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Research Article

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On minimum algebraic connectivity of graphs whose complements are bicyclic

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Abstract: The second smallest eigenvalue of the Laplacian matrix of a graph (network) is called its algebraic connectivity which is used to diagnose Alzheimer's disease, distinguish the group differences, measure the robustness, construct multiplex model, synchronize the stability, analyze the diffusion processes and find the connectivity of the graphs (networks). A connected graph containing two or three cycles is called a bicyclic graph if its number of edges is equal to its number of vertices plus one. In this paper, firstly the unique graph with a minimum algebraic connectivity is characterized in the class of connected graphs whose complements are bicyclic with exactly three cycles. Then, we find the unique graph of minimum algebraic connectivity in the class of connected graphs $\Omega_n^c = \Omega_{1,n}^c \cup \Omega_{2,n}^c$, where $\Omega_{1,n}^c$ and $\Omega_{2,n}^c$ are classes of the connected graphs in which the complement of each graph of order n is a bicyclic graph with exactly two and three cycles, respectively.

Keywords: Laplacian matrix, eigenvalues, algebraic connectivity

MSC: 15A18, 05C50, 05C40, 05D05

1 Introduction

Let $G = (V(G), E(G))$ be a graph having $V(G) = \{v_i : 1 \le i \le n\}$ and $E(G)$ as the sets of vertices and edges. The graph G^c is complement of G with same vertex-set and edge-set $E(G^c) = \{uv : u, v \in V(G), uv \notin E(G)\}$. The number of first neighbors of $v \in V(G)$ is called its degree denoted by $d(v)$. The adjacency matrix (A-matrix) of G is $A(G) = [a_{i,j}]_{n\times n}$ such that $a_{i,j} = 1$ if v_i is adjacent to v_j and $a_{i,j} = 0$ otherwise. By $D(G) = [a_{i,j}]_{n\times n}$, we denote the degree matrix such that $a_{i,i} = d(v_i)$ and zero otherwise. The Laplacian matrix (L-matrix) of the graph G is

$$L(G) = D(G) - A(G).$$

For $1 \le i \le n$, eigenvalues $\mu_i = \mu_i(G)$ and eigenvectors $Z_i = Z_i(G)$ of L-matrix ($L(G)$) are the L-eigenvalues and the L-eigenvectors of G. For n-dimensional column-vectors $Z_i \neq 0$, we have $L(G)Z_i = \mu_i Z_i$. Since $L(G)$ is real and symmetric therefore we have $\mu_1 \ge \mu_2 \ge \ldots \ge \mu_{n-1} \ge \mu_n$, where $\mu_n = 0$ is a minimum L-eigenvalue and $\mu_{n-1}(G) = a(G)$ is algebraic connectivity of G that remains positive if and only if G is connected. Moreover, eigenvectors corresponding to $a(G)$ are called Fiedler vectors. For further study, we refer to [1-7].

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The algebraic connectivity plays an important role in studies of communication and control theory to increase efficiency in air transportation system [8], measure connectivity, convergence speed & synchronization ability of the networks [9-11] generate or absorb the bipartition among the links [12] and construct multiplex model for the inter connected networks [13]. It is also used in brain networks to study the group differences and complex changes in Alzheimer's disease, see [14].

A connected graph is called k-cyclic if $m = n-1+k$, where n denotes the number of vertices, m the number of edges and k non-negative integers. In particular, if $k = 0, 1, 2$ or 3, then G is called a tree, unicyclic, bicyclic or tricyclic graph, respectively. Let $H^{'}(n, 2)$ be a bicyclic graph with two cycles which is obtained from the star $K_{1,n-1}$ by the addition of two edges such that each edge joins two different pendant vertices. Similarly, $H(n, 2)$ is a bicyclic graph with three cycles obtained from the star $K_{1,n-1}$ by the addition of two edges in the pendant vertices such that both edges have one common end point. Let $\Omega_{1,n}$ and $\Omega_{2,n}$ be two classes of the bicyclic graphs with n vertices having exactly two and three cycles other than $H^{'}(n, 2)$ and $H(n, 2)$ respectively. Moreover, assume that $\Omega^c_{1,n}$ and $\Omega^c_{2,n}$ be the classes of the graphs whose complements are bicyclic with exactly two and three cycles respectively i.e $\Omega^c_{1_n} = \{G^c : |G^c| = n \text{ and } G \in \Omega_{1,n}\}$ and $\Omega^c_{2,n} = \{G^c : |G^c| = n \text{ and } G \in \Omega_{2,n}\}$. The condition to exclude $H^{'}(n, 2)$ and $H(n, 2)$ from $\Omega_{1,n}$ and $\Omega_{2,n}$ respectively ensures that $\Omega^c_{1,n}$ and $\Omega^c_{2,n}$ are families of the connected graphs.

Many authors studied the algebraic connectivity for different families of graphs such as connected graphs with certain girth, lollipop graphs and caterpillar unicyclic graphs, see [15-17]. Moreover, the operation of complement in graphs has important role, especially when structures of the simple graphs become more complex than their complements. Recently, Jiang et al. [18], Li et al. [19] and Javaid et al. [20, 21] found the graphs with minimum algebraic connectivity among all the connected graphs whose complements are trees, unicyclic, and bicyclic with exactly two cycles. In this paper, firstly we characterize the unique graph with minimum algebraic connectivity in the class of connected graphs whose complements are bicyclic with three cycles. Then, we find the unique graph with minimum algebraic connectivity in the complete class of connected graphs whose complements are bicyclic with two or three cycles.

The rest of the paper is managed as; In Section 2, some basic definitions and results are given. Section 3 and Section 4 cover the main results. Section 5 includes the conclusion and some new directions of the problem.

2 Preliminaries

For any vector $Z \in R^n$, define a one-one map $\mu : V(G) \to Z$ such that $\mu(u) = Z_u$, where Z_u is entry of Z corresponding to $u \in V(G)$. Then, for $Z \neq \mathbf{0}$, we have

$$Z^T L(G)Z = \sum_{uv\in E(G)} (\mu(u) - \mu(v))^2 = \sum_{uv\in E(G)} (Z_u - Z_v)^2. \tag{1}$$

Moreover, if λ is a L-eigenvalue of G corresponding to $Z \neq \mathbf{0}$ then Laplacian eigenvalue equation (LE-equation) is

$$(d_G(v) - \lambda)Z_v = \sum_{u\in N_G(v)} Z_u \quad \text{for each vertex} \quad v \in V(G), \tag{2}$$

where $N_G(v)$ is set of neighbors of $v \in V(G)$.

Assume that $Z \in R^n$ is a unit vector and perpendicular to all-ones vector, then by Courant-Fisher theorem [3], we have

$$a(G) \leq Z^T L(G)Z, \tag{3}$$

where $a(G)$ achieves the upper bound if Z is a Fiedler vector. If J is all-ones matrix, I is identity matrix and $L(G^c)$ is L-matrix of the complement of G, then for any vector $Z \in R^n$

$$Z^T L(G^c)Z = Z^T(nI - J)Z - Z^T L(G)Z. \tag{4}$$

Suppose that C_4, C_5 and C_6 are cycles of length 4, 5 and 6 respectively. Now, some graphs are defined which are used in the main results.

