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Fixed Points of Meromorphic Functions and Their Higher Order Differences and Shifts

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Abstract: In this paper, we investigate the relationships between fixed points of meromorphic functions, and their higher order differences and shifts, and generalize the case of fixed points into the more general case for first order difference and shift. Concretely, some estimates on the order and the exponents of convergence of special points of meromorphic functions and their differences and shifts are obtained.

Keywords: Meromorphic function; higher order difference; shift; fixed point

MCS: 30D35; 39B32; 39A10

1 Introduction and main results

In this paper, a meromorphic function $f(z)$ means being meromorphic in the whole complex plane $\mathbb{C}$, and the notations are standard ones in the Nevanlinna theory (see e.g. [1-4]). Especially, we use $\rho(f)$ to denote the order of $f(z)$, and use $\lambda(f)$ and $\lambda(\frac{1}{f})$ to denote respectively the exponents of convergence of zeros and poles of $f(z)$. Moreover, we use $\tau(f)$ to denote the exponent of convergence of fixed points of $f(z)$, and use $\sigma(f)$ to denote the type of a transcendental $f(z)$. In addition, a small meromorphic function $\alpha(z)$ with respect to $f(z)$ means it satisfies $T(r,\alpha) = S(r,f)$, where $S(r,f) = o(T(r,f))$ outside a possible exceptional set of finite logarithmic measure.

In the past sixty years, numerous mathematicians have studied fixed points, which is an important topic in the theory of meromorphic functions (see e.g. [5, 6]). In 2002, Chen [6], the first person who studied fixed points of solutions of differential equations, defined the exponent of convergence of fixed points by $\tau(f)$ firstly. After that, many scholars investigated the topic on fixed points and got some interesting fruits. For example, Bergweiler and Pang [7] studied the zeros of $f'(z) - R(z)$ and obtained the following result.

Theorem 1.A ([7]) Let $f(z)$ be a meromorphic function and let $R(z)(\not\equiv 0)$ be a rational function. Suppose that all but finitely many zeros and poles of $f(z)$ are multiple. Then $f'(z) - R(z)$ has infinitely many zeros. (In particular, if $R(z) \equiv z$, then $f'(z)$ has infinitely many fixed points.)

The topic on fixed points can be also investigated in the field of complex differences. Here, the forward differences (see [8]) are defined by

$$\Delta_c^1 f(z) = \Delta_c f(z) = f(z+c) - f(z),$$

$$\Delta_c^n f(z) = \Delta_c(\Delta_c^{n-1} f(z)) = \Delta_c^{n-1} f(z+c) - \Delta_c^{n-1} f(z), \quad n \in \mathbf{N}_+ \backslash \{1\},$$

where $c \in \mathbb{C}\backslash\{0\}$. For example, Chen and shon [9-11] have got some results on the zeros and fixed points of transcendental entire functions and meromorphic functions. Recently, Chen [12] and Zhang [13] studied the

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relationships between fixed points of meromorphic functions and their differences and shifts. Their results are stated as follows.

Theorem 1.B ([12]) Let $f(z)$ be a finite order meromorphic function such that $\lambda(\frac{1}{f(z)}) < \rho(f)$, and let $c \in \mathbb{C}\backslash\{0\}$ be a constant such that $f(z+c) \not\equiv f(z)+c$. Then

$$\max\{\tau(f(z)), \tau(\Delta_c f(z))\} = \rho(f),$$

$$\max\{\tau(f(z)), \tau(f(z+c))\} = \rho(f),$$

$$\max\{\tau(\Delta_c f(z)), \tau(f(z+c))\} = \rho(f).$$

Theorem 1.C ([13]) Let $a \in \mathbb{C}$ and let $f(z)$ be a finite order meromorphic function such that $\lambda(f(z)-a) < \rho(f)$. Let $c \in \mathbb{C}\backslash\{0\}$ be a constant. Then

$$\max\{\tau(f(z)), \tau(\Delta_c f(z))\} = \rho(f),$$

$$\max\{\tau(f(z)), \tau(f(z+c))\} = \rho(f),$$

$$\max\{\tau(\Delta_c f(z)), \tau(f(z+c))\} = \rho(f).$$

Inspired by the previous results, especially Theorems 1.B and 1.C, we proceed to study the relationships between fixed points of meromorphic functions and their differences and shifts. Firstly, we consider higher order differences and shifts instead of first order ones, and obtain the following result.

Theorem 1.1 Let $c \in \mathbb{C}\backslash\{0\}$, $n \in \mathbf{N}_+$, and let $f(z)$ be a finite order transcendental meromorphic function. If $f(z)$ has a Borel exceptional value $a \in \mathbb{C}$, then

$$\max\{\tau(f(z)), \tau(\Delta_c^n f(z))\} = \rho(f),$$

$$\max\{\tau(f(z)), \tau(f(z+nc))\} = \rho(f),$$

$$\max\{\tau(\Delta_c^n f(z)), \tau(f(z+nc))\} = \rho(f).$$

Secondly, we generalize the case of fixed points into the more general case for $n = 1$, and obtain the following result.

