Research Article

Abdul Hameed and Mikhail Tyaglov*

Integral Laplacian graphs with a unique repeated Laplacian eigenvalue, I

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Abstract: The set $S_{i,n} = \{0, 1, 2, \dots, n-1, n\}\backslash\{i\}$, $1 \leqslant i \leqslant n$, is called Laplacian realizable if there exists an undirected simple graph whose Laplacian spectrum is $S_{i,n}$. The existence of such graphs was established by Fallat et al. (*On graphs whose Laplacian matrices have distinct integer eigenvalues*, J. Graph Theory **50** (2005), 162–174). In this article, we consider graphs whose Laplacian spectra have the form

$$S_{\{i,j\}_n^m} = \{0, 1, 2, \dots, m-1, m, m, m+1, \dots, n-1, n\}\backslash\{i, j\}, \quad 0 < i < j \leqslant n,$$

and completely describe those with $m = n - 1$ and $m = n$. We also show close relations between graphs realizing $S_{i,n}$ and $S_{\{i,j\}_n^m}$ and discuss the so-called $S_{n,n}$-conjecture and the corresponding conjecture for $S_{\{i,n\}_n^m}$.

Keywords: Laplacian integral graphs, Laplacian matrix, Laplacian spectrum, integer eigenvalues

MSC 2020: 05C50, 05C76, 15A18

1 Introduction

Let $G = (V(G), E(G))$ be an undirected simple graph with a vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ and the edge set $E(G) = \{e_1, e_2, \dots, e_r\}$. We denote the degree of vertex v_i, $i = 1, 2, \dots, n$, by $d_i = d(v_i)$. A *Laplacian matrix* of G is a matrix defined as follows: $L(G) = D(G) - A(G)$, where $D(G)$ is the vertex degree diagonal matrix, i.e., $D(G) = \operatorname{diag}(d_1, d_2, \dots, d_{n-1}, d_n)$ and $A(G)$ is the $(0, 1)$ adjacency matrix of G. Thus, the entries of the Laplacian matrix have the following form:

$$l_{ij} = \begin{cases} d_i, & \text{if } i = j, \\ -1, & \text{if } i \neq j \quad \text{and} \quad v_i \sim v_j, \\ 0, & \text{otherwise}. \end{cases}$$

The study of graphs whose adjacency matrix has integer eigenvalues was probably introduced by Harary and Schwenk [17]. The same kind of problem has been addressed with the eigenvalues of the Laplacian matrix. A graph G whose Laplacian matrix has integer eigenvalues is called *Laplacian integral*. The most well-known examples of such graphs are the complete graph K_n, the star graph S_n, and the Petersen graph having eigenvalues of its Laplacian matrix $\{0, n^{n-1}\}$, $\{0, 1^{n-2}, n\}$, and $\{0, 2^5, 5^4\}$, respectively (the exponent indicates the multiplicity of the eigenvalue).

Several explicit constructions of Laplacian integral graphs of special types appear in the literature. For example, Merris [22] showed that degree maximal graphs are Laplacian integral, see also [24,26]. A connected

*** Corresponding author: Mikhail Tyaglov**, School of Mathematical Sciences, MOE-LSC, and CMA-Shanghai, Shanghai Jiao Tong University, 800 Dongchuan Road, Shanghai, 200240, P. R. China; Department of Mathematics and Computer Sciences, St. Petersburg State University, St.Petersburg, 199178, Russia, e-mail: tyaglov@mail.ru

Abdul Hameed: School of Mathematical Sciences, Shanghai Jiao Tong University, 800 Dongchuan Road, Shanghai, 200240, P. R. China, e-mail: abdul-hameed_211@sjtu.edu.cn

class of graphs, namely $S(a,b)$, with n vertices and m edges where m is a function of $a, b \in \frac{\mathbb{N}}{2}$, is described to be Laplacian integral [8]. Kirkland [21] characterized Laplacian integral graphs of maximum degree 3. He also established that the addition of a sequence of edges to an integral graph preserves integrality with each addition until the complete graph has been constructed and also characterized constructably Laplacian integral graphs [19,20]. In the articles [16,25,26], the authors discussed the integrality of threshold graphs though the latter have a direct relation to the degree maximal graphs and cographs. For their applications and generalizations, see [2,3]. The family of graphs whose eigenvalues are distinct integer numbers and indecomposable Laplacian integral graphs can be seen in [9,13]. Stanić recently dealt with the Laplacian controllability of cographs that have integer Laplacian spectra [29]. His results are based on the realizability properties of the sets $S_{i,n}$, see Definition 1.1. In the study by Andelic et al. [1], integral Laplacian graphs are applied to study signed graphs with main Laplacian eigenvalues. Thus, Laplacian integral graphs are of key importance in various fields of science. Even though the study of integral graphs has come from a theoretical issue since the beginning, this object has recently been associated with many applications in mathematics, physics, and chemistry. In mathematics, integral graphs are closely related to number theory [28]. In physics, it is used in multiprocessor interconnection networks, whereas in chemistry, it is used in the conjugated molecules, see, e.g., [6,7]. In addition, Laplacian controllability plays an important role in many control problems, such as stabilization of unstable systems or optimal control [29].

One way to describe and study all the integral Laplacian graphs (if any) is to study graphs having certain kinds of Laplacian spectra. Thus, starting with graphs with simple integer Laplacian eigenvalues, one can then consider integral Laplacian graphs with all simple Laplacian eigenvalues excluding one that is repeated. On the next step, it is allowed for graphs to have two Laplacian eigenvalues of multiplicity 2 or a unique multiple eigenvalue of multiplicity 3.

The first step in this scheme was undertaken by Fallat et al. in their work [9] on the integral Laplacian graphs with simple Laplacian eigenvalues.

Definition 1.1. A set S consisting of elements of $\{0, 1, \dots, n\}$ (maybe repeated) such that $|S| = n$ and $0 \in S$ is called Laplacian realizable if there exists a connected simple graph on n vertices whose Laplacian spectrum coincides with S. In this situation, the graph G is said to realize S.

Fallat et al. [9] considered the sets of the form

$$S_{i,n} = \{0, 1, 2, \dots, n-1, n\} \backslash \{i\} \tag{1}$$

and found out when such sets are Laplacian realizable and completely described the graphs realizing $S_{i,n}$. The only problem that remained open in this case is whether the set $S_{n,n}$ is Laplacian realizable or not. This problem is now known as the $S_{n,n}$-conjecture and states that $S_{n,n}$ is *not* Laplacian realizable for every $n \geqslant 2$. Fallat et al. [9] proved that this conjecture is true for $n \leqslant 11$, for prime n, and for $n \equiv 2, 3 \bmod 4$. Later, Goldberger and Neumann [11] showed that the conjecture is true for $n \geqslant 6, 649, 688, 933$. The authors of the present work established that a Cartesian product of two graphs (Definition 2.2) does not realize $S_{n,n}$ [14].

Following the aforementioned scheme, in this work, we study Laplacian integral graphs having only one multiple (non-zero) Laplacian eigenvalue whose multiplicity equals exactly 2. Namely, we describe the graphs realizing the multisets of the form

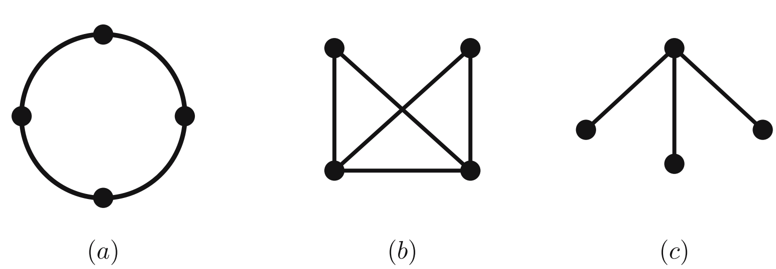

Figure 1: Graph (a), (b), and (c) realizing $S_{\{1,3\}_4^2}$, $S_{\{1,3\}_4^4}$, and $S_{\{2,3\}_4^1}$, respectively.

$$S_{\{i,j\}_n^m} = \{0, 1, 2, \ldots, m-1, m, m, m+1, \ldots, n-1, n\} \backslash \{i, j\}, \quad (2)$$

where $0 < i < j \leqslant n$. So $S_{\{i,j\}_n^m}$ does not contain the numbers i and j, while some number m (and only this number) is repeated. Obviously, if $m = 0$, then the graph is disconnected, so we exclude this case from the further considerations.

