase transitions in the 2D SWNs. The ferromagnetic Ising model built on the networks is discussed with the competing Glauber and Kawasaki dynamics. To make it easier to follow, we first provide the involved theoretical formulation and the simulation method in Sec. 2. By analyzing the data, the phase transition behaviors and the diagrams are discussed in Sec. 3 and Sec. 4 is the conclusions.

2 The model and simulation method

2.1 Theoretical model

In this work, we study a specific network, in which every site is additionally linked to a randomly selected one with the probability p_A on a 2 dimensional square lattice. To make the simulation simpler, we restrict the coordination number of each site lower than 5. If $p_A = 1$, the coordination number equals 5. As the addition of long-range links take a small portion of all the possible links, the network stays in the small-world regime.

The ferromagnetic Hamiltonian of the Ising model built on this network is as follows,

$$H = -\sum_{ij} J_{ij} s_i s_j, \tag{1}$$

where J_{ij} is the coupling constant which equals $J(J > 0)$ if there is a link between the nodes i and j, and equals zero if the nodes i and j are not connected.

Various dynamic processes in critical phenomena are believed to be governed by two basic mechanisms, *i.e.*, the Glauber-type with order parameter non-conserved and the Kawasaki-type with order parameter conserved. With the competing dynamics, the master equation can be written as:

$$\frac{dP(\{s\},t)}{dt} = pG + (1-p)K. \tag{2}$$

Here, pG denotes the Glauber mechanism with probability p, and $(1-p)K$ denotes the Kawasaki mechanism with probability $1-p$.

$$G = \sum_{i,\{s'\}} \left[W(s_i \to \hat{s_i}) P(\{s\},t) - W(\hat{s_i} \to s_i) P(\{s'\},t) \right],$$

$$K = \sum_{i,j,\{s'\}} \left[W(s_i s_j \to \hat{s_i}\hat{s_j}) P(\{s\},t) - W(\hat{s_i}\hat{s_j} \to s_i s_j) P(\{s'\},t) \right],$$

where $\{s\}$ and $\{s'\}$ represent the spin configurations before and after the spin flipping or exchanging respectively. G and K are determined respectively, by the Glauber-type single-spin flip probability $W_i(s_i \to \hat{s}_i)$ and by the Kawasaki-type pair-spin exchange probability $W_{ij}(s_i s_j \to \hat{s}_i \hat{s}_j)$.

In their original form, the Glauber-type dynamics and the Kawasaki-type dynamics both favor a lower energy state. However, the competing dynamics are usually used to describe a system in contact with a heat bath while exposed to an external energy flux. Naturally one requires competition between one process favoring lower system-energy and the other one favoring higher system-energy. Usually, the Glauber-type mechanism is used to simulate the contact of the system with a heat bath and it prefers a lower-energy state. On the other hand, the Kawasaki-type mechanism can be modified to simulate an external energy flux that drives the system towards a higher energy state.

2.2 Simulation method

We apply the standard Metropolis Monte Carlo method to this model using a $N \times N$ square lattice with random addition of long-range interactions, of which the addition mechanism is described in Sec 2.1. Let (k, l) be a node's coordinate of the two-dimensional SWN. The periodic boundary conditions are used in all of our simulations. For a given temperature T and a competing probability p, a randomly uniform number r_1 between zero and unity is generated. If $r_1 \leq p$, the Glauber dynamics are chosen. In this process, a randomly chosen spin s_{kl} may flip to $-s_{kl}$. The flipping probability is dependent on $W_{kl}(s_{kl} \to \hat{s}_{kl})$, which is given by

$$W_{kl}(s_{kl} \to \hat{s}_{kl}) = \begin{cases} e^{-\Delta E_{kl}/KT} & if \quad \Delta E_{kl} > 0 \\ 1 & if \quad \Delta E_{kl} \leq 0 \end{cases} \tag{3}$$