Theorem 1.2 Let $c \in \mathbb{C}\backslash\{0\}$, $m \in \mathbf{N}_+$, $p(z) = p_m z^m + p_{m-1}z^{m-1} + \cdots + p_0$ be a nonzero polynomial such that $p_i \in \mathbb{C}$, $i = 0, 1, \cdots, m$ and $p_m \neq 0$, and let $f(z)$ be a finite order transcendental meromorphic function. If $f(z)$ has a Borel exceptional value $a \in \mathbb{C}$, then

$$\max\{\lambda(f(z)-p(z)), \lambda(\Delta_c f(z)-p(z))\} = \rho(f),$$

$$\max\{\lambda(f(z)-p(z)), \lambda(f(z+c)-p(z))\} = \rho(f),$$

$$\max\{\lambda(\Delta_c f(z)-p(z)), \lambda(f(z+c)-p(z))\} = \rho(f).$$

2 Lemmas for proofs of main results

Lemma 2.1 ([14]) Let $f(z)$ be a meromorphic function with $\rho(f) < \infty$, and let η be a fixed nonzero complex number. Then for each $\varepsilon > 0$, we have

$$T(r, f(z+\eta)) = T(r, f(z)) + O(r^{\rho(f)-1+\varepsilon}) + O(\log r).$$

Lemma 2.2 ([14]) Let $f(z)$ be a meromorphic function with $\lambda(\frac{1}{f(z)}) < \infty$, and let η be a fixed nonzero complex number. Then for each $\varepsilon > 0$, we have

$$N(r, f(z+\eta)) = N(r, f(z)) + O(r^{\lambda(\frac{1}{f(z)})-1+\varepsilon}) + O(\log r).$$

Lemma 2.3 ([13]) Let $a \in \mathbb{C}$ and let $f(z)$ be a meromorphic function with $\lambda(f(z)-a) < \infty$, and let η be a fixed nonzero complex number. Then for each $\varepsilon > 0$, we have

$$N(r, \frac{1}{f(z+\eta)-a}) = N(r, \frac{1}{f(z)-a}) + O(r^{\lambda(f(z)-a)-1+\varepsilon}) + O(\log r).$$

Lemma 2.4 ([15]) Let $A_0(z), \cdots, A_n(z)$ be entire functions of finite order such that among those having the maximal order $\rho = \max\{\rho(A_k) : 0 \le k \le n\}$, exactly one has its type strictly greater than the others. Then for any meromorphic solution $f(z)$ of

$$A_n(z)f(z+c_n) + \cdots + A_1(z)f(z+c_1) + A_0(z)f(z) = 0,$$

we have $\rho(f) \ge \rho + 1$.

Lemma 2.5 ([3]) Let $f(z)$ be a meromorphic function. Then for all irreducible rational functions in $f(z)$,

$$R(z, f(z)) = \frac{\sum\limits_{i=0}^{p} a_i(z)f(z)^i}{\sum\limits_{j=0}^{q} b_j(z)f(z)^j}$$

with meromorphic coefficients $a_i(z), i = 0, 1, \cdots, p$ and $b_j(z), j = 0, 1, \cdots, q$ such that

$$\begin{cases} T(r, a_i) = S(r, f), & i = 0, 1, \cdots, p, \\ T(r, b_j) = S(r, f), & j = 0, 1, \cdots, q, \end{cases}$$

the characteristic function of $R(z, f(z))$ satisfies

$$T(r, R(z, f(z))) = \max\{p, q\}T(r, f) + S(r, f).$$

Lemma 2.6 ([13]) Suppose that $h(z)$ is a nonconstant meromorphic function satisfying

$$\overline{N}(r, h) + \overline{N}(r, \frac{1}{h}) = S(r, h).$$

Let

$$F(z) = \frac{a_0(z)h(z)^p + a_1(z)h(z)^{p-1} + \cdots + a_p(z)}{b_0(z)h(z)^q + b_1(z)h(z)^{q-1} + \cdots + b_q(z)},$$

where $a_i(z), i = 0, 1, \cdots, p, b_j(z), j = 0, 1, \cdots, q$ are small functions of $h(z)$ and $a_0 b_0 a_p \not\equiv 0$. If $q \le p$ and $T(r, F) \ge T(r, h) + S(r, h)$, then $\lambda(F) = \rho(h)$.

Lemma 2.7 ([2, 4]) Suppose that $f_1(z), f_2(z), \cdots, f_n(z)$ are meromorphic functions and that $g_1(z), g_2(z), \cdots, g_n(z)$ are entire functions satisfying the following conditions.

(1) $\sum\limits_{j=1}^{n} f_j(z)e^{g_j(z)} \equiv 0$;

(2) $g_j(z) - g_k(z)$ are not constants for $1 \le j < k \le n$;

(3) for $1 \le j \le n$, $1 \le h < k \le n$,

$$T(r, f_j) = o\{T(r, e^{g_h - g_k})\} \quad n.e. \quad \text{as} \quad r \to \infty.$$

Then $f_j(z) \equiv 0, j = 1, 2, \cdots, n$.

Lemma 2.8 ([13]) Let c be a nonzero constant, $H(z)$ be a meromorphic function, and let $h(z)$ be a polynomial with $\deg h(z) \ge 1$. If $\rho(H) < \rho(e^h)$, then

$$T(r, H(z)) = S(r, e^{h(z)}), T(r, H(z+c)) = S(r, e^{h(z)}), T(r, e^{h(z+c)-h(z)}) = S(r, e^{h(z)}).$$

Remark 2.9 From the proof of Lemma 2.8, we can also obtain

$$T(r, H(z+jc)) = S(r, e^{h(z)}), \quad j = 1, 2, \cdots, n$$

and

$$T(r, e^{h(z+kc)-h(z+sc)}) = S(r, e^{h(z)}), \quad k \in \mathbf{N}_+, s \in \mathbf{N}, k > s,$$

under the conditions in Lemma 2.8.

3 Proofs of Theorems 1.1 and 1.2

Proof of Theorem 1.1 As Theorem 1.C shows, the conclusions hold for $n = 1$. Next, we prove the conclusions for $n \geq 2$.