The graphs realizing $S_{\{i,j\}_n^m}$ exist even for small n. Cvetković et al. [5, p. 286–289] found the Laplacian spectra of all graphs of order up to 5, so it is easy to observe that for $n = 3$, only the multiset $S_{\{1,2\}_3^3}$ is Laplacian realizable, and it is realized by the complete graph K_3. For $n = 4$, there are exactly three multisets $S_{\{1,3\}_4^2}$, $S_{\{1,3\}_4^4}$, and $S_{\{2,3\}_4^1}$ that are Laplacian realizable. They are depicted on Figure 1. For $n = 5$, the only Laplacian realizable multisets of the form $S_{\{i,j\}_n^m}$ are $S_{\{2,3\}_5^4}$, $S_{\{1,3\}_5^5}$, $S_{\{2,4\}_5^3}$, $S_{\{1,4\}_5^2}$, and $S_{\{2,4\}_5^1}$.

Note that the graphs (b) and (c) in Figure 1 are threshold, see, e.g., [22,25]. The explicit constructions of graphs realizing $S_{\{i,j\}_n^m}$ for $n = 4, 5, 6$ are given in Tables A1, A2, A3 of Appendix A. To make these tables, we used the complete lists of all connected graphs of the order less than 7, see [4] and [5, pp. 286–289].

In this study, we consider graphs realizing $S_{\{i,j\}_n^m}$ with $m = n - 1$ and $m = n$. In Theorems 4.2 and 4.3, we list all Laplacian realizable multisets $S_{\{i,j\}_n^n}$ and describe the structure of the graphs realizing them. For the case $m = n - 1$, we show that only the multisets of the form $S_{\{1,j\}_n^{n-1}}$ and $S_{\{2,j\}_n^{n-1}}$ can be Laplacian realizable for certain j, see Theorem 5.1, and in Theorems 5.2 and 5.5, we list all such j for given n. Theorems 5.3 and 5.4 describe the structure of graphs realizing these multisets.

As mentioned above, for $n \geqslant 2$, the sets $S_{n,n}$ (without n) are not Laplacian realizable. But the multisets $S_{\{i,n\}_n^m}$ (i.e., without n like $S_{n,n}$) can be Laplacian realizable, at least for small n. In Section 6, we show that the ladder graphs on 6 and a graph on 8 vertices and their complements realize such multisets (Figures 2 and 3).

However, there are no other Laplacian realizable multisets $S_{\{i,n\}_n^m}$ for $n \leqslant 7$, and we conjecture that there are no such multisets for $n \geqslant 6$ except the four graphs depicted in Figures 2 and 3, see Conjecture 6.4. In attempts to prove this conjecture, we found out that $S_{\{i,n\}_n^m}$ are not Laplacian realizable for prime integer n (Proposition 6.2), and that any graph realizing $S_{\{i,n\}_n^m}$ (if any) cannot be the Cartesian product of two graphs for $n \geqslant 9$ (Theorem 3.6). To establish some results of the present work, we used the new Laplacian spectral properties of the join of graphs, see Definition 2.1 and Theorems 3.1 and 3.2.

The article is organized as follows: in Section 2, we list basic definitions and notations; several important results that we use in our work are stated in Section 3: where we also prove some auxiliary statements and theorems; in Sections 4 and 5, we list all the Laplacian realizable multisets of the form $S_{\{i,j\}_n^n}$ and $S_{\{i,j\}_n^{n-1}}$, respectively, and develop an algorithm for constructing the graphs realizing those multisets; Section 6 is devoted to the Conjecture 6.4; some concluding remarks are presented in Section 7; and finally, in Appendix A, we list all the Laplacian realizable multisets $S_{\{i,j\}_n^m}$ for $n = 4, 5, 6$ and a few ones for $n = 7$. The corresponding graphs realizing those multisets are presented.

2 Preliminaries

Let G be an undirected simple graph. The *complement* $\overline{G}$ of the graph G is a graph $\overline{G}$ on the same vertices such that two distinct vertices of $\overline{G}$ are adjacent if and only if they are not adjacent in G.

Definition 2.1. Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, the *union* of graphs G_1 and G_2, denoted by $G_1 \cup G_2$, is the graph $G = (V, E)$ for which $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$. The union of k copies of the same graph G is denoted by kG. The *join* $G_1 \vee G_2$ of the graphs G_1 and G_2 is the graph obtained from $G_1 \cup G_2$ by joining every vertex of G_1 with every vertex of G_2, i.e., $G_1 \vee G_2 = \overline{(\overline{G_1} \cup \overline{G_2})}$.

Definition 2.2. The *Cartesian product* of the graphs G_1 and G_2 is the graph $G_1 \times G_2$ whose vertex set is the Cartesian product $V(G_1) \times V(G_2)$, and for $v_1, v_2 \in V(G_1)$ and $u_1, u_2 \in V(G_2)$, the vertices (v_1, u_1) and (v_2, u_2) are adjacent in $G_1 \times G_2$ if and only if either

- $v_1 = v_2$ and $\{u_1, u_2\} \in E(G_1)$;
- $\{v_1, v_2\} \in E(G_2)$ and $u_1 = u_2$.

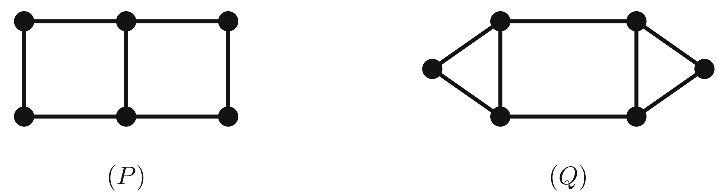

Figure 2: Graph P and its complement Q realizing $S_{\{4,6\}_6^3}$ and $S_{\{2,6\}_6^3}$, respectively.

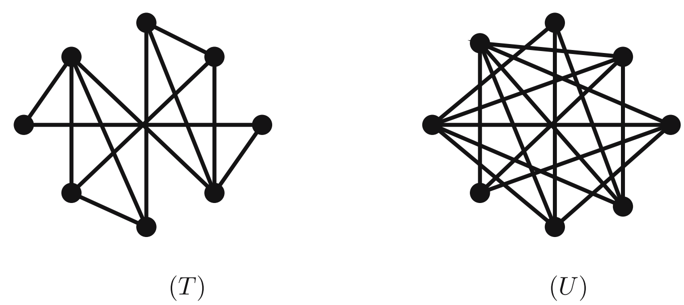

Figure 3: Graph T and its complement U realizing $S_{\{7,8\}_8^3}$ and $S_{\{1,8\}_8^5}$, respectively.

The ladder graph is an example of a graph which is obtained from the Cartesian product of two path graphs: $L_{n,1} = P_n \times P_2$.