Let H_1 and H_2 be two bicyclic graphs with exactly three cycles which are obtained by joining any single non adjacent pair of vertices with an edge in C_4 and C_5, respectively. The bicyclic graphs with exactly three cycles H_3 is obtained from C_6 by joining a pair of vertices with an edge such that H_3 consists on an outer cycle of length 6 and two inner cycles both of lengths 4. If we insert a vertex in an edge which is incident on two vertices of degree 3 in H_1, then we obtain a bicyclic graph with three cycles H_4 such that its all the cycles (one outer and two inner) are of lengths 4.

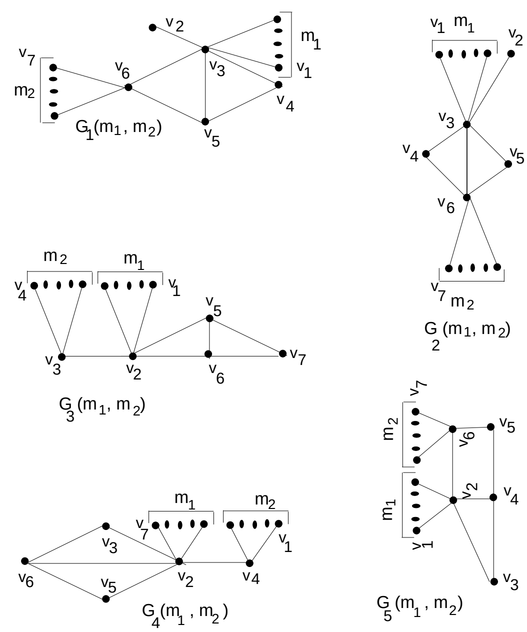

Figure 1: Bicycle graphs with three cycles $G_1(m_1, m_2)$, $G_2(m_1, m_2)$, $G_3(m_1, m_2)$, $G_4(m_1, m_2)$. and $G_5(m_1, m_2)$.

Let $G_1(m_1, m_2)$ be obtained by attaching m_1 and m_2 pendant vertices with two adjacent vertices of degree 3 and 2 in H_1 respectively and $G_2(m_1, m_2)$ is obtained by attaching m_1 and m_2 pendant vertices with both the vertices of degree 3 in H_1. If we identify H_1 by a vertex of degree 2 with an edge of length one and attach m_2 and m_1 pendant vertices with a pendant vertex and a vertex adjacent to it respectively, then we obtain $G_3(m_1, m_2)$. Similarly, $G_4(m_1, m_2)$ is obtained when we identify H_1 by a vertex of degree 3 with an edge of length one and attach m_2 and m_1 pendant vertices with a pendant vertex and a vertex adjacent to it, respectively (see Figure 1).

By attaching m_1 and m_2 pendant vertices with two adjacent vertices of degrees 2 in H_2, $G_6(m_1, m_2)$ is obtained and if we attach m_1 and m_2 pendant vertices with vertices of degree 3 in H_2, we obtain $G_8(m_1, m_2)$. The graph $G_9(m_1, m_2)$ is obtained by joining m_1 and m_2 pendant vertices with vertices of degree 3 in H_3. Finally, we obtain $G_7(m_1, m_2)$ from H_4 by attaching m_1 and m_2 pendant vertices with two adjacent vertices of degree 2 and 3 in H_4 such both are on the outer cycle of H_4 (see Figure 2).

Moreover, $G_1^{'}(m_1, m_2)$ is a bicyclic graph with exactly two cycles which is obtained by attaching m_2 pendant vertices with a vertex of degree 2 of the graph $H^{'}(m_1 + 5, 2)$ (see Figure 3). Now, we state some results which are frequently used in main results.

Lemma 2.1. [3] Let G be a simple graph. Then $a(G) \leq \delta(G)$, where $\delta(G) = min\{d_G(v), v \in V(G)\}$.

Lemma 2.2. [18] If Z_i for $1 \leq i \leq n$ is a non-increasing sequence, then, for any $1 \leq i, j \leq n$, $(Z_i - Z_j)^2 \leq max\{(Z_i - Z_1)^2, (Z_i - Z_n)^2\} \leq (Z_1 - Z_n)^2$.

Theorem 2.3. [21] Let n, m_1 and m_2 be any positive integers such that $m_1 \geq m_2 \geq 1$, $n \geq 11$ and $n = m_1+m_2+5$, then for any bicyclic graph with exactly two cycles $G \in \Omega_{1,n}$,

$$a(G_1^{'}(n-6, 1)^c) \leq a(G_1^{'}(m_1, m_2)^c) \leq a(G^c),$$

where equalities hold if and only if $G_1^{'}(n-6, 1)^c \cong G_1^{'}(m_1, m_2) \cong G$.

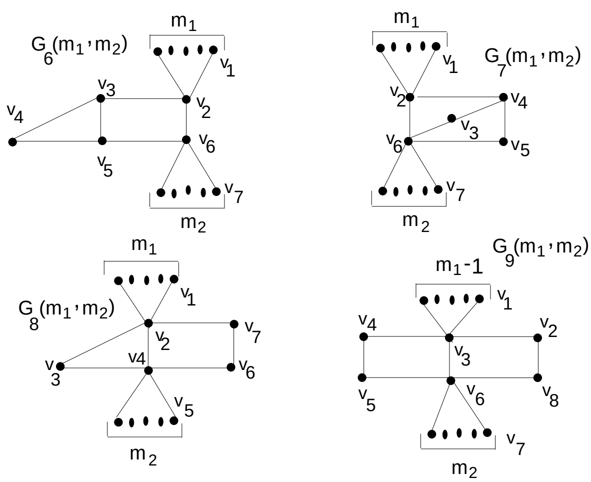

Figure 2: Bicycle graphs with three cycles $G_6(m_1, m_2)$, $G_7(m_1, m_2)$, $G_8(m_1, m_2)$, and $G_9(m_1, m_2)$.

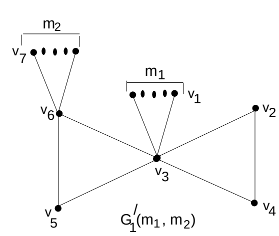

Figure 3: Bicycle graphs with two cycles $G_1^{'}(m_1, m_2)$.

3 Computational results of minimum algebraic connectivity

The computational results of the algebraic connectivity are presented in this section.

Lemma 3.1. Let m_1, m_2 and n be positive integers, such that $m_1 \geq m_2 \geq 1$, $n \geq 11$ and $m_1 + m_2 + 5 = n$. Then $a(G_1(n-6, 1)^c) < \ldots < a(G_1(m_1+1, m_2-1)^c) < a(G_1(m_1, m_2)^c)$.