Firstly, we prove the conclusions for $n = 2$. Suppose that $\tau(f(z)) < \rho(f)$, and we prove $\tau(\Delta_c^2 f(z)) = \tau(f(z + 2c)) = \rho(f)$ next. Denote

$$g_1(z) = \frac{f(z) - z}{f(z) - a}. \tag{3.1}$$

Obviously, $g(z)$ satisfies $\rho(g_1) = \rho(f) < \infty$. Then we have

$$\lambda(\frac{1}{g_1(z)}) = \lambda(f(z) - a) < \rho(f) = \rho(g_1)$$

and

$$\lambda(g_1(z)) = \lambda(f(z) - z) = \tau(f(z)) < \rho(f) = \rho(g_1),$$

which means that 0 and ∞ are Borel exceptional values of $g_1(z)$. By Hadamard's factorization theory, $g_1(z)$ can be written as

$$g_1(z) = H_1(z)e^{h_1(z)}, \tag{3.2}$$

where $H_1(z)(\not\equiv 0)$ is a meromorphic function such that $\rho(H_1) < \rho(g_1) = \rho(f)$ and

$$h_1(z) = a_k z^k + a_{k-1}z^{k-1} + \cdots + a_0 \tag{3.3}$$

is a polynomial such that $\rho(f) = \rho(g_1) = \deg h_1(z) = k$, where $k \in \mathbf{N}_+$ and $a_i \in \mathbb{C}, i = 0, 1, \cdots, k, a_k \neq 0$. Substituting (3.2) into (3.1), we obtain

$$f(z) = \frac{z - a}{1 - g_1(z)} + a = \frac{z - a}{1 - H_1(z)e^{h_1(z)}} + a. \tag{3.4}$$

We get from (3.4) that

$$\begin{aligned} \Delta_c^2 f(z) &= f(z + 2c) - 2f(z + c) + f(z) \\ &= \frac{z + 2c - a}{1 - g_1(z + 2c)} - \frac{2(z + c - a)}{1 - g_1(z + c)} + \frac{z - a}{1 - g_1(z)} \\ &= \frac{z + 2c - a}{1 - H_1(z + 2c)e^{h_1(z+2c)}} - \frac{2(z + c - a)}{1 - H_1(z + c)e^{h_1(z+c)}} + \frac{z - a}{1 - H_1(z)e^{h_1(z)}} \\ &= \frac{P_{2,2}(z)e^{2h_1(z)} + P_{2,1}(z)e^{h_1(z)}}{P_{2,5}(z)e^{3h_1(z)} + P_{2,4}(z)e^{2h_1(z)} + P_{2,3}(z)e^{h_1(z)} + 1}, \end{aligned} \tag{3.5}$$

where

$$\begin{aligned} P_{2,1}(z) &= (z + 2c - a)e^{h_1(z+2c)-h_1(z)}H_1(z + 2c) - 2(z + c - a)e^{h_1(z+c)-h_1(z)}H_1(z + c) \\ &\quad +(z - a)H_1(z), \\ P_{2,2}(z) &= (z - a)e^{h_1(z+2c)+h_1(z+c)-2h_1(z)}H_1(z + 2c)H_1(z + c) - 2(z + c - a)e^{h_1(z+2c)-h_1(z)} \\ &\quad H_1(z + 2c)H_1(z) + (z + 2c - a)e^{h_1(z+c)-h_1(z)}H_1(z + c)H_1(z), \\ P_{2,3}(z) &= -e^{h_1(z+2c)-h_1(z)}H_1(z + 2c) - e^{h_1(z+c)-h_1(z)}H_1(z + c) - H_1(z), \\ P_{2,4}(z) &= e^{h_1(z+2c)+h_1(z+c)-2h_1(z)}H_1(z + 2c)H_1(z + c) + e^{h_1(z+2c)-h_1(z)}H_1(z + 2c)H_1(z) \\ &\quad +e^{h_1(z+c)-h_1(z)}H_1(z + c)H_1(z), \\ P_{2,5}(z) &= -e^{h_1(z+2c)+h_1(z+c)-2h_1(z)}H_1(z + 2c)H_1(z + c)H_1(z). \end{aligned}$$

By (3.5), we obtain

$$\Delta_c^2 f(z) - z = \frac{-zP_{2,5}(z)e^{3h_1(z)} + (P_{2,2}(z) - zP_{2,4}(z))e^{2h_1(z)} + (P_{2,1}(z) - zP_{2,3}(z))e^{h_1(z)} - z}{P_{2,5}(z)e^{3h_1(z)} + P_{2,4}(z)e^{2h_1(z)} + P_{2,3}(z)e^{h_1(z)} + 1}$$

$$= \frac{-zP_{2,5}(z)e^{3h_1(z)} + P_{2,7}(z)e^{2h_1(z)} + P_{2,6}(z)e^{h_1(z)} - z}{P_{2,5}(z)e^{3h_1(z)} + P_{2,4}(z)e^{2h_1(z)} + P_{2,3}(z)e^{h_1(z)} + 1}, \tag{3.6}$$

where

$$\begin{aligned} P_{2,6}(z) &= (2z + 2c - a)e^{h_1(z+2c)-h_1(z)}H_1(z + 2c) - (z + 2c - 2a)e^{h_1(z+c)-h_1(z)} \\ &\quad H_1(z + c) + (2z - a)H_1(z), \\ P_{2,7}(z) &= -ae^{h_1(z+2c)+h_1(z+c)-2h_1(z)}H_1(z + 2c)H_1(z + c) - (3z + 2c - 2a)e^{h_1(z+2c)-h_1(z)} \\ &\quad H_1(z + 2c)H_1(z) + (2c - a)e^{h_1(z+c)-h_1(z)}H_1(z + c)H_1(z). \end{aligned}$$