If $r_1 > p$, the Kawasaki dynamics is chosen. In this process, a randomly chosen spin s_{kl} and its coupled spin $s_{k'l'}$, which is also randomly chosen, may exchange. The exchanging probability is dependent on $W_{kl,k'l'}(s_{kl}s_{k'l'} \rightarrow \hat{s}_{kl}\hat{s}_{k'l'})$, which is given by

$$W_{kl,k'l'}(s_{kl}s_{k'l'} \rightarrow \hat{s}_{kl}\hat{s}_{k'l'}) = \begin{cases} 0 & if \quad \Delta E_{kl,k'l'} \leq 0 \\ 1 & if \quad \Delta E_{kl,k'l'} > 0 \end{cases} \tag{4}$$

Here, ΔE_{kl} is the energy change resulting from flipping spin s_{kl}, and $\Delta E_{kl,k'l'}$ is the energy change resulting form exchanging the linked spins s_{kl} and $s_{k'l'}$.

Typically $2{\times}10^6$ MC steps per spin (MCS) are discarded to reach the stationary state, and 2×10^6 MCS are retained for the averages to obtain the bulk properties. We take 50 network samples for the average and $3 - 5$ independent computations with different initial configurations to obtain the error bars. If error bars are not visible then they are smaller than the size of the symbols.

The measured thermodynamic quantities in our simulations are the magnetization per spin, the anti-ferro magnetization, the specific heat, the Binder cumulant and reduced fourth order energy cumulant:

$$m_f = [< M >] = \frac{1}{N^2}[< \sum_{kl} s_{kl} >], \tag{5}$$

$$m_A = [< M' >] = \frac{1}{N^2}[< \sum_{kl} (-1)^{(k+l)} s_{kl} >], \tag{6}$$

$$c_V = \frac{[< E^2 > - < E >^2]}{N^2 k T^2}, \tag{7}$$

$$U_4 = 1 - \frac{[< m^4 >]}{3[< m^2 >^2]}, \tag{8}$$

$$V_4 = 1 - \frac{[< E^4 >]}{3[< E^2 >^2]}. \tag{9}$$

$[\cdots]$ denotes the different network realizations taken over 50 configurations, and$< \cdots >$ denotes the thermal average taken over 2×10^5 to 10^6 MCS. m_f and m_A give the ferromagnetic and anti-ferromagnetic order parameters respectively. In Eq.(8), if $m = m_f$, U_4 represents the Binder cumulant of the ferromagnetic order parameter, while, if $m = m_A$, U_4 represents the Binder cumulant of the anti-ferromagnetic order parameter. Lattice sizes from $N = 16$ to $N = 64$ are simulated and the data are analyzed via the finite-size scaling.

2.3 Finite-size scaling analysis

Although the original finite-size scaling was used to analyze the equilibrium phase transitions [23], many researchers [4, 6] still proposed that the bulk properties obey the scaling relation near the stationary critical point:

$$c_{VL} = L^{\alpha/\nu} c_V^0(\epsilon L^{1/\nu}), \tag{10}$$

where α and ν are critical exponents, and $\epsilon = (T - T_c)/T_c$, where T_c is the critical temperature. The exponent ν can be obtained via the derivative of the fourth order parameter cumulant U_4 [24]:

$$(\frac{dU_4}{dK_T})_{max} = aL^{1/\nu}(1 + bL^{-\omega}). \tag{11}$$

where K_T is the inverse temperature with the definition $K_T = J/KT$, and $bL^{-\omega}$ is a small correction term which is not taken into account in this article.

If the phase transition is first order, the peak of the specific heat scales proportional to L^d, where d is the dimension of the system. Moreover, V_4 has a minimum at an effective transition temperature which approaches the infinite lattice value as the inverse volume of the system [23]. If the phase transition is of the second order, for large enough lattice size, the curves for the cumulate U_4 cross as a function of temperature at a fix point value and the location of the fixed point is the critical point.