Proof. Let $G_1(m_1, m_2)$ be a graph with labeled vertices as shown in Figure 1 and Z be a unit Fiedler vector of $G_1(m_1, m_2)^c$. By Lemma 2.1 and LE-equation (3), $a(G_1(m_1, m_2)^c) \neq d_{G_1(m_1,m_2)^c}(v) + 1$ for any

$v \in V(G_1(m_1, m_2)^c)$ and all the pendant vertices attached with same vertex have same values given by Z. Therefore, $Z_i := Z_{v_i}$ for $1 \le i \le 7$ and we have the following equations for $a = a(G_1(m_1, m_2)^c)$,

$$(m_1 + m_2 + 3 - a)Z_1 = (m_1 - 1)Z_1 + Z_2 + Z_4 + Z_5 + Z_6 + m_2 Z_7,$$

$$(m_1 + m_2 + 3 - a)Z_2 = m_1 Z_1 + Z_4 + Z_5 + Z_6 + m_2 Z_7,$$

$$(m_2 - a)Z_3 = m_2 Z_7,$$

$$(m_1 + m_2 + 2 - a)Z_4 = m_1 Z_1 + Z_2 + Z_6 + m_2 Z_7, (A_1)$$

$$(m_1 + m_2 + 2 - a)Z_5 = m_1 Z_1 + Z_2 + Z_4 + m_2 Z_7,$$

$$(m_1 + 2 - a)Z_6 = m_1 Z_1 + Z_2 + Z_4,$$

$$(m_1 + m_2 + 3 - a)Z_7 = m_1 Z_1 + Z_2 + Z_3 + Z_4 + Z_5 + (m_2 - 1)Z_7.$$

Transform (A_1) into a matrix equation $(M - aI)Z = 0$, where $Z = (Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7)^T$ and

$$M = \begin{bmatrix} m_2+4 & -1 & 0 & -1 & -1 & -1 & -m_2 \\ -m_1 & m_1+m_2+3 & 0 & -1 & -1 & -1 & -m_2 \\ 0 & 0 & m_2 & 0 & 0 & 0 & -m_2 \\ -m_1 & -1 & 0 & m_1+m_2+2 & 0 & -1 & -m_2 \\ -m_1 & -1 & 0 & -1 & m_1+m_2+2 & 0 & -m_2 \\ -m_1 & -1 & 0 & -1 & 0 & m_1+2 & 0 \\ -m_1 & -1 & -1 & -1 & -1 & 0 & m_1+4 \end{bmatrix}$$

Let $g(\lambda; m_1, m_2) := det(\lambda I - M)$, then we have

$g(\lambda; m_1, m_2) = \lambda(-4 - m_1 - m_2 + \lambda)(-3 - m_1 - m_2 + \lambda)(15m_2 + 23m_1m_2 + 9m_1^2m_2 + m_1^3m_2 + 8m_2^2 + 10m_1m_2^2 + 2m_1^2m_2^2 + m_2^3 + m_1m_2^3 - 16\lambda - 24m_1\lambda - 9m_1^2\lambda - m_1^3\lambda - 32m_2\lambda - 28m_1m_2\lambda - 5m_1^2m_2\lambda - 11m_2^2\lambda - 5m_1m_2^2\lambda - m_2^3\lambda + 24\lambda^2 + 18m_1\lambda^2 + 3m_1^2\lambda^2 + 19m_2\lambda^2 + 7m_1m_2\lambda^2 + 3m_2^2\lambda^2 - 9\lambda^3 - 3m_1\lambda^3 - 3m_2\lambda^3 + \lambda^4)$.

Since $g(0; m_1, m_2) = 0 = g(a; m_1, m_2)$, thus a is the second smallest root of $g(\lambda; m_1, m_2)$. Observe that

$$\begin{aligned} g(\lambda; m_1+1, m_2-1) - g(\lambda; m_1, m_2) &= -\lambda(2 + m_1 - m_2)(3 + m_1 + m_2 - \lambda)^2(4 + m_1 + m_2 - \lambda)(5 + m_1 + m_2 - \lambda) \\ &= -\lambda(2 + m_1 - m_2)(3 + m_1 + m_2 - \lambda)^2(4 + m_1 + m_2 - \lambda)(n - \lambda). \end{aligned}$$

By Lemma 2.1, $a = a(G_1(m_1, m_2)^c) < n$. Then by $m_1 \ge m_2$, $a > 0$ and $g(a; m_1, m_2) = 0$, we have $g_1(a; m_1+1, m_2-1) = -a(2 + m_1 - m_2)(3 + m_1 + m_2 - a)^2(4 + m_1 + m_2 - a)(n - a) < 0$. This shows that $a(G_1(m_1+1, m_2-1)^c) < a(G_1(m_1, m_2)^c)$. Similarly, we can prove $a(G_1(m_1+2, m_2-2)^c) < a(G_1(m_1+1, m_2-1)^c)$. Consequently, $a(G_1(n-6, 1)^c) < \dots < a(G_1(m_1+1, m_2-1)^c) < a(G_1(m_1, m_2)^c)$.

Lemma 3.2. Let m_1, m_2 and n be positive integers, such that $m_1 \ge m_2 \ge 1$, $n \ge 11$, $m_1 + m_2 + 5 = n$ and $2 \le i \le 9$. Then $a(G_1(m_1, m_2)^c) \le a(G_i(m_1, m_2)^c)$, where equality holds if $G_1(m_1, m_2) \cong G_i(m_1, m_2)$.

Proof. Using Lemma 2.1 and (3) (as in Lemma 3.1), we find the following polynomials of the graphs $G_i(m_1, m_2)^c$ for $2 \le i \le 9$.

$g_2(a; m_1+1, m_2-1) = -a(n-a)(4 + m_1 + m_2 - a)(14 + 20m_1 + 8m_1^2 + m_1^3 + 10m_2 + 6m_1m_2 + m_1^2m_2 - 2m_2^2 - m_1m_2^2 - m_2^3 - 15a - 11m_1a - 2m_1^2a - m_2a + 2m_2^2a + 3a^2 + m_1a^2 - m_2a^2)$,

$g_3(a; m_1+1, m_2-1) = (m_1 + m_2 - a + 1)(-23 - 25m_1 - 9m_1^2 - m_1^3 - 11m_2 - 6m_1m_2 - m_1^2m_2 + 3m_2^2 + m_1m_2^2 + m_2^3 - 50a - 97m_1a - 57m_1^2a - 13m_1^3a - m_1^4a - 69m_2a - 60m_1m_2a - 19m_1^2m_2a - 2m_1^3m_2a - 3m_2^2a + m_1m_2^2a + 7m_2^3a + 2m_1m_2^3a + m_2^4a + 86a^2 + 88m_1a^2 + 29m_1^2a^2 + 3m_1^3a^2 + 32m_2a^2 + 18m_1m_2a^2 + 3m_1^2m_2a^2 - 11m_2^2a^2 -$

$3m_1m_2^2a^2 - 3m_2^3a^2 - 30a^3 - 19m_1a^3 - 3m_1^2a^3 + m_2a^3 + 3m_2^2a^3 + 3a^4 + m_1a^4 - m_2a^4)$,

$g_4(a; m_1+1, m_2-1) = -a(4+m_1-m_2)(1+m_1+m_2-a)(3+m_1+m_2-a)(4+m_1+m_2-a)(5+m_1+m_2-a)$,

$g_5(a, m_1+1, m_2-1) = -a(-4-m_1-m_2+a)(-50-86m_1-50m_1^2-12m_1^3-m_1^4-54m_2-53m_1m_2-18m_1^2m_2-2m_1^3m_2-3m_2^2+6m_2^3+2m_1m_2^3+m_2^4+71a+77m_1a+27m_1^2a+3m_1^3a+30m_2a+18m_1m_2a+3m_1^2m_2a-9m_2^2a-3m_1m_2^2a-3m_2^3a-27a^2-18m_1a^2-3m_1^2a^2+3m_2^2a^2+3a^3+m_1a^3-m_2a^3)$,