By (3.5) and (3.6), we can see $\Delta_c^2 f(z)$ and $\Delta_c^2 f(z) - z$ as rational functions in $e^{h_1(z)}$. Since $\rho(H_1) < \rho(e^{h_1}) = k$, by Lemma 2.8 and Remark 2.9, we get that the coefficients $P_{2,j}(z)$, $j = 1, 2, \cdots, 7$ are small functions with respect to $e^{h_1(z)}$, that is, $T(r, P_{2,j}(z)) = S(r, e^{h_1(z)})$, $j = 1, 2, \cdots, 7$. Next, we assert $P_{2,1}(z) \not\equiv 0$. Since

$$\begin{aligned} h_1(z + 2c) - h_1(z) &= a_k(z + 2c)^k + a_{k-1}(z + 2c)^{k-1} + a_{k-2}(z + 2c)^{k-2} + \cdots + a_0 \\ &\quad -(a_k z^k + a_{k-1}z^{k-1} + a_{k-2}z^{k-2} + \cdots + a_0) \\ &= (a_k \cdot c_k^1 \cdot 2c)z^{k-1} + (a_k \cdot c_k^2 \cdot 4c^2 + a_{k-1} \cdot c_{k-1}^1 \cdot 2c)z^{k-2} + \cdots, \\ h_1(z + c) - h_1(z) &= a_k(z + c)^k + a_{k-1}(z + c)^{k-1} + a_{k-2}(z + c)^{k-2} + \cdots + a_0 \\ &\quad -(a_k z^k + a_{k-1}z^{k-1} + a_{k-2}z^{k-2} + \cdots + a_0) \\ &= (a_k \cdot c_k^1 \cdot c)z^{k-1} + (a_k \cdot c_k^2 \cdot c^2 + a_{k-1} \cdot c_{k-1}^1 \cdot c)z^{k-2} + \cdots, \end{aligned}$$

we get from $k \in \mathbf{N}_+$, $a_k \neq 0$ and $c \neq 0$ that

$$\rho(e^{h_1(z+2c)-h_1(z)}) = \rho(e^{h_1(z+c)-h_1(z)}) = \rho(e^{h_1(z+2c)+h_1(z+c)-2h_1(z)}) = k - 1$$

and

$$\sigma(e^{h_1(z+2c)-h_1(z)}) = \frac{1}{\pi}|a_k \cdot c_k^1 \cdot 2c| > \frac{1}{\pi}|a_k \cdot c_k^1 \cdot c| = \sigma(e^{h_1(z+c)-h_1(z)}).$$

Thus, if $P_{2,1}(z) \equiv 0$, then by Lemma 2.4, we have $\rho(H_1) \geq k - 1 + 1 = k$, which contradicts with $\rho(H_1) < \rho(g_1) = \rho(f) = k$. So, the assertion $P_{2,1}(z) \not\equiv 0$ holds. Consequently, by Lemma 2.7 and $P_{2,1}(z) \not\equiv 0$, we have

$$P_{2,2}(z)e^{2h_1(z)} + P_{2,1}(z)e^{h_1(z)} \not\equiv 0.$$

Then by (3.5) and the obvious fact that $P_{2,5}(z) \not\equiv 0$, we have

$$T(r, \Delta_c^2 f(z)) \geq T(r, e^{h_1(z)}) + S(r, e^{h_1(z)}),$$

and consequently

$$T(r, \Delta_c^2 f(z) - z) \geq T(r, e^{h_1(z)}) + S(r, e^{h_1(z)}). \tag{3.7}$$

By (3.6), (3.7) and Lemma 2.6, we have

$$\tau(\Delta_c^2 f(z)) = \lambda(\Delta_c^2 f(z) - z) = \rho(e^{h_1}) = \rho(f) = k, \tag{3.8}$$

that is,

$$\max\{\tau(f(z)), \tau(\Delta_c^2 f(z))\} = \rho(f).$$

Meanwhile, we have by (3.4) that

$$f(z + 2c) - z = \frac{z + 2c - a}{1 - H_1(z + 2c)e^{h_1(z+2c)}} + a - z = \frac{(z - a)H_1(z + 2c)e^{h_1(z+2c)} + 2c}{1 - H_1(z + 2c)e^{h_1(z+2c)}}, \tag{3.9}$$

where $(z - a)H_1(z + 2c) + 2cH_1(z + 2c) = (z - a + 2c)H_1(z + 2c) \not\equiv 0$. Thus, $f(z + 2c) - z$ can be seen as an irreducible rational function in $e^{h_1(z+2c)}$. Since $\rho(H_1(z + 2c)) = \rho(H_1) < \rho(e^{h_1}) = \rho(e^{h_1(z+2c)})$, by Lemma 2.8 and Remark 2.9, we have

$$T(r, H_1(z + 2c)) = S(r, e^{h_1(z+2c)}). \tag{3.10}$$

By (3.9), (3.10) and Lemma 2.5, we have

$$T(r, f(z+2c)-z) = T(r, e^{h_1(z+2c)}) + S(r, e^{h_1(z+2c)}).$$

Then, by Lemma 2.6, we have

$$\tau(f(z+2c)) = \lambda(f(z+2c)-z) = \rho(e^{h_1(z+2c)}) = \rho(e^{h_1}) = \rho(f),$$

that is,

$$\max\{\tau(f(z)), \tau(f(z+2c))\} = \rho(f).$$

Suppose that $\tau(f(z+2c)) < \rho(f)$, and we prove $\tau(\Delta_c^2 f(z)) = \rho(f)$ next. Denote