3 Results and discussion

The purpose of this work is to reveal the SWN effect on non-equilibrium phase transitions. The order of the phase transitions, the transition temperatures, and the critical exponents are discussed in this part. In contrast to Szolnoki's work [6], in which they determined the critical points p_c at fixed temperatures, we locate the transition points T_c at fixed competing probabilities p instead. We revisit the case of the regular 2D lattice. There exist the anti-ferromagnetic(AF), paramagnetic(PM), and ferromagnetic(FM) in the phase diagrams of the regular lattice systems.

We use the specific heat and the fourth order cumulant of energy to distinguish the first and second phase transitions rather than use V_4 only. Typical data for the specific heat and the fourth order cumulant are shown in Figure 1. The specific heat peaks diverge with increasing lattice sizes, following the relations: $c_{max} \sim N^{\alpha}$. If $p = 0.5$, we get $\alpha = 2$, which implies that the system undergoes a first order transition. $\alpha = 0$ (logarithm) when $p = 0.7$ which implies a second order phase transition. Both results are also confirmed by the curves of the fourth order cumulant.

We simulate the cases of $p = 0.1$ to 0.9 and get the tricritical point: $T_t = 2.17 \pm 0.01$ and $p_t = 0.67 \pm 0.01$,

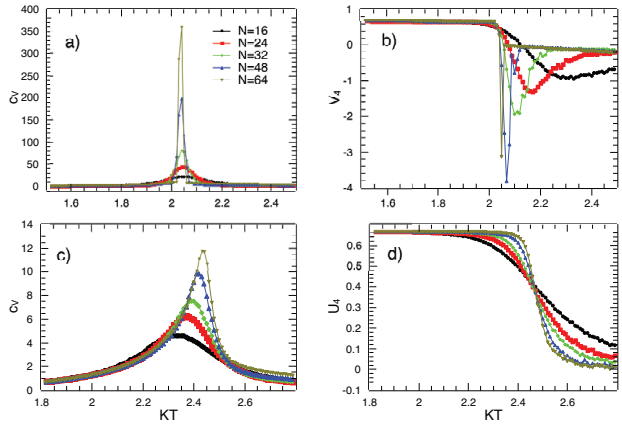

Figure 1: The bulk properties as functions of temperature in regular square lattice. a) specific heat and b) reduced fourth order energy cumulant at $p = 0.5$, c) specific heat and b) Binder cumulant at $p = 0.7$

which agree well with the theoretical results of Szolnoki's work [6]. We estimate the exponent from finite-size behavior of specific heat peaks, $c_{max} \sim N^{\alpha/\nu}$. Near the tricritical point, we get $\alpha/\nu = 1.59$. It is clear that the stationary second order transition has the same universality class as the 2D equilibrium Ising model.

We then introduce the random interactions to the system, and the regular lattice becomes the SWN. Many researchers suggested that the equilibrium critical behaviors have the mean-field nature of such systems. While Newman assumed that the effective dimension is higher than regular lattices [25], and it is an interesting problem to explore how random interactions affect the non-equilibrium stationary state transitions. So we simulate the cases of $p_A = 0.1$,0.5 and 1. Firstly, we study the FM to PM phase transitions. Figure 2 gives the specific heat and the fourth order cumulants of energy as a function of temperature at the competing probability $p = 0.5$. Table 1 gives the phase transition temperatures(T_f) obtained from Figure 2.

Table 1: Transition temperatures of FM to PM phases at $p = 0.5$ with different link addition probability

p_A:	0.1	0.5	1
T_f:	2.22	2.84	3.14

The simulations show that the transition temperatures become higher as the link addition probability p_A increases. Moreover, the peak of the specific heat scales proportional to L^2, and V_4 has a minimum at the effective transition temperature. Both of these suggest that the system undergoes a first order phase transition at $p = 0.5$.