$g_6(a, m_1+1, m_2-1) = (16+56m_1+53m_1^2+18m_1^3+2m_1^4+56m_2+113m_1m_2+67m_1^2m_2+15m_1^3m_2+m_1^4m_2+53m_2^2+67m_1m_2^2+26m_1^2m_2^2+3m_1^3m_2^2+18m_2^3+15m_1m_2^3+3m_1^2m_2^3+2m_2^4+m_1m_2^4-171a-353m_1a-260m_1^2a-90m_1^3a-15m_1^4a-m_1^5a-271m_2a-378m_1m_2a-186m_1^2m_2a-39m_1^3m_2a-3m_1^4m_2a-105m_2^2a-88m_1m_2^2a-24m_1^2m_2^2a-2m_1^3m_2^2a+8m_2^3a+9m_1m_2^3a+2m_1^2m_2^3a+9m_2^4a+3m_1m_2^4a+m_2^5a+265a^2+394m_1a^2+210m_1^2a^2+48m_1^3a^2+4m_1^4a^2+239m_2a^2+231m_1m_2a^2+75m_1^2m_2a^2+8m_1^3m_2a^2+14m_2^2a^2+3m_1m_2^2a^2-24m_2^3a^2-8m_1m_2^3a^2-4m_2^4a^2-152a^3-158m_1a^3-54m_1^2a^3-6m_1^3a^3-60m_2a^3-37m_1m_2a^3-6m_1^2m_2a^3+18m_2^2a^3+6m_1m_2^2a^3+6m_2^3a^3+36a^4+24m_1a^4+4m_1^2a^4-4m_2^2a^4-3a^5-m_1a^5+m_2a^5)$,

$g_7(a, m_1+1, m_2-1) = -a(2+m_1-m_2)(-5-m_1-m_2+a)(-4-m_1-m_2+a)(-3-m_1-m_2+a)(-1-m_1-m_2+a)$,

$g_8(a, m_1+1, m_2-1) = -a(1+m_1-m_2)(2+m_1+m_2-a)(3+m_1+m_2-a)(4+m_1+m_2-a)(5+m_1+m_2-a)$,

$g_9(a, m_1+1, m_2-1) = (2+m_1+m_2-a)(4+m_1+m_2-a)(-8m_1-6m_1^2-m_1^3+8m_2-m_1^2m_2+6m_2^2+m_1m_2^2+m_2^3+4a+32m_1a+32m_1^2a+10m_1^3a+m_1^4a-30m_2a+10m_1^2m_2a+2m_1^3m_2a-32m_2^2a-10m_1m_2^2a-10m_2^3a-2m_1m_2^3a-m_2^4a-a^2-31m_1a^2-20m_1^2a^2-3m_1^3a^2+31m_2a^2-3m_1^2m_2a^2+20m_2^2a^2+3m_1m_2^2a^2+3m_2^3a^2+10m_1a^3+3m_1^2a^3-10m_2a^3-3m_2^2a^3-m_1a^4+m_2a^4)$.

Now, consider
$g_1(a, m_1+1, m_2-1) - g_2(a, m_1+1, m_2-1) = a(1+a-2m_2)(-5+a-m_1-m_2)(-4+a-m_1-m_2)^2 > 0$.
Thus,

$$a(G_1(m_1+1, m_2-1)^c) < a(G_2(m_1+1, m_2-1)^c). \tag{5}$$

$g_1(a, m_1+1, m_2-1) - g_3(a, m_1+1, m_2-1) = 23-333a+266a^2-50a^3-3a^4+a^5+48m_1-460am_1+262a^2m_1-36a^3m_1+34m_1^2-222am_1^2+82a^2m_1^2-6a^3m_1^2+10m_1^3-44am_1^3+8a^2m_1^3+m_1^4-3am_1^4+34m_2-114am_2-56a^2m_2+54a^3m_2-8a^4m_2+42m_1m_2-112am_1m_2-18a^2m_1m_2+12a^3m_1m_2+16m_1^2m_2-38am_1^2m_2+2m_1^3m_2-4am_1^3m_2+8m_2^2+110am_2^2-100a^2m_2^2+18a^3m_2^2+2m_1m_2^2+56am_1m_2^2-24a^2m_1m_2^2+6am_1^2m_2^2-4m_2^3+50am_2^3-16a^2m_2^3-2m_1m_2^3+12am_1m_2^3-m_2^4+5am_2^4 < 0$.

Thus,

$$a(G_1(m_1+1, m_2-1)^c) < a(G_3(m_1+1, m_2-1)^c). \tag{6}$$

$g_1(a, m_1+1, m_2-1) - g_4(a, m_1+1, m_2-1) = 2a(1+a-2m_2)(-5+a-m_1-m_2)(-4+a-m_1-m_2)(-3+a-m_1-m_2) > 0$.

Thus,

$$a(G_1(m_1+1, m_2-1)^c) < a(G_4(m_1+1, m_2-1)^c). \tag{7}$$

$g_1(a, m_1+1, m_2-1) - g_5(a, m_1+1, m_2-1) = a(-4+a-m_1-m_2)(40-7a-5a^2+a^3+37m_1-6am_1-a^2m_1+11m_1^2-am_1^2+m_1^3-21m_2+25am_2-5a^2m_2-9m_1m_2+6am_1m_2-m_1^2m_2-20m_2^2+7am_2^2-5m_1m_2^2-3m_2^3) > 0$.

Thus,

$$a(G_1(m_1+1, m_2-1)^c) < a(G_5(m_1+1, m_2-1)^c). \tag{8}$$

$g_1(a, m_1+1, m_2-1) - g_6(a, m_1+1, m_2-1) = -16-189a+137a^2-14a^3-6a^4+a^5-56m_1-229am_1+139a^2m_1-15a^3m_1-a^4m_1-53m_1^2-107am_1^2+46a^2m_1^2-3a^3m_1^2-18m_1^3-23am_1^3+5a^2m_1^3-2m_1^4-2am_1^4-$

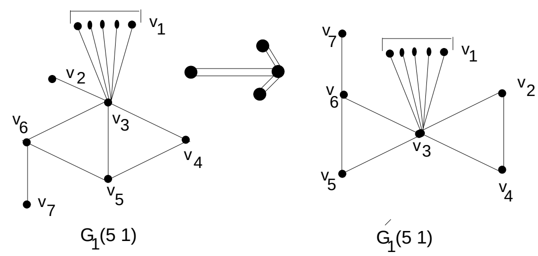

Figure 4: $a(G_1^{'}(5,1)^c) < a(G_1(5,1)^c)$.