$$g_2(z) = \frac{f(z+2c)-z}{f(z+2c)-a}. \tag{3.11}$$

From (3.11) we get

$$f(z+2c) = \frac{z-a}{1-g_2(z)} + a$$

and

$$f(z) = \frac{z-2c-a}{1-g_2(z-2c)} + a. \tag{3.12}$$

By (3.11), (3.12) and Lemma 2.1, we have $\rho(g_2) = \rho(g_2(z-2c)) = \rho(f(z+2c)) = \rho(f)$. By Lemma 2.3 and the assumption that $\lambda(f(z)-a) < \rho(f)$, we have

$$\lambda(\frac{1}{g_2(z)}) = \lambda(f(z+2c)-a) = \lambda(f(z)-a) < \rho(f) = \rho(g_2)$$

and

$$\lambda(g_2(z)) = \lambda(f(z+2c)-z) = \tau(f(z+2c)) < \rho(f) = \rho(g_2),$$

which means that 0 and ∞ are Borel exceptional values of $g_2(z)$. Then following the steps similar to (3.2)-(3.8), we have $\tau(\Delta_c^2 f(z)) = \lambda(\Delta_c^2 f(z)-z) = \rho(f)$, that is,

$$\max\{\tau(\Delta_c^2 f(z)), \tau(f(z+2c))\} = \rho(f).$$

Secondly, we prove the conclusions for $n \geq 3$. Suppose that $\tau(f(z)) < \rho(f)$, and we prove $\tau(\Delta_c^n f(z)) = \tau(f(z+nc)) = \rho(f)$ next. By (3.1), (3.2) and (3.4), we have

$$\begin{aligned}
\Delta_c^n f(z) &= \sum_{j=0}^{n} (-1)^j c_n^j f(z+(n-j)c) \\
&= \sum_{j=0}^{n} (-1)^j c_n^j [\frac{z+(n-j)c-a}{1-H_1(z+(n-j)c)e^{h_1(z+(n-j)c)}} + a] \\
&= \sum_{j=0}^{n} (-1)^j c_n^j \frac{z+(n-j)c-a}{1-H_1(z+(n-j)c)e^{h_1(z+(n-j)c)}} + a\sum_{j=0}^{n}(-1)^j c_n^j \\
&= \sum_{j=0}^{n} (-1)^j c_n^j \frac{z+(n-j)c-a}{1-H_1(z+(n-j)c)e^{h_1(z+(n-j)c)}} \\
&= \frac{\sum_{j=0}^{n}[(-1)^j c_n^j [z+(n-j)c-a]\prod_{i\neq j}(1-H_1(z+(n-i)c)e^{h_1(z+(n-i)c)})]}{\prod_{j=0}^{n}(1-H_1(z+(n-j)c)e^{h_1(z+(n-j)c)})}
\end{aligned} \tag{3.13}$$

$$= \frac{\sum_{j=1}^{n} P_{n,j}(z)e^{jh_1(z)} + \sum_{j=0}^{n}(-1)^j c_n^j [z+(n-j)c-a]}{\sum_{j=1}^{n+1} P_{n,n+j}(z)e^{jh_1(z)} + 1}$$

$$= \frac{\sum_{j=1}^{n} P_{n,j}(z)e^{jh_1(z)} + c\sum_{j=0}^{n}(-1)^j c_n^j (n-j)}{\sum_{j=1}^{n+1} P_{n,n+j}(z)e^{jh_1(z)} + 1}$$

$$= \frac{\sum_{j=1}^{n} P_{n,j}(z)e^{jh_1(z)}}{\sum_{j=1}^{n+1} P_{n,n+j}(z)e^{jh_1(z)} + 1},$$

where

$$P_{n,1}(z) = -\sum_{j=0}^{n}(-1)^j c_n^j [z+(n-j)c-a]\sum_{i\neq j} e^{h_1(z+(n-i)c)-h_1(z)} H_1(z+(n-i)c),$$

$$P_{n,2}(z) = \sum_{j=0}^{n}(-1)^j c_n^j [z+(n-j)c-a]\sum_{\substack{\mu,\nu\neq j\\ 0\le\mu<\nu\le n}} e^{h_1(z+(n-\mu)c)+h_1(z+(n-\nu)c)-2h_1(z)}$$
$$H_1(z+(n-\mu)c)H_1(z+(n-\nu)c),$$

$$\cdots\cdots$$

$$P_{n,n-1}(z) = (-1)^{n-1}\sum_{j=0}^{n}(-1)^j c_n^j [z+(n-j)c-a]\sum_{\mu\neq j}\prod_{i\neq j,\mu} e^{h_1(z+(n-i)c)-h_1(z)} H_1(z+(n-i)c),$$

$$P_{n,n}(z) = (-1)^n\sum_{j=0}^{n}(-1)^j c_n^j [z+(n-j)c-a]\prod_{i\neq j} e^{h_1(z+(n-i)c)-h_1(z)} H_1(z+(n-i)c),$$

$$P_{n,n+1}(z) = -\sum_{j=0}^{n} e^{h_1(z+(n-j)c)-h_1(z)} H_1(z+(n-j)c),$$

$$P_{n,n+2}(z) = \sum_{0\le\mu<\nu\le n} e^{h_1(z+(n-\mu)c)+h_1(z+(n-\nu)c)-2h_1(z)} H_1(z+(n-\mu)c)H_1(z+(n-\nu)c),$$

$$\cdots\cdots$$

$$P_{n,2n}(z) = (-1)^n\sum_{j=0}^{n}\prod_{i\neq j} e^{h_1(z+(n-i)c)-h_1(z)} H_1(z+(n-i)c),$$

$$P_{n,2n+1}(z) = (-1)^{n+1}\prod_{j=0}^{n} e^{h_1(z+(n-j)c)-h_1(z)} H_1(z+(n-j)c).$$