If we increase the probability p, the order of the transition will change. The results of $p = 0.9$ are shown in Figure 3. It is obvious that the exponent $\alpha/\nu = 0$ (logarithm), which is the typical feature of the second phase transitions. The locations of the crossing fixed point of U_4 are the critical points. And the critical temperatures (T_c) increase as p_A increases. According to the scaling relation, the critical exponent ν and the effective dimensions d_{eff} satisfy the formula $\nu d_{eff} = 2 - \alpha$ [26]. By analyzing the data of U_4, we obtain the critical exponents $1/\nu = 1.02\pm0.01, 1.05\pm0.02$ and 1.13 ± 0.01, corresponding to $p_A = 0.1, 0.5, 1.0$ respectively. For ν is smaller than 1, d_{eff} is larger than 2. The mean-field behavior ($d_{eff} \geqslant 4$) does not appear on the SWNs. The reason for this phenomenon is that we restrict the number of extra links. Besides, we find that the effective dimensions increase as p_A increases. Hence we speculated that the SWNs would show the mean-field nature ($1/\nu = 2$ and $d_{eff} = 4$) if enough extra random links exist in the networks.

The previous simulations indicate there should be a tricritical point, which connects the first and second phase transitions. To get this point, we simulate the cases of $p = 0.01$ to 0.9 and estimate the exponent α/ν via finite size analyzing and locate the tricritical point (p_t, T_t). The results of $p_A = 0.1, 0.5, 1.0$ are shown in Table 2. An-

Table 2: Tricritical points with different link addition probability

p_A:	0.1	0.5	1
p_t:	0.65	0.63	0.58
T_t:	2.46	3.02	3.24

other interesting phenomenon is how the random links affect the self-organization under the competition between the Glauber and Kawasaki processes. If the flux of energy $(1 - p)$ is further increased, the Ising model on the regular square lattice self-organizes into an antiferromagnetic phase [6]. To locate the transition point of the antiferromagnetic phase to other phases on the SWNs we use the order parameters and the cumulants of the antiferromagnetic order parameter. The quantities as a function of competing probabilities are simulated at different constant temperatures.

The phase diagrams are shown in Figure 4. The data plotted in the diagrams are quite different to the one of the regular lattice. There are PM, AF and FM phases in the figures. The more random links give rise to the higher