$56m_2+49am_2-108a^2m_2+53a^3m_2-7a^4m_2-113m_1m_2+46am_1m_2-51a^2m_1m_2+13a^3m_1m_2-67m_1^2m_2+13am_1^2m_2-6a^2m_1^2m_2-15m_1^3m_2+am_1^3m_2-m_1^4m_2-53m_2^2+140am_2^2-90a^2m_2^2+15a^3m_2^2-67m_1m_2^2+81am_1m_2^2-24a^2m_1m_2^2-26m_1^2m_2^2+12am_1^2m_2^2-3m_1^3m_2^2-18m_2^3+45am_2^3-13a^2m_2^3-15m_1m_2^3+13am_1m_2^3-3m_1^2m_2^3-2m_2^4+4am_2^4-m_1m_2^4>0.$

Thus,

$$a(G_1(m_1+1,m_2-1)^c) < a(G_6(m_1+1,m_2-1)^c). \tag{9}$$

$g_1(a,m_1+1,m_2-1)-g_7(a,m_1+1,m_2-1)=2a(-5+a-m_1-m_2)(-4+a-m_1-m_2)(-3+a-m_1-m_2)(2+m_1-m_2)>0.$

Thus,

$$a(G_1(m_1+1,m_2-1)^c) < a(G_7(m_1+1,m_2-1)^c). \tag{10}$$

$g_1(a,m_1+1,m_2-1)-g_8(a,m_1+1,m_2-1)=-a(-4+a-2m_1)(-5+a-m_1-m_2)(-4+a-m_1-m_2)(-3+a-m_1-m_2)<0.$

Thus,

$$a(G_1(m_1+1,m_2-1)^c) < a(G_8(m_1+1,m_2-1)^c). \tag{11}$$

$g_1(a,m_1+1,m_2-1)-g_9(a,m_1+1,m_2-1)=(-4+a-m_1-m_2)(98a-84a^2+23a^3-2a^4-16m_1+199am_1-178a^2m_1+68a^3m_1-13a^4m_1+a^5m_1-20m_1^2+163am_1^2-131a^2m_1^2+39a^3m_1^2-4a^4m_1^2-8m_1^3+66am_1^3-39a^2m_1^3+6a^3m_1^3-m_1^4+13am_1^4-4a^2m_1^4+am_1^5+16m_2-31am_2+86a^2m_2-56a^3m_2+13a^4m_2-a^5m_2+46am_1m_2-12a^2m_1m_2-8m_1^2m_2+70am_1^2m_2-39a^2m_1^2m_2+6a^3m_1^2m_2-2m_1^3m_2+26am_1^3m_2-8a^2m_1^3m_2+3am_1^4m_2+20m_2^2-117am_2^2+119a^2m_2^2-39a^3m_2^2+4a^4m_2^2+8m_1m_2^2-58am_1m_2^2+39a^2m_1m_2^2-6a^3m_1m_2^2+2am_1^3m_2^2+8m_2^3-62am_2^3+39a^2m_2^3-6a^3m_2^3+2m_1m_2^3-26am_1m_2^3+8a^2m_1m_2^3-2am_1^2m_2^3+m_2^4-13am_2^4+4a^2m_2^4-3am_1m_2^4-am_2^5)>0.$

Thus,

$$a(G_1(m_1+1,m_2-1)^c) < a(G_9(m_1+1,m_2-1)^c). \tag{12}$$

From (5) to (12), we have $a(G_1(m_1+1,m_2-1)^c) \le a(G_i(m_1+1,m_2-1)^c)$. Consequently, for $2 \le i \le 9$, $a(G_1(m_1,m_2)^c) \le a(G_i(m_1,m_2)^c)$, where equality holds if $G_1(m_1,m_2) \cong G_i(m_1,m_2)$.

Lemma 3.3. Let m_1, m_2 and n be positive integers, such that $m_1 \ge m_2 \ge 1$, $n \ge 11$, and $m_1+m_2+5=n$. Then $a(G_1^{'}(m_1,m_2)^c) < a(G_1(m_1,m_2)^c)$.

Proof. Let $Z=\{Z_{v_i}\}$ be a unit Fiedler vector of the graph $G_1(m_1,m_2)^c$ for $1 \le i \le n$. After deleting the edge v_4v_5 and adding v_4v_2 in $G_1(m_1,m_2)$, we obtain $G_1^{'}(m_1,m_2)$ (see Figure 1 and Figure 3, for particular values of $m_1=5$ and $m_2=1$ see Figure 4). Now, by (2) and Lemma 2.2, we have

$$Z^TL(G_1(m_1,m_2))Z=\sum_{v_iv_j\in E(G_1(m_1,m_2))}(Z_{v_i}-Z_{v_j})^2 \le \sum_{v_iv_j\in E(G_1^{'}(m_1,m_2))}(Z_{v_i}-Z_{v_j})^2=Z^TL(G_1^{'}(m_1,m_2))Z.$$

Using (5), we obtain $a(G_1(m_1,m_2)^c)=Z^TL(G_1(m_1,m_2)^c)Z=Z^T(nI-J)Z-Z^TL(G_1(m_1,m_2))Z > Z^T(nI-J)Z-Z^TL(G_1^{'}(m_1,m_2))Z > a(G_1^{'}(m_1,m_2)^c)$. Consequently, $a(G_1^{'}(m_1,m_2)^c) < a(G_1(m_1,m_2)^c)$.

4 Characterization

This section consists on the main results of the paper.

Theorem 4.1. Let n, m_1 and m_2 be any positive integers such that $m_1 \geq m_2 \geq 1$, $n \geq 11$ and $n = m_1 + m_2 + 5$, then for any bicyclic graph with three cycles $G \in \Omega_{2,n}$,

$$a(G_i(m_1, m_2)^c) \leq a(G^c),$$

where $1 \leq i \leq 9$.

Proof. Let G be a bicyclic graph with three cycles $C_1(l_1)$, $C_2(l_2)$ and $C_3(l_3)$ with lengths $l_1 \geq 3$, $l_2 \geq 3$ and $l_3 \geq 4$, respectively. The cycles $C_1(l_1)$ and $C_2(l_2)$ are inner cycles with at least one common edge and $C_3(l_3)$ is an outer cycle such that $l_3 = l_1 + l_2 - 2k$, where k are common edges in $C_1(l_1)$ and $C_2(l_2)$. Let Z be a unit Fiedler vector of G^c. Then, we have a sequence $\{Z_{v_n}\}$ such that

$$Z_{v_1} \geq Z_{v_2} \geq \ldots \geq Z_{v_n}.$$

For $d_G(v_1, v_n) > 1$, we can assume the path $v_1 G v_n = v_1 w_1 \ldots w_2 v_n$. In the path $v_1 G v_n$, $d_G(v_1, v_n) = 2$ if $w_1 = w_2$. Add the edge $v_1 v_n$ and delete $v_1 w_1$ or $w_2 v_n$ such that the resulting bicyclic graph G_α is not $H(n, 2)$. Then by (2) and Lemma 2.2, we have

$$Z^T L(G) Z = \sum_{v_i v_j \in E(G)} (Z_{v_i} - Z_{v_j})^2 \leq \sum_{v_i v_j \in E(G_\alpha)} (Z_{v_i} - Z_{v_j})^2 = Z^T L(G_\alpha) Z, \tag{13}$$

where G_α is a bicyclic graph with three cycles $C_1(l_1^{'})$, $C_2(l_2^{'})$ and $C_3(l_3^{'})$ having some trees attached with the vertices of one or both the cycles $C_1(l_1^{'})$ and $C_2(l_2^{'})$. The lengths $l_1^{'}$, $l_2^{'}$ and $l_2^{'}$ may or may not different from l_1, l_2 and l_3 respectively. Most importantly, we note that $d_{G_\alpha}(v_1, v_n) = 1$. If $G_\alpha \notin \{G_i : 1 \leq i \leq 9\}$, then we have the following three cases for G_α:

Case a. Both the vertices v_1 and v_n are cycle vertices. In this case, we discuss further four possibilities (1) both the vertices are on exactly one inner cycle, (2) one of v_1 and v_n is a common vertex of both the inner cycles, (3) both the vertices v_1 and v_n are common vertices of the inner cycles, and (4) each inner cycle contains exactly one of v_1 and v_n.