We denote the eigenvalues of the Laplacian matrix $L(G)$ arranged in a nondecreasing order as follows:

$$0 = \mu_1 \leqslant \mu_2 \leqslant \ldots \leqslant \mu_{n-1} \leqslant \mu_n.$$

It is well known that $L(G)$ has a zero eigenvalue corresponding to the eigenvector with equal entries, while other eigenvalues are nonnegative [26]. Fiedler [10, p. 298] showed that $\mu_2 > 0$ if and only if the graph G is connected. The *Laplacian spectrum* $\sigma_L(G)$ of a graph G is the spectrum of the Laplacian matrix of G. The *Laplacian spectral radius* denoted by $\rho(G)$ of G is the absolute value of the largest eigenvalue of the Laplacian matrix $L(G)$, i.e., $\rho(G) = \max_{1\leqslant i\leqslant n}|\mu_i| = \mu_n$. The Laplacian spectrum of a graph is bounded from above by the order of the graph, see [23].

Proposition 2.3. *Let G be a simple graph on n vertices. Then,* $\rho(G) \leqslant n$.

The Laplacian spectrum of the graph operations mentioned above is related to the Laplacian spectra of the initial graphs. Thus, the Laplacian spectrum of $\bigcup_{j=1}^{k} G_j$ is the union of the Laplacian spectra of the graphs $G_1, \ldots, G_k$, see, e.g., [5]. The Laplacian spectra of the complement of a graph and the join of two graphs are given by the following theorems, see, e.g., [5,23].

Theorem 2.4. *Let G be a graph with n vertices. If* $\sigma_L(G) = \{0, \mu_2, \ldots, \mu_{n-1}, \mu_n\}$, *then*

$$\sigma_L(\overline{G}) = \{0, n - \mu_n, n - \mu_{n-1}, \ldots, n - \mu_2\}.$$

The spectrum of the join of two graphs was obtained by Kel'mans [18] in terms of the characteristic polynomials of Laplacian matrices. The following form of Kel'mans' theorem can be found, e.g., in the study by Merris [26].

Theorem 2.5. *Let G and H be two graphs of order n and m, respectively, whose eigenvalues are as follows*:

$$0 = \mu_1 \leqslant \mu_2 \leqslant \mu_3 \ldots \leqslant \mu_{n-1} \leqslant \mu_n \quad \text{and} \quad 0 = \lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \ldots \leqslant \lambda_{m-1} \leqslant \lambda_m.$$

Then, the Laplacian spectrum of the join $G \vee H$ *has the form*

$$\{0, m + \mu_2, m + \mu_3, \ldots, m + \mu_n, n + \lambda_2, n + \lambda_3, \ldots, n + \lambda_m, n + m\}. \tag{3}$$

Note that the eigenvalues here are not in increasing order.

As one can see from equation (3), the order of the join of two graphs is a Laplacian eigenvalue of the join. It turns out that this fact is a necessary and sufficient condition for a graph to be a join, see, e.g., [27].

Theorem 2.6. *Let G be a connected graph of order n. Then, n is a Laplacian eigenvalue of G if and only if G is the join of two graphs.*

The Laplacian spectrum of the Cartesian product of two graphs is given in the following theorem, see, e.g., [23].

Theorem 2.7. *Let F and H be graphs having spectrum,*

$$\sigma_L(F) = \{\lambda_1, \lambda_2, \ldots, \lambda_{n-1}, \lambda_n\}, \quad \sigma_L(H) = \{\mu_1, \mu_2, \ldots, \mu_{p-1}, \mu_p\},$$

then the Laplacian spectrum of the Cartesian product of F and H is

$$\sigma_L(F \times H) = \{\lambda_i + \mu_k\}, \quad 1 \leqslant i \leqslant n, \quad 1 \leqslant k \leqslant p. \tag{4}$$

From equation (4), it is easy to see that the Laplacian spectrum of the Cartesian product of two Laplacian integral graphs is Laplacian integral.

As we mentioned in Introduction, the graphs whose Laplacian spectra are of the form $S_{i,n}$ defined in equation (1), were introduced and completely investigated by Fallat et al. [9]. In the sequel, we use the following results established by Fallat et al. [9, Theorems 2.3 and 2.6].

Proposition 2.8.

(*i*) *If* $n \equiv 0 \bmod 4$, *then for each* $i = 1, 2, 3, \ldots, \frac{n-2}{2}$, $S_{2i,n}$ *is Laplacian realizable*;

(*ii*) *If* $n \equiv 1 \bmod 4$, *then for each* $i = 1, 2, 3, \ldots, \frac{n-1}{2}$, $S_{2i-1,n}$ *is Laplacian realizable*;

(*iii*) *If* $n \equiv 2 \bmod 4$, *then for each* $i = 1, 2, 3, \ldots, \frac{n}{2}$, $S_{2i-1,n}$ *is Laplacian realizable*;

(*iv*) *If* $n \equiv 3 \bmod 4$, *then for* $i = 1, 2, \ldots, \frac{n-1}{2}$, $S_{2i,n}$ *is Laplacian realizable.*

Proposition 2.9. *Suppose that $n \geqslant 6$ and that G is a graph on n vertices. Then, G realizes $S_{1,n}$ if and only if G is formed in one of the following two ways*:

(*i*) $G = (K_1 \cup K_1) \vee (K_1 \cup G_1)$, *where G_1 is a graph on $n - 3$ vertices that realizes $S_{n-4,n-3}$*;

(*ii*) $G = K_1 \vee H$, *where H is a graph on $n - 1$ vertices that realizes $S_{n-1,n-1}$*.

Proposition 2.10. *Suppose that $n \geqslant 6$ and that G is a graph on n vertices. Then, G realizes $S_{i,n}$ with $2 \leqslant i \leqslant n - 2$ if and only if $G = K_1 \vee (K_1 \cup H)$, where H is a graph on $n - 2$ vertices that realizes $S_{i-1,n-2}$.*

Proposition 2.11. *Suppose that $n \geqslant 6$ and that G is a graph on n vertices. G realizes $S_{n-1,n}$ if and only if G is formed in one of the following two ways*:

(*i*) $G = K_1 \vee (K_2 \cup G_1)$, *where G_1 is a graph on $n - 3$ vertices that realizes $S_{2,n-3}$*

(*ii*) $G = K_1 \vee (K_1 \cup H)$, *where H is a graph on $n - 2$ vertices that realizes $S_{n-2,n-2}$*.

Proposition 2.8 completely resolves the existence of graphs realizing the spectrum $S_{i,n}$, where $1 \leqslant i < n$. Furthermore, it is easily deduced from Propositions 2.9–2.11 that if the set $S_{n,n}$ were not realizable for any n, then there is a unique graph, which realizes $S_{i,n}$, for $1 \leqslant i < n$. Furthermore, in Sections 4–5, we prove some analogs of these results for the multisets $S_{\{i,j\}_n^n}$ and $S_{\{i,j\}_n^{n-1}}$.

3 Laplacian spectral properties of the join of graphs and auxiliary properties of multisets $S_{\{i,j\}_n^m}$

Theorem 2.6 characterizes graphs with the largest Laplacian eigenvalue equal to the order of the graphs. However, if the order of the graph is a repeated eigenvalue, then we can obtain more information about the structure as well as spectrum of the corresponding graph.

Theorem 3.1. *Let G be a connected graph of order n, and let n be a Laplacian eigenvalue of G of multiplicity* 2. *Then, $G = F \vee H$, where F is a join of two graphs and H is not a join. Moreover, the number* 1 *does not belong to the Laplacian spectrum of G in this case.*

Proof. Since n is a Laplacian eigenvalue of G, the graph G is a join of two graphs according to Theorem 2.6, i.e., $G = F \vee H$. Suppose that $|F| = p$ and $|H| = n - p$ for some integer number p, $1 \leqslant p \leqslant n - 1$.