By (3.13), we have

$$\Delta_c^n f(z) - z = \frac{\sum_{j=1}^{n} P_{n,j}(z)e^{jh_1(z)} - z\sum_{j=1}^{n+1} P_{n,n+j}(z)e^{jh_1(z)} - z}{\sum_{j=1}^{n+1} P_{n,n+j}(z)e^{jh_1(z)} + 1}$$

$$= \frac{-zP_{n,2n+1}(z)e^{(n+1)h_1(z)} + \sum_{j=1}^{n}(P_{n,j}(z) - zP_{n,n+j}(z))e^{jh_1(z)} - z}{\sum_{j=1}^{n+1} P_{n,n+j}(z)e^{jh_1(z)} + 1}. \tag{3.14}$$

By (3.13) and (3.14), We can see $\Delta_c^n f(z)$ and $\Delta_c^n f(z) - z$ as rational functions in $e^{h_1(z)}$. By Lemma 2.8 and Remark 2.9, we have

$$\rho(P_{n,j}) \le \max\{\rho(H_1), \rho(e^{h_1(z+(n-i)c)-h_1(z)})\} < k, j = 1, 2, \cdots, 2n+1, i = 1, 2, \cdots, n.$$

So, the coefficients $P_{n,j}(z), j = 1, 2, \cdots, 2n+1$ are small functions with respect to $e^{h_1(z)}$. Next, we assert $P_{n,1}(z) \not\equiv 0$. We rewrite $P_{n,1}(z)$ as

$$P_{n,1}(z) = -\sum_{j=0}^{n}\sum_{i\neq j}(-1)^i c_n^i [z + (n-i)c - a] e^{h_1(z+(n-j)c)-h_1(z)} H_1(z+(n-j)c).$$

Clearly, we have

$$\rho(e^{h_1(z+(n-j)c)-h_1(z)}) = k-1, \quad j = 0, 1, \cdots, n-1$$

and

$$\sigma(e^{h_1(z+nc)-h_1(z)}) = \frac{1}{\pi}|a_k \cdot c_k^1 \cdot nc| > \frac{1}{\pi}|a_k \cdot c_k^1 \cdot (n-j)c| = \sigma(e^{h_1(z+(n-j)c)-h_1(z)}), j = 1, 2, \cdots, n-1.$$

Obviously, $\sum_{i=0}^{n-1}(-1)^i c_n^i (z + (n-i)c - a) = \sum_{i=0}^{n}(-1)^i c_n^i (z + (n-i)c - a) - (-1)^n (z-a) = (-1)^{n+1}(z-a) \not\equiv 0$. Thus, if $P_{n,1}(z) \equiv 0$, then by Lemma 2.4, we have $\rho(H_1) \geq k - 1 + 1 = k$, which contradicts with $\rho(H_1) < k$. So, the assertion $P_{n,1}(z) \not\equiv 0$ holds. Consequently, by Lemma 2.7 and $P_{n,1}(z) \not\equiv 0$, we have

$$\sum_{j=1}^{n} P_{n,j}(z) e^{jh_1(z)} \not\equiv 0.$$

Then by (3.13) and the obvious fact that $P_{n,2n+1}(z) \not\equiv 0$, we have

$$T(r, \Delta_c^n f(z)) \geq T(r, e^{h_1(z)}) + S(r, e^{h_1(z)}),$$

and consequently

$$T(r, \Delta_c^n f(z) - z) \geq T(r, e^{h_1(z)}) + S(r, e^{h_1(z)}). \tag{3.15}$$

By (3.14), (3.15) and Lemma 2.6, we have

$$\tau(\Delta_c^n f(z)) = \lambda(\Delta_c^n f(z) - z) = \rho(e^{h_1}) = \rho(f) = k, \tag{3.16}$$

that is,

$$\max\{\tau(\Delta_c^n f(z)), \tau(f(z))\} = \rho(f).$$

Meanwhile, we have by (3.4) that

$$\begin{aligned} f(z+nc) - z &= \frac{z + nc - a}{1 - H_1(z+nc)e^{h_1(z+nc)}} + a - z \\ &= \frac{(z-a)H_1(z+nc)e^{h_1(z+nc)} + nc}{1 - H_1(z+nc)e^{h_1(z+nc)}}, \end{aligned} \tag{3.17}$$

where $(z-a)H_1(z+nc) + ncH_1(z+nc) = (z - a - nc)H_1(z+nc) \not\equiv 0$. Thus, $f(z+nc) - z$ can be seen as an irreducible rational function in $e^{h_1(z+nc)}$. Since $\rho(H_1(z+nc)) = \rho(H_1) < \rho(e^{h_1(z+nc)}) = \rho(e^{h_1})$, by Lemma 2.8 and Remark 2.9, we have

$$T(r, H_1(z+nc)) = S(r, e^{h_1(z+nc)}). \tag{3.18}$$

By (3.17), (3.18) and Lemma 2.5, we have

$$T(r, f(z+nc) - z) = T(r, e^{h_1(z+nc)}) + S(r, e^{h_1(z+nc)}).$$

Then, by Lemma 2.6, we have get

$$\tau(f(z+nc)) = \lambda(f(z+nc) - z) = \rho(e^{h_1(z+nc)}) = \rho(e^{h_1}) = \rho(f),$$

that is,

$$\max\{\tau(f(z)), \tau(f(z+nc))\} = \rho(f).$$

Suppose that $\tau(f(z+nc)) = \lambda(f(z+nc)-z) < \rho(f)$, and we prove $\tau(\Delta_c^n f(z)) = \rho(f)$ next. Denote