(1) We assume without loss of generality that both the vertices v_1 and v_n are on the cycle $C_1(l_1^{'})$. Since, for $l_1^{'} \geq 4$ and $l_2^{'} = 3$ the cycles $C_1(l_1^{'})$ and $C_2(l_2^{'})$ have two common vertices, therefore, we can assume $C_1(l_1^{'}) = v_1 v_n w_1 w_2 w_3 \ldots w_i w_{i+1} \ldots w_m v_1$, where $m = l_1^{'} - 2$ and two vertices other than v_1 and v_n are also of $C_2(l_2^{'})$.

(i) Suppose that w_{m-1} and w_m are common vertices of the inner cycles. If $(Z_{w_{m-1}} - Z_{v_1})^2 \geq (Z_{w_{m-1}} - Z_{v_n})^2$, we delete the edge $w_{m-2} w_{m-1}$ and add $w_{m-1} v_1$ (as (b) is obtained from (a) in Figure 5). The resulting graph $G_{\alpha\alpha}$ is a bicyclic graph with two inner cycles $C_1(l_1^{'})$ and $C_2(l_2^{'})$, and an outer cycle $C_3(l_3^{'})$ such that $l_1^{'} = 3 = l_2^{'}$, $l_3^{'} = 4$, some trees are attached on v_1 in $C_1(l_1^{'})$ and some trees are attached on v_n which is non cycle. Thus, $G_{\alpha\alpha}$ is a bicycle graph $G_3(m_1, m_2)$ which is obtained when we identify B_1 by a vertex of degree 2 with end point say v_1 of an edge $v_1 v_n$ having some trees on v_1 and v_n (see Figure 1 with $v_1 v_n = v_2 v_3$).

If $(Z_{w_{m-1}} - Z_{v_1})^2 < (Z_{w_{m-1}} - Z_{v_n})^2$, we delete the edge $w_{m-2} w_{m-1}$ and add $w_{m-1} v_n$ (as (c) is obtained from (a) in Figure 5). The resulting graph $G_{\alpha\alpha}$ is a bicyclic graph with two inner cycles $C_1(l_1^{'})$ and $C_2(l_2^{'})$, and an outer cycle $C_3(l_3^{'})$ such that $l_1^{'} = 4$, $l_2^{'} = 3$, $l_3^{'} = 5$ and some trees are attached on both v_1 and v_n in $C_1(l_1^{'})$. Thus, $G_{\alpha\alpha}$ is a bicycle graph which is infect B_2 with some trees on the two adjacent vertices of degree 2 i.e $G_6(m_1, m_2)$ (see Figure 2). If we proceed from the other side of the path, then for $(Z_{w_2} - Z_{v_1})^2 \geq (Z_{w_2} - Z_{v_n})^2$, we delete

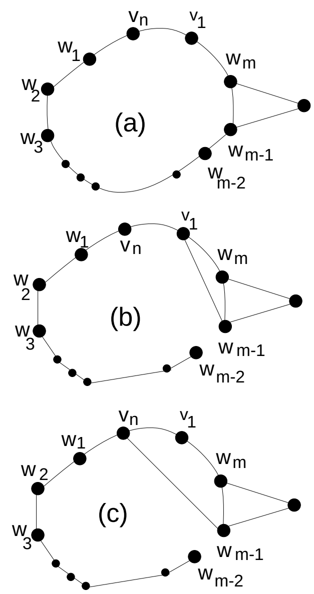

Figure 5

the edge w_1w_2 and add the edge w_2v_1 otherwise, we delete w_1w_2 and add w_2v_n. Thus, the resulting graph $G_{\alpha\alpha}$ is a bicyclic graph with three cycles such that the lengths of the inner cycles are $l_1^{'}-2$ and $l_2^{'}=3$ or $l_1^{'}-1$ and $l_2^{'}=3$. Now, repeat the process for the vertex w_3 and continue up to the vertex w_{m-1}. Thus, we obtain the same graphs $G_3(m_1,m_2)$ and $G_6(m_1,m_2)$.

(ii) Suppose that w_i and w_{i+1} are common vertices of the inner cycles, where $2\le i\le m-2$. If $(Z_{w_i}-Z_{v_1})^2\ge(Z_{w_i}-Z_{v_n})^2$, we delete $w_{i-1}w_i$ and add w_iv_1. Now, if $(Z_{w_{i+1}}-Z_{v_1})^2\ge(Z_{w_{i+1}}-Z_{v_n})^2$, we delete $w_{i+1}w_{i+2}$ and add $w_{i+1}v_1$, otherwise delete $w_{i+1}w_{i+2}$ and add $w_{i+1}v_n$. Thus, the resulting graphs are $G_3(m_1,m_2)$ or $G_6(m_1,m_2)$, respectively.

If $(Z_{w_i}-Z_{v_1})^2<(Z_{w_i}-Z_{v_n})^2$, we delete $w_{i-1}w_i$ and add w_iv_n. Now, if $(Z_{w_{i+1}}-Z_{v_1})^2\ge(Z_{w_{i+1}}-Z_{v_n})^2$, we delete $w_{i+1}w_{i+2}$ and add $w_{i+1}v_1$, otherwise delete $w_{i+1}w_{i+2}$ and add $w_{i+1}v_n$. Thus, the resulting graphs are $G_6(m_1,m_2)$ or $G_3(m_1,m_2)$, respectively.

(iii) Suppose that w_1 and w_2 are common vertices of the inner cycles, then we repeat (i) and the obtained graphs are same as there.

If in (1(i)-(iii)), $l_2^{'}\ge 4$, then we can assume $C_2(l_2^{'})=u_1u_2...u_lw_iw_{i+1}$, where w_i and w_{i+1} are two common vertices of the cycles $C_1(l_1^{'})$ and $C_2(l_2^{'})$ for $1\le i\le m-1$ and $l=l_2^{'}-2$. By the use of (1(i)-(iii)), we have $C_1(l_1^{'})$ with $l_1^{'}=3$, some trees attached on $v_1\in C_1(l_1^{'})$ and some trees attached on v_n (pendant vertex) or with $l_1^{'}=4$ and some trees attached on v_1 and v_n (both are in $C_1(l_1^{'})$). Now, for $C_2(l_2^{'})$, if $(Z_{u_1}-Z_{v_1})^2\ge(Z_{u_1}-Z_{v_n})^2$, delete the edge u_1u_2 and add the edge u_1v_1, otherwise delete u_1u_2 and add u_1v_n. Thus, the resulting graphs are $G_4(m_1,m_2)$ and $G_5(m_1,m_2)$ or $G_5(m_1,m_2)$ and $G_7(m_1,m_2)$, (see Figure 1 and Figure 2) respectively. Moreover, $l_1^{'}=3=l_2^{'}$ is not possible as both the vertices v_1 and v_n can not appear on only $C_1(l_1^{'})$.