Let us denote the Laplacian spectra of the graphs F and H as follows:

$$0 = \mu_1 \leqslant \mu_2 \ldots \leqslant \mu_{p-1} \leqslant \mu_p \quad \text{and} \quad 0 = \lambda_1 \leqslant \lambda_2 \ldots \leqslant \lambda_{n-p-1} \leqslant \lambda_{n-p}. \tag{5}$$

Then, by Theorem 2.5, the Laplacian spectrum of G has the form

$$\sigma_L(G) = \{0, (n-p) + \mu_2, \ldots, (n-p) + \mu_p, p + \lambda_2, \ldots, p + \lambda_{n-p-1}, p + \lambda_{n-p}, n\}. \tag{6}$$

Here, the eigenvalues are not in increasing order.

Since the graph G has exactly two eigenvalues n by assumption, one of the numbers $n - p + \mu_p$ or $p + \lambda_{n-p}$ equals n while the other one is less than n. Without loss of generality, we can suppose that $n - p + \mu_p = n$ and $p + \lambda_{n-p} < n$ which implies that $\mu_p = p$ and $\lambda_{n-p} < n - p$.

Now, from equation (6), we obtain that $\mu_p = p$ is a simple Laplacian eigenvalue of G. Indeed, if μ_p is not simple, then n has multiplicity at least 3 that contradicts the assumption of the theorem. Consequently, the graph F is a join with simple maximal Laplacian eigenvalue, whereas the graph H is not a join by Theorem 2.6, since $\lambda_{n-p} < n - p$.

Since the graph F of order p is a join, we have $p \geqslant 2$ and $n - p \geqslant 1$, so both numbers $(n-p) + \mu_2$ and $p + \lambda_2$ pretending to be the minimal positive Laplacian eigenvalues of G are greater than 1. □

On the contrary, the following theorem provides a necessary and sufficient condition on a graph G to have a Laplacian eigenvalue 1 provided G is a join.

Theorem 3.2. *Let a graph G be a join. The number* 1 *is a Laplacian eigenvalue of G if and only if $G = F \vee K_1$, where F is a disconnected graph of order at least* 2.

Proof. Let G be a join that has a Laplacian eigenvalue 1 and let $G = F \vee H$, where the graph F is of order p while the graph H is of order $n - p$. According to equations (5) and (6), one of the numbers $(n-p) + \mu_2$ or $p + \lambda_2$ equals 1.

If both the numbers $n - p + \mu_2$ and $p + \lambda_2$ equal 1, then $\mu_2 = \lambda_2 = 0$ and $n - p = p = 1$. So, in this case, G is a join of two isolated vertices, i.e., $G = K_1 \vee K_1$, and the Laplacian spectrum of G is equal to $\{0, 2\}$. But this contradicts to the assumption that G has a Laplacian eigenvalue 1.

Thus, we have that one of the numbers $n - p + \mu_2$ and $p + \lambda_2$ is greater than 1. Without loss of generality, we can suppose that $n - p + \mu_2 = 1$ and $p + \lambda_2 > 1$ (as p is a positive integer number). This implies that $n - p = 1$ and $\mu_2 = 0$. So the graph F has at least two Laplacian eigenvalues equal zero. Consequently, $p \geqslant 2$ and F is disconnected [10, p. 298]. As $n - p = 1$, the graph H consists of a single vertex K_1.

Conversely, let $G = F \vee K_1$, where F is a disconnected graph of order at least 2. Since F is disconnected, the multiplicity of the Laplacian eigenvalue 0 is at least 2, so by Theorem 2.5, the Laplacian spectrum of G contains 1 as an eigenvalue of G. □

Now we are in a position to establish a few general facts on the realizability of multisets $S_{\{i,j\}_n^m}$.

Lemma 3.3. *Suppose that $n \geqslant 3$ and G is a graph of order n realizing $S_{\{i,j\}_n^m}$ for $i < j$. Then,*
(*i*) *for $n \equiv 0$ or $3 \bmod 4$, the numbers $(i + j)$ and m are of the same parity;*
(*ii*) *for $n \equiv 1$ or $2 \bmod 4$, the numbers $(i + j)$ and m are of opposite parity.*

Proof. If a graph G realizes $S_{\{i,j\}_n^m}$, then the sum of elements in the multiset $S_{\{i,j\}_n^m}$ equals the sum of degrees of vertices in the graph G (the trace of $L(G)$). So

$$\mathrm{Tr}\ (L(G)) = 2|E(G)| = \frac{n(n+1)}{2} + m - (i + j). \tag{7}$$

Here, m is the repeated eigenvalue of $L(G)$.

Now if $n \equiv 0$ or $3 \bmod 4$, then $\frac{n(n+1)}{2}$ is an even number. Since $\mathrm{Tr}\ (L(G))$ is even as well, one can see that $m - (i + j)$ must be even by equation (7). Therefore, m and $i + j$ are of the same parity. Analogously, if $n \equiv 1$ or $2 \bmod 4$, then $\frac{n(n+1)}{2}$ is odd, so $(i + j)$ and m are of the opposite parity, as required. □

The following lemma is of frequent use in the sequel.

Lemma 3.4. *Let $n \geqslant 3$, if $S_{\{i,j\}_n^m}$ is Laplacian realizable, then so is $S_{\{i+1,j+1\}_{n+2}^{m+1}}$.*

Proof. By Theorem 2.5, one easily obtains that if a graph G realizes the multiset $S_{\{i,j\}_n^m}$, then the Laplacian spectrum of the graph $K_1 \vee (K_1 \cup G)$ is exactly $S_{\{i+1,j+1\}_{n+2}^{m+1}}$. □

Remark 3.5. Note that the converse statement of Lemma 3.4 does not hold, in general. For instance, from Table A2 in Appendix A, it follows that the multiset $S_{\{2,4\}_5^3}$ is Laplacian realizable. However, $S_{\{1,3\}_3^2}$ is not Laplacian realizable, since the only realizable multiset $S_{\{i,j\}_3^m}$ is $S_{\{1,2\}_3^3}$. Analogously, $S_{\{4,6\}_6^3}$ is Laplacian realizable, but $S_{\{3,5\}_4^2}$ is not.

We conclude this section with the following theorem.

Theorem 3.6. *Let G realize $S_{\{i,j\}_n^m}$ for $n \geqslant 9$. Then, G is not a Cartesian product of two graphs.*

Proof. Let G realize $S_{\{i,j\}_n^m}$ with $n \geqslant 9$, and suppose that $G = G_1 \times G_2$. By Theorem 2.7, all the eigenvalues of the graphs G_1 and G_2 belong to the Laplacian spectrum of G, so G_1 or G_2 must have only integer eigenvalues. Moreover, both graphs must have simple eigenvalues. Otherwise, G has at least two multiple eigenvalues according to equation (4).

Thus, the graphs G_1 and G_2 have simple integer Laplacian eigenvalues, so that they realize some multisets $S_{l,r}$ and $S_{q,p}$, respectively, and $n = rp \geqslant 9$. By equation (4) the spectrum of G contains the eigenvalues $(l - 1) + (q + 1) = q + l$ and $(l + 1) + (q - 1) = q + l$, so the Laplacian eigenvalue $q + l$ of the graph G has multiplicity at least 2. Let us show that the graph G has at least one more multiple eigenvalue.

Indeed, if $r = p$, then the graph G has a multiple eigenvalue r because $0 + r$ and $r + 0$ are both the Laplacian eigenvalues of G by equation (4). If $r \neq p$ (say, $r < p$) and $r \in S_{q,p}$, then the graph G again has two equal Laplacian eigenvalues $0 + r$ and $r + 0$.