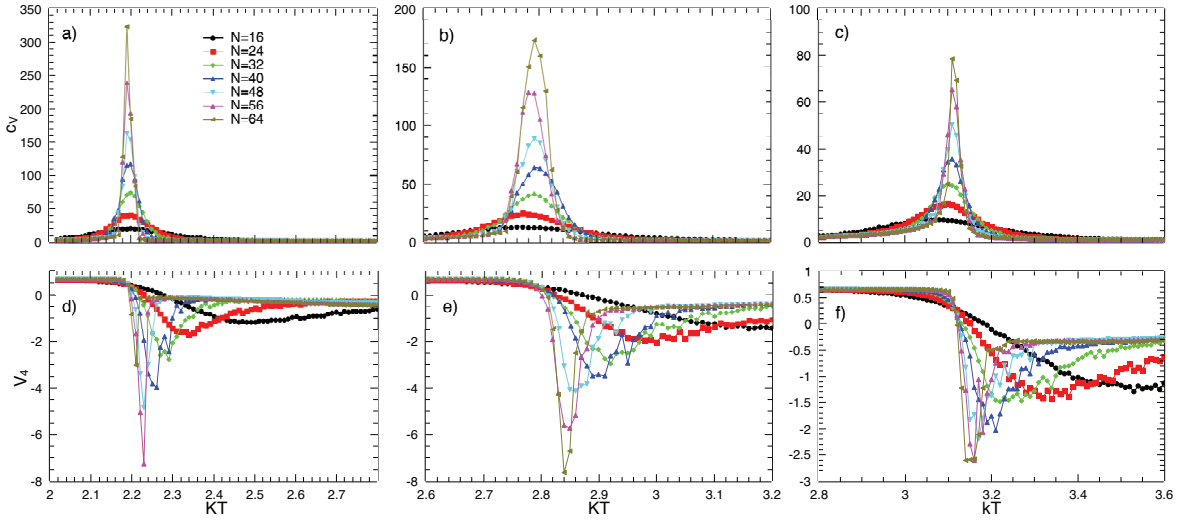

Figure 2: The bulk properties as functions of temperature at $p = 0.5$. a) to c) specific heat of the SWNs with the adding probability $p_A = 0.1, 0.5, 1.0$, respectively, d) to f) reduced fourth order energy cumulant of the SWNs with the adding probability $p_A = 0.1, 0.5, 1.0$, respectively

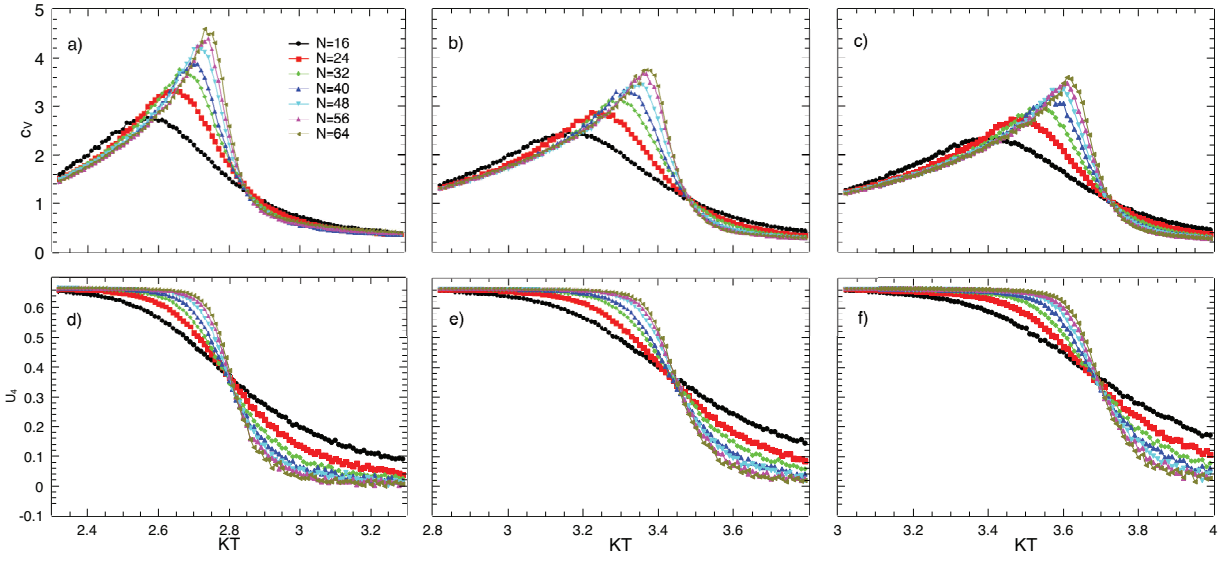

Figure 3: The bulk properties as functions of temperature at $p = 0.9$. a) to c) specific heat of the SWNs with the adding probability $p_A = 0.1, 0.5, 1.0$, respectively, d) to f) Binder cumulant of the SWNs with the adding probability $p_A = 0.1, 0.5, 1.0$, respectively

transition temperatures of FM to PM. More ferromagnetic coupling makes the system like a ferromagnetic state. The higher temperature is needed to break the ferromagnetic order. Compared to the regular lattice, the region of the FM phase becomes larger, and especially at small p (large energy flux), the systems undergo FM to AF transitions, which does not exist in the regular lattice. The transition of FM to AF is first order. As a result, the critical line of AF to PM meets and is truncated by a first order transition line. The truncated point should be the critical endpoint.