(2) Without loss of generality suppose that v_n is on the cycle $C_1(l_1^{'})$ and v_1 is a common vertex of the inner cycles. Assume that $l_1^{'}\ge 4$, $l_2^{'}=3$ and $C_1(l_1^{'})=v_1v_nw_1w_2w_3...w_mv_1$, where w_m is also a common vertex of

the inner cycles and $m = l_1^{'} - 2$.

If $(Z_{w_{m-1}} - Z_{v_1})^2 \geq (Z_{w_{m-1}} - Z_{v_n})^2$, we delete the edge $w_{m-2}w_{m-1}$ and add the edge $w_{m-1}v_1$. The resulting graph $G_{\alpha\alpha}$ is a bicyclic graph with two inner cycles $C_1(l_1^{'})$ and $C_2(l_2^{'})$, and an outer cycle $C_3(l_3^{'})$ such that $l_1^{'} = 3 = l_2^{'}$, $l_3^{'} = 4$, some trees are attached on v_1 which is a common vertex of both the inner cycles and some trees are attached on v_n in $C_2(l_2^{'})$. Thus, $G_{\alpha\alpha}$ is a bicycle graph $G_4(m_1, m_2)$ which is obtained when we identify B_1 by a vertex of degree 3 with an end point v_1 of an edge v_1v_n having some trees on v_1 and v_n (see Figure 1 with $v_1v_n = v_2v_4$).

If $(Z_{w_{m-1}} - Z_{v_1})^2 < (Z_{w_2} - Z_{v_n})^2$, we delete the edge $w_{m-2}w_{m-1}$ and add the edge $w_{m-1}v_n$. The resulting graph $G_{\alpha\alpha}$ is a bicyclic graph with two inner cycles $C_1(l_1^{'})$ and $C_2(l_2^{'})$, and an outer cycle $C_3(l_3^{'})$ such that $l_1^{'} = 4$, $l_2^{'} = 3$, $l_3^{'} = 5$, some trees are attached on v_1 which is a common vertex of both the inner cycles and some trees are attached on v_n in $C_2(l_2^{'})$. Thus, $G_{\alpha\alpha}$ is a bicycle graph $G_5(m_1, m_2)$ (see Figure 1) which is infect B_2 with some trees which are attached on two adjacent vertices of degree 2 and 3 in $C_4 \subseteq B_2$.

If we proceed from the other side of the path, then for $(Z_{w_2} - Z_{v_1})^2 \geq (Z_{w_2} - Z_{v_n})^2$, we delete the edge w_1w_2 and add the edge w_2v_1 otherwise, we delete the edge w_1w_2 and add the edge w_2v_n. Thus, the resulting graph $H_{\alpha\alpha}$ is a bicyclic graph with three cycles such that the lengths of the inner cycles are $l_1^{'} - 2$ and $l_2^{'} = 3$ or $l_1^{'} - 1$ and $l_2^{'} = 3$. Now, repeat the process for the vertex w_3 and continue up to the vertex w_{m-1}. Thus, we obtain the same graphs $G_4(m_1, m_2)$ and $G_5(m_1, m_2)$.

If in (2), $l_2^{'} \geq 4$, then we can assume $C_2(l_2^{'}) = u_1u_2...u_lw_iv_1u_1$, where w_i and v_1 are two common vertices of the cycles $C_1(l_1^{'})$ and $C_2(l_2^{'})$ for $1 \leq i \leq m$ and $l = l_2^{'} - 2$. By the use of (2), we have $C_1(l_1^{'})$ with $l_1^{'} = 3$, some trees attached on $v_1 \in C_1(l_1^{'})$ and some trees attached on v_n (pendant vertex) or with $l_1^{'} = 4$ and some trees attached on v_1 and v_n (both are in $C_1(l_1^{'})$). Now, for $C_2(l_2^{'})$, if $(Z_{u_l} - Z_{v_1})^2 \geq (Z_{u_l} - Z_{v_n})^2$, delete the edge $u_{l-1}u_l$ and add the edge u_lv_1, otherwise delete the edge $u_{l-1}u_l$ and add the edge u_lv_n. Thus, the resulting graphs $G_{\alpha\alpha}$ are $G_4(m_1, m_2)$ and $G_5(m_1, m_2)$ or $G_5(m_1, m_2)$ and $G_7(m_1, m_2)$, respectively. Moreover, if $l_1^{'} = 3 = l_2^{'}$, then we obtain $G_2(m_1, m_2)$.

(3) Suppose that v_1 and v_n both are common vertices of the inner cycles. Assume that $l_1^{'} \geq 4$, $l_2^{'} = 3$ and $C_1(l_1^{'}) = v_1v_nw_1w_2w_3...w_mv_1$, where $m = l_1^{'} - 2$.
If $(Z_{w_{m-1}} - Z_{v_n})^2 \geq (Z_{w_{m-1}} - Z_{v_1})^2$, we delete the edge $w_{m-2}w_{m-1}$ and add the edge $w_{m-1}v_n$. The resulting graph $G_{\alpha\alpha}$ is a bicyclic graph with two inner cycles $C_1(l_1^{'})$ and $C_2(l_2^{'})$, and an outer cycle $C_3(l_3^{'})$ such that $l_1^{'} = 4$, $l_2^{'} = 3$, $l_3^{'} = 5$, some trees are attached on v_1 and v_n which are common vertices of the inner cycles. Thus, $G_{\alpha\alpha}$ is a bicycle graph $G_8(m_1, m_2)$ (see Figure 2) which is obtained from B_2 by attaching some trees on both the vertices of degree 3.

If $(Z_{w_{m-1}} - Z_{v_1})^2 > (Z_{w_{m-1}} - Z_{v_n})^2$, we delete the edge $w_{m-1}w_m$ and add the edge $w_{m-1}v_1$. Then, check for w_{m-2}, if $(Z_{w_{m-2}} - Z_{v_n})^2 \geq (Z_{w_{m-2}} - Z_{v_1})^2$, delete the edge $w_{m-3}w_{m-2}$ and add the edge $w_{m-2}v_n$. The resulting graph is $H_8(m_1, m_2)$. If $(Z_{w_{m-2}} - Z_{v_1})^2 > (Z_{w_{m-2}} - Z_{v_n})^2$, delete $w_{m-2}w_{m-1}$ and add $w_{m-2}v_1$. Repeat this process until we reach on the vertex w_2. If $(Z_{w_2} - Z_{v_n})^2 \geq (Z_{w_2} - Z_{v_1})^2$, delete the edge w_2w_1 and add the edge w_2v_n, otherwise delete w_2w_3 and add w_2v_1. The resulting graph is $G_8(m_1, m_2)$.

If in (3), $l_2^{'} \geq 4$, then we can assume $C_2(l_2^{'}) = u_1u_2...u_lv_nv_1u_1$. By the use of (3), we have $C_1(l_1^{'})$ with $l_1^{'} = 4$ and some trees attached on v_1 and v_n (both are common vertices). Now, again repeat (3) for $C_2(l_2^{'})$ and we obtain $G_9(m_1, m_2)$. If $l_1^{'} = 3 = l_2^{'}$, then we obtain $G_1(m_1, m_2)$.