If $r \neq p$ (say, $r < p$) and $r \notin S_{q,p}$, then $r = q$ and $r - 1 \in S_{q,p}$. Now if $l \neq r - 1$, then by Theorem 2.7, the number $r - 1$ is a Laplacian eigenvalue of G of multiplicity at least 2. If $l = r - 1$ and $r > 2$, then $r - 2$ belongs to $S_{l,r}$ and $S_{q,p}$, so $r - 2 + 0$ and $0 + r - 2$ both belong to the Laplacian spectrum of G. Finally, if $l = r - 1$ and $r = 2$, we have that $p \geqslant 5$ as G is of order $rp \geqslant 9$. Therefore, the eigenvalues 3 and 5 are in the Laplacian spectrum of the graph G_2 realizing $S_{2,p}$ in this case. According to Theorem 2.7, the graph G has a multiple Laplacian eigenvalue 5, since $0 + 5 = 2 + 3 = 5$ belongs to the Laplacian spectrum of G.

Thus, if a graph G realizing $S_{\{i,j\}_n^m}$ is a Cartesian product of two graphs, then it must have at least two multiple Laplacian eigenvalues. □

Remark 3.7. For $n \leqslant 8$, the only multisets $S_{\{i,j\}_n^m}$ realized by Cartesian products are $S_{\{1,3\}_4^2}$, $S_{\{4,6\}_6^3}$, and $S_{\{7,8\}_8^3}$. For $n = 4, 6$, it follows from Tables A1 and A3 in Appendix A. For $n = 8$, we have only five Cartesian products K_2 with a Laplacian integral graph on four vertices. Among them, only $S_{\{7,8\}_8^3}$ is of kind $S_{\{i,n\}_n^m}$.

4 Graphs realizing the multisets $S_{\{i,j\}_n^n}$

In this section, we describe the graphs realizing the multisets $S_{\{i,j\}_n^n}$, and first, we note that if G is a connected graph of order n realizing a set $S_{j,n}$, then according to Theorem 2.5, the graph $K_1 \vee G$ realizes the multiset

$$S_{\{1,j+1\}_{n+1}^{n+1}} = \{0, 2, 3, \dots, j-1, j, j+2, \dots, n, n+1, n+1\}.$$

Thus, we come to the following lemma (cf. [9, Lemma 2.5]).

Lemma 4.1. *If the multiset $S_{j,n}$ is Laplacian realizable, then so is $S_{\{1,j+1\}_{n+1}^{n+1}}$.*

Now we are in a position to describe all the Laplacian realizable multisets $S_{\{i,j\}_n^n}$ (cf. Proposition 2.8).

Theorem 4.2. *Suppose $n \geqslant 3$. The only Laplacian realizable multisets $S_{\{i,j\}_n^n}$, $i < j$ are the following:*
(*i*) *If $n \equiv 0 \bmod 4$, then for each $k = 1, 2, \dots, \frac{n-2}{2}$, $S_{\{1,2k+1\}_n^n}$ is Laplacian realizable;*
(*ii*) *If $n \equiv 1 \bmod 4$, then for each $k = 1, 2, \dots, \frac{n-3}{2}$, $S_{\{1,2k+1\}_n^n}$ is Laplacian realizable;*
(*iii*) *If $n \equiv 2 \bmod 4$, then for each $k = 1, 2, \dots, \frac{n-2}{2}$, $S_{\{1,2k\}_n^n}$ is Laplacian realizable;*
(*iv*) *If $n \equiv 3 \bmod 4$, then for each $k = 1, 2, \dots, \frac{n-1}{2}$, $S_{\{1,2k\}_n^n}$ is Laplacian realizable.*

Proof. The case of $n < 6$ was explained in Section 1 (Tables A1 and A2). Suppose now that $n \geqslant 6$.

By Theorem 3.1, if $S_{\{i,j\}_n^n}$ is Laplacian realizable, then $i = 1$, so the multisets $S_{\{i,j\}_n^n}$ are not Laplacian realizable for $i > 1$.
(i) If $n \equiv 0 \bmod 4$, then $n - 1 \equiv 3 \bmod 4$. Therefore, by Proposition 2.8 (iv), the multiset $S_{2k,n-1}$ is Laplacian realizable for $k = 1, 2, \dots, \frac{n-2}{2}$. Now from Lemma 4.1, we have that $S_{\{1,2k+1\}_n^n}$ is Laplacian realizable for any $k = 1, 2, \dots, \frac{n-2}{2}$. At the same time, since n is even and $m = n$, $S_{\{1,2k\}_n^n}$ are not Laplacian realizable by Lemma 3.3 (i).
(ii) If $n \equiv 1 \bmod 4$, then $n - 1 \equiv 0 \bmod 4$, so for each $k = 1, 2, \dots, \frac{n-3}{2}$, $S_{2k,n-1}$ is Laplacian realizable by Proposition 2.8 (i). Lemma 4.1 implies that $S_{\{1,2k+1\}_n^n}$ is Laplacian realizable for any $k = 1, 2, \dots, \frac{n-3}{2}$, while Lemma 3.3 (ii) gives that $S_{\{1,2k\}_n^n}$ is not Laplacian realizable, since n is odd, so is $m = n$.

The cases (iii) and (iv) can be proved analogously. □

Our next result discusses the structure of graphs realizing $S_{\{1,j\}_n^n}$ for various possible j.

Theorem 4.3. *Let G be a graph of order n, $n \geqslant 5$.*
(*a*) *The graph G realizes $S_{\{1,2\}_n^n}$ if and only if G is formed in one of the following two ways:*
  (*i*) *$G = P_3 \vee (K_1 \cup H)$, where H realizes $S_{n-5,n-4}$ and P_3 is the path graph on three vertices;*
  (*ii*) *$G = K_2 \vee H$, where H realizes $S_{n-2,n-2}$.*
(*b*) *If $3 \leqslant j \leqslant n-2$, then G realizes $S_{\{1,j\}_n^n}$ if and only if $G = K_2 \vee (K_1 \cup H)$, where the graph H realizes $S_{j-2,n-3}$.*
(*c*) *The graph G realizes $S_{\{1,n-1\}_n^n}$ if and only if G is formed in one of the following two ways:*
  (*i*) *$G = K_2 \vee (K_2 \cup H)$, where H realizes $S_{2,n-4}$;*
  (*ii*) *$G = K_2 \vee (K_1 \cup H)$, where H realizes $S_{n-3,n-3}$.*

Proof. For $n = 5$, there is only one graph realizing $S_{\{1,j\}_n^n}$, according to Table A2. This graph is $G = K_2 \vee (K_1 \cup K_2) = K_2 \vee \overline{P_3}$ whose Laplacian spectrum is $S_{\{1,3\}_5^5}$. This graph satisfies condition (b) of the theorem.

For $n = 6$, there are only two graphs realizing $S_{\{1,j\}_n^n}$ according to Table A3. The graph $G_1 = P_3 \vee (K_1 \cup K_2) = P_3 \vee \overline{P_3}$ has the Laplacian spectrum $S_{\{1,2\}_6^6}$ and satisfies condition (a)(i), while the graph $G_2 = K_2 \vee (K_1 \cup P_3)$ with Laplacian spectrum $S_{\{1,4\}_6^6}$ satisfies condition (b). Thus, the theorem is true for $n = 5$ and 6.

Let $n \geqslant 7$, and suppose that G realizes $S_{\{1,2\}_n^n}$. Then, by Theorem 2.4, one has $\sigma_L(\overline{G}) = \{0\} \cup \{0\} \cup S_{n-2,n-2}$. Therefore, one can represent the complement of the graph G as follows: $\overline{G} = K_1 \cup \overline{F}$, where $\overline{F}$ is disconnected and $\sigma_L(\overline{F}) = \{0\} \cup S_{n-2,n-2}$. Again by Theorem 2.4, we obtain $\sigma_L(F) = S_{1,n-1}$. According to Proposition 2.9, there are only two possibilities to construct the graph F.