$$g_3(z) = \frac{f(z+nc)-z}{f(z+nc)-a}, \tag{3.19}$$

then we have

$$f(z) = \frac{z-nc-a}{1-g_3(z-nc)} + a. \tag{3.20}$$

By (3.19), (3.20) and Lemma 2.1, we have $\rho(g_3) = \rho(g_3(z+nc)) = \rho(f(z+nc)) = \rho(f)$. By Lemma 2.3 and the assumption that $\lambda(f(z)-a) < \rho(f)$, we have

$$\lambda(\frac{1}{g_3(z)}) = \lambda(f(z+nc)-a) = \lambda(f(z)-a) < \rho(f) = \rho(g_3)$$

and

$$\lambda(g_3(z)) = \lambda(f(z+nc)-z) = \tau(f(z+nc)) < \rho(f) = \rho(g_3),$$

which means that 0 and ∞ are Borel exceptional values of $g_3(z)$. Then following the steps similar to (3.2)-(3.4) and (3.13)-(3.16), we have $\tau(\Delta_c^n f(z)) = \lambda(\Delta_c^n f(z)-z) = \rho(f)$, that is,

$$\max\{\tau(\Delta_c^n f(z)), \tau(f(z+nc))\} = \rho(f).$$

Therefore, the proof of Theorem 1.1 is complete.

Proof of Theorem 1.2 Suppose that $\lambda(f(z)-p(z)) < \rho(f)$, and we prove $\lambda(\Delta_c f(z)-p(z)) = \lambda(f(z+c)-p(z)) = \rho(f)$ next. Denote

$$g_4(z) = \frac{f(z)-p(z)}{f(z)-a}. \tag{3.21}$$

Obviously, $\rho(g_4) = \rho(f)$. Then we have

$$\lambda(\frac{1}{g_4(z)}) = \lambda(f(z)-a) < \rho(f) = \rho(g_4)$$

and

$$\lambda(g_4(z)) = \lambda(f(z)-p(z)) < \rho(f) = \rho(g_4),$$

which means that 0 and ∞ are Borel exceptional values of $g_4(z)$. By Hadamard's factorization theory, $g_4(z)$ can be written as

$$g_4(z) = H_2(z)e^{h_2(z)}, \tag{3.22}$$

where $H_2(z)(\not\equiv 0)$ is a meromorphic function such that $\rho(H_2) < \rho(g_4) = \rho(f)$ and

$$h_2(z) = b_l z^l + b_{l-1}z^{l-1} + \cdots + b_0,$$

is a polynomial such that $\rho(f) = \rho(g_4) = \deg h_2(z) = l$, where $l \in \mathbf{N}_+$ and $b_i \in \mathbb{C}, i = 0, 1, \cdots, l, b_l \neq 0$. Substituting (3.22) into (3.21), we obtain

$$f(z) = \frac{p(z)-a}{1-g_4(z)} + a = \frac{p(z)-a}{1-H_2(z)e^{h_2(z)}} + a. \tag{3.23}$$

We get from (3.23) that

$$\begin{aligned} &\Delta_c f(z) = f(z+c) - f(z) \\ = &\frac{p(z+c)-a}{1-g_4(z+c)} - \frac{p(z)-a}{1-g_4(z)} \end{aligned}$$

$$\begin{aligned} &= \frac{p(z+c)-a}{1-H_2(z+c)e^{h_2(z+c)}} - \frac{p(z)-a}{1-H_2(z)e^{h_2(z)}} \\ &= \frac{[(a-p(z+c))H_2(z)+(p(z)-a)H_2(z+c)e^{h_2(z+c)-h_2(z)}]e^{h_2(z)}+p(z+c)-p(z)}{[H_2(z+c)H_2(z)e^{h_2(z+c)-h_2(z)}]e^{2h_2(z)}+[-H_2(z+c)e^{h_2(z+c)-h_2(z)}-H_2(z)]e^{h_2(z)}+1} \\ &= \frac{Q_1(z)e^{h_2(z)}+p(z+c)-p(z)}{Q_3(z)e^{2h_2(z)}+Q_2(z)e^{h_2(z)}+1}, \end{aligned} \tag{3.24}$$

where

$$\begin{aligned} Q_1(z) &= (a-p(z+c))H_2(z)+(p(z)-a)H_2(z+c)e^{h_2(z+c)-h_2(z)}, \\ Q_2(z) &= -H_2(z+c)e^{h_2(z+c)-h_2(z)}-H_2(z), \\ Q_3(z) &= H_2(z+c)H_2(z)e^{h_2(z+c)-h_2(z)}, \\ p(z+c)-p(z) &= p_m(z+c)^m+p_{m-1}(z+c)^{m-1}+p_{m-2}(z+c)^{m-2}+\cdots+p_0 \\ &\quad -(p_m z^m+p_{m-1}z^{m-1}+p_{m-2}z^{m-2}+\cdots+p_0) \\ &= (p_m\cdot c_m^1\cdot c)z^{m-1}+(p_m\cdot c_m^2\cdot c^2+p_{m-1}\cdot c_{m-1}^1\cdot c)z^{m-2}+\cdots . \end{aligned}$$