The diagrams reveal the fact that adding random links breaks the AF order and leads to the appearance of the FM phase even at considerable energy flux if the temperature is low. However, the Kawasaki mechanism, which prefers the high temperature, dominates the system when p is small. Hence, when the temperature is high enough, the system will self-organize to the AF phase again. However, the axis $p = 0$ is a unique zone in the phase diagrams. We cannot identify the system simply as FM or AF, because in this case, it depends on the initial state. If $m_f \neq 0$ and

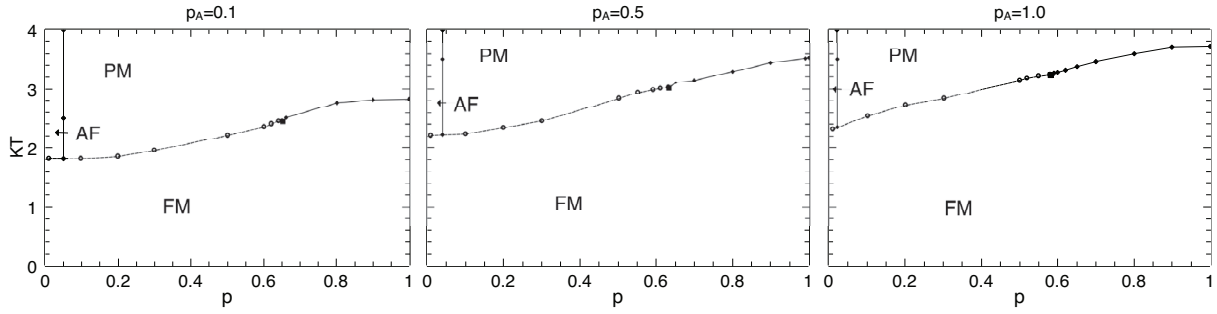

Figure 4: Phase diagrams of the SWNs with the adding probability $p_A = 0.1, 0.5, 1.0$, respectively. The boundary of the first order transitions are shown by dashed lines and open circles, and the second order transitions are shown by solid lines and closed diamonds. The solid squares represent the tricritical points

$m_A = 0$ at the initial state, the system will stay in a FM state; otherwise, the system will be AF.

The tricritical points are found in the diagrams, and this was not reported in the analytical work of 1D networks [22]. If the energy flux is large enough, the transition from FM to PM is the first order; otherwise, the transition is the second order. When the energy flux is small, the Glauber mechanism dominates the system. Consequently, critical phenomena exist in these systems. However, if the energy flux is not sufficiently small, the Glauber and the Kawasaki mechanism competitively govern the system. There is no spontaneous symmetry breaking, so the transition should be the first order. The transition probability p_t of tricritical points decreases if the adding probability p_A increases. This means that the region of the spontaneous symmetry breaking becomes larger because of the extra random FM coupling.

4 Conclusions

Using extensive MC simulations, we have studied the non-equilibrium phase transitions in 2D SWNs governed by the competing Glauber and Kawasaki mechanism. The Ising model is used, and the phase diagrams are obtained. We find that adding random interactions leads to four effects. First, they make ferro- to paramagnetic phase transition temperatures increase, which is the same effect as the equilibrium transitions in SWNs. Second, the tricritical competing probability p_t decreases as the p_A increases. Third, the self-organizing behavior is restrained because of the randomness, and the AF to FM phase transition appears in the networks. Finally, the critical endpoints exist in the phase diagrams. Via finite size analyzing, we obtain the critical exponent $1/\nu > 1$, which implies that the effect dimension is larger than 2. Our results further indicate that the method for analyzing equilibrium phase transitions could be used to analyze studies of the non-equilibrium stationary transitions. We believe that the method can be applied to many real systems in nature and society.

Acknowledgement: This work is supported by the National Natural Science Foundation of China under Grant No. 11405127 and 11647022.

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