If $F = (K_1 \cup K_1) \vee (K_1 \cup H)$, where H is a graph on $n-4$ vertices realizing $S_{n-5,n-4}$ (Proposition 2.9 (i)), then we obtain $\overline{G} = K_1 \cup \overline{K_1 \cup K_1} \cup \overline{K_1 \cup H}$. It is clear that $K_1 \cup \overline{K_1 \cup K_1} = K_1 \cup K_2 = \overline{P_3}$, where P_3 is the path graph on three vertices. Thus, we obtain that $G = P_3 \vee (K_1 \cup H)$, where H realizes the set $S_{n-5,n-4}$.

If $F = K_1 \vee H$, where H is a graph on $n-2$ vertices that realizes $S_{n-2,n-2}$ (Proposition 2.9 (ii)), then $\overline{G} = K_1 \cup \overline{K_1 \vee H} = (K_1 \cup K_1) \cup \overline{H}$. Therefore, $G = K_2 \vee H$, since $K_2 = \overline{K_1 \cup K_1}$. It can be easily checked that the graphs mentioned in conditions $(a)(i)$ and $(a)(ii)$ realize $S_{\{1,2\}_n^n}$, so the statement (a) of the theorem is proved.

Let now $n \geqslant 7$ and $3 \leqslant j \leqslant n-2$. If G realizes $S_{\{1,j\}_n^n}$, then from Theorem 2.4, we obtain $\sigma_L(\overline{G}) = \{0\} \cup \{0\} \cup S_{n-j,n-2}$. So the complement of the graph G can be represented as follows: $\overline{G} = K_1 \cup \overline{F}$, where $\overline{F}$ is disconnected and $\sigma_L(\overline{F}) = \{0\} \cup S_{n-j,n-2}$. By Theorem 2.4, we obtain $\sigma_L(F) = S_{j-1,n-1}$. According to Proposition 2.10, we have $F = K_1 \vee (K_1 \cup H)$, where H is a graph on $n-3$ vertices that realizes $S_{j-2,n-3}$. Therefore, $\overline{G} = K_1 \cup K_1 \cup \overline{K_1 \cup H} = \overline{K_2} \cup \overline{K_1 \cup H}$. Thus, $G = K_2 \vee (K_1 \cup H)$, where H is a graph realizing $S_{j-2,n-3}$. Converse statement (b) can be easily checked.

Finally, let $n \geqslant 7$, and suppose that G realizes $S_{\{1,n-1\}_n^n}$. Then, by Theorem 2.4, one has $\sigma_L(\overline{G}) = \{0\} \cup \{0\} \cup S_{1,n-2}$. Therefore, one can represent the complement of the graph G as follows: $\overline{G} = K_1 \cup \overline{F}$, where $\overline{F}$ is disconnected and $\sigma_L(\overline{F}) = \{0\} \cup S_{1,n-2}$. Again from Theorem 2.4, we have $\sigma_L(F) = S_{n-2,n-1}$. According to Proposition 2.11, there are only two possibilities to construct the graph F.

If $F = K_1 \vee (K_2 \cup H)$, where H is a graph on $n-4$ vertices realizing $S_{2,n-4}$ (Proposition 2.11 (i)), then we obtain $\overline{G} = K_1 \cup K_1 \cup \overline{K_2 \cup H}$. Thus, we obtain that $G = K_2 \vee (K_2 \cup H)$, where H realizes the set $S_{2,n-4}$.

If $F = K_1 \vee (K_1 \cup H)$, where H is a graph on $n-3$ vertices that realizes $S_{n-3,n-3}$ (Proposition 2.11 (ii)), then $\overline{G} = K_1 \cup \overline{K_1 \vee (K_1 \cup H)} = (K_1 \cup K_1) \cup \overline{K_1 \cup H}$. Therefore, $G = K_2 \vee (K_1 \cup H)$. It can be easily checked that the graphs mentioned in conditions $(c)(i)$ and $(c)(ii)$ realize $S_{\{1,n-1\}_n^n}$, so the statement (c) of the theorem is proved. □

Remark 4.4. Note that the only graphs of orders 3 and 4 realizing $S_{\{1,j\}_n^n}$ are K_3 and $K_2 \vee 2K_1$, which satisfy the conditions of Theorem 4.3, as well, if we formally specify that the isolated vertex has the Laplacian spectrum $S_{1,1} \coloneqq \{0\}$.

Figure 4 illustrates the graphs realizing the multisets $S_{\{1,j\}_n^n}$ for $n = 6$.

It is easily deduced from Theorem 4.3 that if the sets $S_{n,n}$ were not realizable for any n, then there is a unique graph, which realizes $S_{\{i,j\}_n^n}$, for $1 \leqslant i < j < n$.

5 Graphs realizing multisets $S_{\{i,j\}_n^{n1}}$

In this section, we provide a criterion for $S_{\{i,j\}_n^{n-1}}$ to be Laplacian realizable and describe the graphs realizing them. We start from a necessary condition.

Theorem 5.1. *If $S_{\{i,j\}_n^{n-1}}$ is Laplacian realizable, then the number i is either* 1 *or* 2.

Proof. Suppose, on the contrary, that $S_{\{i,j\}_n^{n-1}}$ is Laplacian realizable for some $i \geqslant 3$ (here $j > i$), and G is a graph realizing this multiset. Then, by Theorem 3, G is a join of two graphs, say, $G = F \vee J$. Let the order of F be p and the order of J be $n-p$. We denote the Laplacian spectra F and J as follows:

$$0 = \mu_1 \leqslant \mu_2 \leqslant \mu_3 \ldots \leqslant \mu_{p-1} \leqslant \mu_p \quad \text{and} \quad 0 = \lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \ldots \leqslant \lambda_{n-p-1} \leqslant \lambda_{n-p}.$$

Then, by Theorem 2.5, the Laplacian spectrum of G has the form

$$\sigma_L(G) = \{0, (n-p) + \mu_2, \ldots, (n-p) + \mu_p, p + \lambda_2, \ldots, p + \lambda_{n-p-1}, p + \lambda_{n-p}, n\}, \tag{8}$$

where the eigenvalues are not in the increasing order.

Since 1 is a simple Laplacian eigenvalue of G, one has either $(n-p) + \mu_2 = 1$ or $p + \lambda_2 = 1$. Without loss of generality, we can suppose that $(n-p) + \mu_2 = 1$ and $p + \lambda_2 \neq 1$. This implies that $n - p = 1$ and $\mu_2 = 0$, so the graph J is an isolated vertex with Laplacian spectrum $\sigma_L(J) = \{0\}$, while the graph F is a union of two disjoint connected graphs, $F = H_1 \cup H_2$. Now we recall that $n-1$ is the only repeated Laplacian eigenvalue of G by assumption, so from equation (8), we have $n - p + \mu_{p-1} = n - p + \mu_p = n - 1$, which implies that $\mu_{p-1} = \mu_p = p - 1 = n - 2$. Thus, from Proposition 2.3, we obtain that $H_1 = K_1$, and the graph H_2 is of order $n-2$ with Laplacian eigenvalue $n-2$ of multiplicity 2.

On the other hand, the number 2 is also a simple Laplacian eigenvalue of G, so $(n-p) + \mu_3 = 2$ and hence $\mu_3 = 1$. Thus, H_2 has a Laplacian eigenvalue 1 that contradicts Theorem 3.1. Therefore, the multisets $S_{\{i,j\}_n^{n-1}}$ are not Laplacian realizable for $i \geqslant 3$. □

5.1 Graphs realizing the multisets $S_{\{2,j\}_n^{n1}}$

First, we describe the multisets $S_{\{2,j\}_n^{n-1}}$ because they are used in the description of the multisets $S_{\{1,j\}_n^{n-1}}$. The following theorem lists all the Laplacian realizable multisets $S_{\{2,j\}_n^{n-1}}$.