By (3.24), we have

$$\begin{aligned} \Delta_c f(z)-p(z) &= \frac{-p(z)Q_3(z)e^{2h_2(z)}+(Q_1(z)-p(z)Q_2(z))e^{h_2(z)}+p(z+c)-2p(z)}{Q_3(z)e^{2h_2(z)}+Q_2(z)e^{h_2(z)}+1} \\ &= \frac{-p(z)Q_3(z)e^{2h_2(z)}+Q_4(z)e^{h_2(z)}+p(z+c)-2p(z)}{Q_3(z)e^{2h_2(z)}+Q_2(z)e^{h_2(z)}+1}, \end{aligned} \tag{3.25}$$

where

$$\begin{aligned} Q_4(z) &= Q_1(z)-p(z)Q_2(z) \\ &= (a-p(z+c)+p(z))H_2(z)+(2p(z)-a)H_2(z+c)e^{h_2(z+c)-h_2(z)}, \\ p(z+c)-2p(z) &= p_m(z+c)^m+p_{m-1}(z+c)^{m-1}+p_{m-2}(z+c)^{m-2}+\cdots+p_0 \\ &\quad -2(p_m z^m+p_{m-1}z^{m-1}+p_{m-2}z^{m-2}+\cdots+p_0) \\ &= -p_m z^m+(p_m\cdot c_m^1\cdot c-p_{m-1})z^{m-1}+(p_m\cdot c_m^2\cdot c^2+p_{m-1}\cdot c_{m-1}^1\cdot c \\ &\quad -p_{m-2})z^{m-2}+\cdots . \end{aligned}$$

By (3.24) and (3.25), we can see $\Delta_c f(z)$ and $\Delta_c f(z)-p(z)$ as rational functions in $e^{h_2(z)}$. Since $\rho(H_2)<\rho(e^{h_2})=l$, by Lemma 2.8 and Remark 2.9, we get that the coefficients $Q_j(z)$, $j=1,2,3,4$ are small functions with respect to $e^{h_2(z)}$, that is, $T(r,Q_j)=S(r,e^{h_2(z)})$, $j=1,2,3,4$. Obviously, $p(z+c)-p(z)\not\equiv 0$, $Q_j(z)\not\equiv 0$, $j=1,2,3,4$. By Lemma 2.7, we have $Q_1(z)e^{h_2(z)}+p(z+c)-p(z)\not\equiv 0$. Then by (3.24) and (3.25), we have

$$T(r,\Delta_c f(z))\geq T(r,e^{h_2(z)})+S(r,e^{h_2(z)}),$$

and consequently

$$T(r,\Delta_c f(z)-p(z))\geq T(r,e^{h_2(z)})+S(r,e^{h_2(z)}). \tag{3.26}$$

By (3.25), (3.26) and Lemma 2.6, we have

$$\lambda(\Delta_c f(z)-p(z))=\rho(e^{h_2})=\rho(f), \tag{3.27}$$

that is,

$$\max\{\lambda(f(z)-p(z)),\lambda(\Delta_c f(z)-p(z))\}=\rho(f).$$

Meanwhile, we have by (3.23) that

$$\begin{aligned} f(z+c)-p(z) &= \frac{p(z+c)-a}{1-H_2(z+c)e^{h_2(z+c)}}+a-p(z) \\ &= \frac{(p(z)-a)H_2(z+c)e^{h_2(z+c)}+p(z+c)-p(z)}{1-H_2(z+c)e^{h_2(z+c)}}, \end{aligned} \tag{3.28}$$

where $(p(z)-a)H_2(z+c)+(p(z+c)-p(z))H_2(z+c)=(p(z+c)-a)H_2(z+c)\not\equiv 0$. Thus, $f(z+c)-p(z)$ can be seen as an irreducible rational function in $e^{h_2(z+c)}$. Since $\rho(H_2(z+c))=\rho(H_2)<\rho(e^{h_2})=\rho(e^{h_2(z+c)})$, by Lemma 2.8 and Remark 2.9, we have

$$T(r, H_2(z+c)) = S(r, e^{h_2(z+c)}). \tag{3.29}$$

By (3.28), (3.29) and Lemma 2.5, we have

$$T(r, f(z+c)-p(z)) = T(r, e^{h_2(z+c)}) + S(r, e^{h_2(z+c)}).$$

Then, by Lemma 2.6, we have

$$\lambda(f(z+c)-p(z)) = \rho(e^{h_2(z+c)}) = \rho(e^{h_2}) = \rho(f),$$

that is,

$$\max\{\lambda(f(z)-p(z)), \lambda(f(z+c)-p(z))\} = \rho(f).$$

Suppose that $\lambda(f(z+c)-p(z))<\rho(f)$, and we prove $\lambda(\Delta_c f(z)-p(z))=\rho(f)$ next. Denote

$$g_5(z) = \frac{f(z+c)-p(z)}{f(z+c)-a}, \tag{3.30}$$

then we have

$$f(z) = \frac{p(z-c)-a}{1-g_5(z-c)} + a. \tag{3.31}$$

By (3.30), (3.31) and Lemma 2.1, we have $\rho(g_5)=\rho(g_5(z-c))=\rho(f(z+c))=\rho(f)$. By Lemma 2.3 and the assumption that $\lambda(f(z)-a)<\rho(f)$, we have

$$\lambda(\frac{1}{g_5(z)}) = \lambda(f(z+c)-a) = \lambda(f(z)-a) < \rho(f) = \rho(g_5)$$

and

$$\lambda(g_5(z)) = \lambda(f(z+c)-p(z)) < \rho(f) = \rho(g_5),$$

which means that 0 and ∞ are Borel exceptional values of $g_5(z)$. Then following the steps similar to (3.22)-(3.27), we have $\lambda(\Delta_c f(z)-p(z))=\rho(f)$, that is,

$$\max\{\lambda(\Delta_c f(z)-p(z)), \lambda(f(z+c)-p(z))\} = \rho(f).$$

Therefore, the proof of Theorem 1.2 is complete.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors drafted the manuscript, read and approved the final manuscript.

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