(4) Suppose that v_1 is on $C_1(l_1^{'})$ and v_n is on $C_2(l_2^{'})$, where $l_1^{'}, l_2^{'} \geq 3$. We note that $d_{G_\alpha}(v_1, v_n) \geq 2$, which is not possible.

Case b. One of v_1, v_n is a cycle vertex.
We assume that v_1 is a cycle vertex and v_n is non cycle vertex without loss of generality. In this case, for v_1, we have three possibilities (1) v_1 is on $C_1(l_1^{'})$, (2) v_1 is a common vertex of both the inner cycles and (3) v_1 is on $C_2(l_2^{'})$.

(1) If v_1 is only on $C_1(l_1^{'})$, then for $l_1^{'} \geq 4$ and $l_2^{'} = 3$, we have $C_1(l_1^{'}) = v_1w_1w_2w_3...w_iw_{i+1}...w_mv_1$. Assume w_i and w_{i+1} are common vertices of the inner cycles, where $1 \leq i \leq m-1$. Now, we repeat (Case a (1)) and obtain $G_3(m_1, m_2)$ or $G_6(m_1, m_2)$. If $l_2^{'} \geq 4$ then again by (Case a (1)), the resulting graphs are $G_4(m_1, m_2)$ and $G_5(m_1, m_2)$ or $G_5(m_1, m_2)$ and $G_7(m_1, m_2)$. Moreover, if $l_1^{'} = 3 = l_2^{'}$, then the resulting graph is $G_3(m_1, m_2)$.

(2) If v_1 is a common vertex of the inner cycles. Assume that w_m is an other common vertex such that $l_1^{'} \geq 4$, $l_2^{'} = 3$, $C_1(l_1^{'}) = v_1v_nw_1w_2w_3...w_mv_1$ and $m = l_1^{'} - 1$. Now, by (Case a (2)), we obtain $G_4(m_1, m_2)$ or $G_5(m_1, m_2)$. If $l_2^{'} \geq 4$ then again by (Case a (2)), the resulting graphs are $G_4(m_1, m_2)$ and $G_5(m_1, m_2)$ or $G_5(m_1, m_2)$ and $G_7(m_1, m_2)$. Moreover, if $l_1^{'} = 3 = l_2^{'}$, then the resulting graph is $G_4(m_1, m_2)$. (3) If v_1 is only on $C_2(l_2^{'})$, then follow (Case b (1)).

Case c. Both v_1 and v_n are non cycle vertices.
Suppose that u and v are common vertices of the inner cycles such that $C_1(l_1^{'}) = uvw_1w_2w_3...w_m$ and $C_2(l_2^{'}) = uvu_1u_2u_3...u_l$, where $l_1^{'} \geq 4$, $m = l_1^{'} - 2$ and $l = l_2^{'} - 2$. Assume that there is a path P containing the vertices v_1 and v_n has one end point either u or v. If v is on P and w_1 is adjacent to v in $C_1(l_1^{'}$ such that $(Z_{w_1} - Z_{v_1})^2 \geq (Z_{w_1} - Z_{v_n})^2$, then we delete w_1v and add w_1v_1, otherwise delete w_1v and add w_1v_n. Then the resulting bicyclic graph $G_{\alpha,\alpha}$ is in Case a or Case b.

Then by equation (2) and Lemma (3), we have

$$Z^TL(G_\alpha)Z = \sum_{v_iv_j \in E(G_\alpha)} (Z_{v_i} - Z_{v_j})^2 \leq \sum_{v_iv_j \in E(G_{\alpha\alpha})} (Z_{v_i} - Z_{v_j})^2 = Z^TL(G_{\alpha\alpha})Z, \tag{14}$$

If $G_{\alpha\alpha} \notin \{G_i : 1 \leq i \leq 9\}$ and there exists a pendant vertex v, whose neighbor a is neither v_1 nor v_n, satisfying $(Z_v - Z_{v_1})^2 \geq (Z_v - Z_{v_n})^2$, then delete av and add vv_1; otherwise delete av and add vv_n. Repeat this rearranging until the resulting graph

$$G_{\alpha\alpha\alpha} \in \{G_i : 1 \leq i \leq 9\}.$$

Then by equation (2) and Lemma (2.2), we have

$$Z^TL(G_{\alpha\alpha})Z = \sum_{v_iv_j \in E(G_{\alpha\alpha})} (Z_{v_i} - Z_{v_j})^2 \leq \sum_{v_iv_j \in E(G_{\alpha\alpha\alpha})} (Z_{v_i} - Z_{v_j})^2 = Z^TL(G_{\alpha\alpha\alpha})Z \tag{15}$$

By (13) – (15), we have

$$\begin{aligned} a(G^c) =& Z^TL(G^c)Z = Z^T(nI - J)Z - Z^TL(G)Z \geq Z^T(nI - J)Z - Z^TL(G_\alpha)Z \geq Z^T(nI - J)Z - Z^TL(G_{\alpha\alpha})Z \\ &\geq Z^T(nI - J)Z - Z^TL(G_{\alpha\alpha\alpha})Z = Z^TL(G^c_{\alpha\alpha\alpha})Z \geq a(G_{\alpha\alpha\alpha}). \end{aligned}$$

Hence we have $a(G^c) \geq a(G^c_{\alpha\alpha\alpha})$. Consequently, $a(G_i(m_1, m_2)^c) \leq a(G^c)$ and equality holds if and only if $G_i(m_1, m_2) \cong G$, where $G \in \Omega_{2,n}$ and $1 \leq i \leq 9$.

Theorem 4.2. Let n, m_1 and m_2 be any positive integers such that $m_1 \geq m_2 \geq 1$, $n \geq 11$ and $n = m_1 + m_2 + 5$, then for any bicyclic graph with three cycles $G \in \Omega_{2,n}$,

$$a(G_1(n-6, 1)^c) \leq a(G_1(m_1, m_2)^c) \leq a(G^c),$$

where equalities hold if and only if $G_1(n-6, 1)^c \cong G_1(m_1, m_2) \cong G$.
Proof. The proof of this theorem follows from Lemma 3.1, Lemma 3.2 and Theorem 4.1.

Theorem 4.3. Let n, m_1 and m_2 be any positive integers such that $m_1 \geq m_2 \geq 1$, $n \geq 11$ and $n = m_1 + m_2 + 5$, then for any bicyclic graph with two or three cycles $G \in \Omega_n = \Omega_{1,n} \cup \Omega_{2,n}$,

$$a(G'_1(n-6,1)^c) \leq a(G'_1(m_1,m_2)^c) \leq a(G^c),$$

where equalities hold if and only if $G'_1(n-6,1)^c \cong G'_1(m_1,m_2) \cong G$.

Proof. The proof of this theorem follows from Theorem 2.3, Lemma 3.3 and Theorem 4.2.

5 Conclusions

In this paper, we have characterized the unique graph in the class of connected graphs whose complements are bicyclic having exactly three cycles with respect to the second least Laplacian eigenvalue (algebraic connectivity) of the Laplacian matrix. Mainly, we found the unique graph with minimum algebraic connectivity in the complete class of connected graphs whose complements are bicyclic with two or three cycles. The problem is still open to discuss the algebraic connectivity of the other families of the connected graphs whose complements are k-cyclic graphs for $k \geq 3$ (tricyclic, tetracyclic and so on.)

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