Theorem 5.2. *Let $n \geqslant 3$, and let G be a connected graph of order n. The only Laplacian realizable multisets $S_{\{2,j\}_n^{n-1}}$ are the following*:

(*i*) *If $n \equiv 0 \bmod 4$, then for each $k = 1, 2, \ldots, \frac{n-4}{2}$, $S_{\{2,2k+1\}_n^{n-1}}$ is Laplacian realizable*;

(*ii*) *If $n \equiv 1 \bmod 4$, then for each $k = 1, 2, \ldots, \frac{n-3}{2}$, $S_{\{2,2k+1\}_n^{n-1}}$ is Laplacian realizable*;

(*iii*) *If $n \equiv 2 \bmod 4$, then for each $k = 1, 2, \ldots, \frac{n-4}{2}$, $S_{\{2,2k+2\}_n^{n-1}}$ is Laplacian realizable*;

(*iv*) *If $n \equiv 3 \bmod 4$, then for each $k = 1, 2, \ldots, \frac{n-5}{2}$, $S_{\{2,2k+2\}_n^{n-1}}$ is Laplacian realizable.*

Proof. For $n = 5$, the only Laplacian realizable $S_{\{i,j\}_n^{n-1}}$ multiset is $S_{\{2,3\}_5^4}$ (Table A2 in Appendix A). This agrees with the condition (ii) of the theorem. Suppose now that $n \geqslant 6$.

(i) If $n \equiv 0 \bmod 4$, then $n - 2 \equiv 2 \bmod 4$, then for $k = 1, 2, \ldots, \frac{n-4}{2}$, the multisets $S_{\{1,2k\}_{n-2}^{n-2}}$ are Laplacian realizable by Theorem 4.2(iii). From Lemma 3.4, it follows that $S_{\{2,2k+1\}_n^{n-1}}$ is Laplacian realizable for each

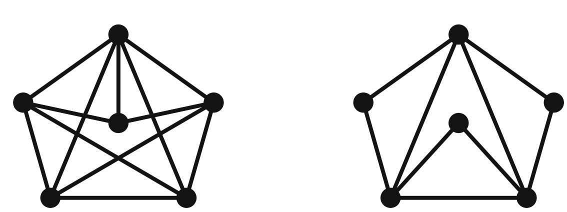

Figure 4: Graphs realizing $S_{\{1,2\}_6^6}$ and $S_{\{1,4\}_6^6}$, respectively.

$k = 1, 2, \ldots, \frac{n-4}{2}$. At the same time, the multisets $S_{\{2,2k\}_n^{n-1}}$ are not Laplacian realizable for any k by Lemma 3.3 (i), since the repeated eigenvalue $m = n - 1$ is an odd number in this case.

(ii) For $n \equiv 1 \bmod 4$, we have $n - 2 \equiv 3 \bmod 4$. Thus, for $k = 1, 2, \ldots, \frac{n-3}{2}$, the multisets $S_{\{1,2k\}_{n-2}^{n-2}}$ are Laplacian realizable by Theorem 4.2(iv). Now Lemma 3.4 implies that $S_{\{2,2k+1\}_n^{n-1}}$ are Laplacian realizable for $k = 1, 2, \ldots, \frac{n-3}{2}$. As $m = n - 1$ is even, from Lemma 3.3(ii), it follows that the multisets $S_{\{2,2k\}_n^{n-1}}$ are not Laplacian realizable for any k.

The cases (iii) and (iv) can be proved analogously with the use of Lemmas 3.3, 4.2 and 3.4, respectively. □

In the following theorem, we discuss construction of graphs realizing the multisets $S_{\{2,j\}_n^{n-1}}$.

Theorem 5.3. *Let $n \geqslant 5$, and let G be a connected graph of order n. Then, G realizes $S_{\{2,j\}_n^{n-1}}$ if and only if $G = K_1 \vee (K_1 \cup H)$, where H is a graph on $n - 2$ vertices realizing $S_{\{1,j-1\}_{n-2}^{n-2}}$;*

Proof. Let G realize $S_{\{2,j\}_n^{n-1}}$. Then, $G = G_1 \vee G_2$ by Theorem 2.6, so $\overline{G} = \overline{G_1} \cup \overline{G_2}$. From Theorem 2.4, it follows that $\sigma_L(\overline{G}) = \{0\} \cup S_{\{n-j,n-2\}_{n-1}^{1}}$. Thus, by Proposition 2.3, we obtain that $G_1 = K_1$, and G_2 is of order $n - 1$, so that $\sigma_L(\overline{G_2}) = S_{\{n-j,n-2\}_{n-1}^{1}}$. Using Theorem 2.4, we obtain $\sigma_L(G_2) = \{0\} \cup S_{\{1,j-1\}_{n-2}^{n-2}}$. Again Proposition 2.3 gives us that $G_2 = K_1 \cup H$, where H is a graph on $n - 2$ vertices realizing $S_{\{1,j-1\}_{n-2}^{n-2}}$. Consequently, $G = K_1 \vee (K_1 \cup H)$, as required.

Conversely, if $G = K_1 \vee (K_1 \cup H)$, where H is a graph on $n - 2$ vertices realizing $S_{\{1,j-1\}_{n-2}^{n-2}}$, then from Theorem 2.5, it follows that G realizes $S_{\{2,j\}_n^{n-1}}$. □

5.2 Graphs realizing the multisets $S_{\{1,j\}_n^{n1}}$

Now we are in a position to study the realizability of the multisets $S_{\{1,j\}_n^{n-1}}$. It turns out that for a fixed n, there is only one Laplacian realizable multiset of this kind.

Theorem 5.4. *Let G be a simple connected graph of order n, $n \geqslant 6$.*
(*i*) *For $n \equiv 0$ or $1 \bmod 4$, the multiset $S_{\{1,j\}_n^{n-1}}$ is Laplacian realizable if and only if $j = 2$.*
(*ii*) *For $n \equiv 2$ or $3 \bmod 4$, the multiset $S_{\{1,j\}_n^{n-1}}$ is Laplacian realizable if and only if $j = 3$.*

Proof. (i) If $n \equiv 0$ or $1 \bmod 4$ (so that $n \geqslant 8$), then from Lemma 3.3, it follows that the multisets $S_{\{1,2k+1\}_n^{n-1}}$ are not Laplacian realizable for any k. Let us now find out which multisets of the form $S_{\{1,2k\}_n^{n-1}}$ are realizable.

Suppose first that $k \geqslant 2$. Let a graph G realize $S_{\{1,2k\}_n^{n-1}}$. Then, we have

$$\sigma_L(\overline{G}) = \{0, 0, 1, 1, 2, \ldots, n - (2k + 1), n - (2k - 1), \ldots, n - 3, n - 2\}. \tag{9}$$

Thus, the graph $\overline{G}$ is a union of two disjoint connected graphs, $\overline{G} = \overline{H_1} \cup \overline{H_2}$. From Proposition 2.3, it follows that one of the components, say H_2, must be of order at least $n - 2$.

If H_2 has $n - 2$ vertices, then H_1 has two vertices, and therefore $\sigma_L(\overline{H_1}) = \{0, 2\}$. By Theorem 2.4, we obtain $\sigma_L(H_1) = \{0, 0\}$, so $H_1 = 2K_1$. Now from equation (9) we obtain $\sigma_L(\overline{H_2}) = S_{\{2,n-2k\}_{n-2}^{1}}$, and Theorem 2.4 implies $\sigma_L(H_2) = \{0, 0, 1, 2, \ldots, 2k - 4, 2k - 3, 2k - 1, 2k, \ldots, n - 5, n - 3, n - 3\}$. By Proposition 2.3, we have that $H_2 = K_1 \cup H_3$, where $\sigma_L(H_3) = S_{\{2k-2,n-4\}_{n-3}^{n-3}}$, so $1 \in \sigma_L(H_3)$. This contradicts Theorem 3.1.

If H_2 has $n - 1$ vertices, then $H_1 = K_1$, so that $\overline{G} = K_1 \cup \overline{H_2}$ with $\sigma_L(\overline{H_2}) = S_{\{n-2k,n-1\}_{n-1}^{1}}$. Now Theorem 2.4 gives us $\sigma_L(H_2) = S_{\{2k-1,n-1\}_{n-1}^{n-2}}$. However, for $k \geqslant 2$, such multisets are not Laplacian realizable according to Theorem 5.1. Thus, we obtained that the multisets $S_{\{1,2k\}_n^{n-1}}$ are not Laplacian realizable for $k \geqslant 2$.

Assume now that the graph G realizes $S_{\{1,2\}_n^{n-1}}$. According to Theorem 2.4,

$$\sigma_L(\overline{G}) = \{0, 0, 1, 1, 2, \dots, n-4, n-3\}.$$

Thus, the graph $\overline{G}$ is a union of two disjoint connected graphs, $\overline{G} = \overline{H_1} \cup \overline{H_2}$. From Proposition 2.3 it follows that one of the components, say, H_2 must be of order at least $n-3$.

Let H_2 have $n-3$ vertices. Then, H_1 has three vertices, and the only possible spectrum[1] of $\overline{H_1}$ is $\{0, 1, 3\}$ which implies $H_1 = K_1 \cup K_2$ by Theorem 2.4. Therefore, $\sigma_L(\overline{H_2}) = S_{3,n-3}$. By Theorem 2.4, one obtains $\sigma_L(H_2) = \{0\} \cup S_{n-6,n-4}$. By Theorem 2.8, these sets are Laplacian realizable for $n \equiv 0$ or $1 \bmod 4$. Thus, the graph $G = (K_1 \cup K_2) \vee (K_1 \cup H)$, where the graph H realizes $S_{n-6,n-4}$.

If H_2 has $n-2$ vertices, then H_1 has two vertices and $\sigma_L(\overline{H_1}) = \{0, 2\}$, so that $H_1 = 2K_1$. Hence, $\sigma_L(\overline{H_2}) = S_{\{2,n-2\}^1_{n-2}}$. By Theorem 2.4, we have that the graph H_2 must realize the multiset $S_{\{n-4,n-2\}^{n-3}_{n-2}}$. However, this multiset is not Laplacian realizable for $n \geqslant 8$ according to Theorem 5.1. Consequently, H_2 cannot be of order $n-2$.

Suppose now that H_2 has $n-1$ vertices. Then, $H_1 = K_1$, and $\sigma_L(\overline{H_2}) = S_{\{n-2,n-1\}^1_{n-1}}$. Theorem 2.4 gives $\sigma_L(H_2) = S_{\{1,n-1\}^{n-2}_{n-1}}$. However, this multiset is not Laplacian realizable as we established above.

Thus, we obtained that if $S_{\{1,j\}^{n-1}_n}$ is Laplacian realizable, then $j = 2$, and the graph $G = (K_1 \cup K_2) \vee (K_1 \cup H)$ with $\sigma_L(H) = S_{n-6,n-4}$ realizes $S_{\{1,2\}^{n-1}_n}$. Converse statement is obvious.

(ii) Let now $n \equiv 2$ or $3 \bmod 4$. Then, from Lemma 3.3, it follows that the multisets $S_{\{1,2k\}^{n-1}_n}$ are not Laplacian realizable for any k. We now find out which multisets of the form $S_{\{1,2k+1\}^{n-1}_n}$ are realizable.

As above, we first consider the case $k \geqslant 2$. Let the graph G realize $S_{\{1,2k+1\}^{n-1}_n}$. Then, we have

$$\sigma_L(\overline{G}) = \{0, 0, 1, 1, 2, \dots, n-2k-1, n-2k+1, \dots, n-3, n-2\}.$$

Thus, the graph $\overline{G}$ is a union of two disjoint connected graphs, $\overline{G} = \overline{H_1} \cup \overline{H_2}$. From Proposition 2.3, it follows that one of the components, say H_2, must be of order at least $n-2$.

If H_2 is of order $n-2$, then the Laplacian spectrum of $\overline{H_1}$ must be $\{0, 2\}$, so that $H_1 = 2K_1$. Consequently, $\sigma_L(\overline{H_2}) = S_{\{2,n-2k-1\}^1_{n-2}}$, and by Theorem 2.4 $\sigma_L(H_2) = \{0\} \cup S_{\{2k-1,n-4\}^{n-3}_{n-3}}$. By Proposition 2.3, we have $H_2 = K_1 \cup H_3$, where $\sigma_L(H_3) = S_{\{2k-1,n-4\}^{n-3}_{n-3}}$. Since the multiset $S_{\{2k-1,n-4\}^{n-3}_{n-3}}$ is not Laplacian realizable for $k \geqslant 2$ (Theorem 3.1), H_2 cannot be of order $n-2$.

Let now the order of H_2 be $n-1$. Then, we have $H_1 = K_1$, and $\sigma_L(\overline{H_2}) = S_{\{n-2k-1,n-1\}^1_{n-1}}$, so that $\sigma_L(H_2) = S_{\{2k,n-1\}^{n-2}_{n-1}}$ by Theorem 2.4. Now, from Theorem 5.1, it follows that for $k \geqslant 2$ such multisets are not realizable. Thus, if $S_{\{1,2k+1\}^{n-1}_n}$ is Laplacian realizable, then k can only be equal 1 (if any).

Suppose now that a graph G realizes the multiset $S_{\{1,3\}^{n-1}_n}$ for $n \equiv 2$ or $3 \bmod 4$. Then, its complement has the following Laplacian spectrum:

$$\sigma_L(\overline{G}) = \{0, 0, 1, 1, 2, \dots, n-4, n-2\}.$$

As above, we obtain that $\overline{G} = \overline{H_1} \cup \overline{H_2}$, where one of the components, say H_2, must be of order at least $n-2$.

If the order of H_2 equals $n-2$, then necessarily $\sigma_L(\overline{H_1}) = \{0, 2\}$, and $H_1 = 2K_1$, while $\sigma_L(\overline{H_2}) = S_{\{2,n-3\}^1_{n-2}}$. Thus, by Theorem 2.4 we have $\sigma_L(H_2) = \{0\} \cup S_{\{1,n-4\}^{n-3}_{n-3}}$. By Proposition 2.3, we have $H_2 = K_1 \cup H_3$, where $\sigma_L(H_3) = S_{\{1,n-4\}^{n-3}_{n-3}}$. Finally, we obtain $G = 2K_1 \vee (K_1 \cup F)$, where $\sigma_L(F) = S_{\{1,n-4\}^{n-3}_{n-3}}$.

Let H_2 be of order $n-1$. Then, $H_1 = K_1$, and $\sigma_L(\overline{H_2}) = S_{\{n-3,n-1\}^1_{n-1}}$, so that $\sigma_L(H_2) = S_{\{2,n-1\}^{n-2}_{n-1}}$. This case is realizable if the multiset $S_{\{2,n-1\}^{n-2}_{n-1}}$ is Laplacian realizable. □

From the proof of Theorem 5.4, one can easily see the structure of graphs realizing $S_{\{1,j\}^{n-1}_n}$.

Theorem 5.5. *Let G be a graph of order n, $n \geqslant 6$.*

(a) *The graph G realizes $S_{\{1,2\}^{n-1}_n}$ if and only if $n \equiv 0$ or $1 \bmod 4$, and $G = (K_1 \cup K_2) \vee (K_1 \cup H)$, where H realizes $S_{n-6,n-4}$;*

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1 The only connected graphs on three vertices are P_3 and K_3, but the Laplacian spectrum of K_3 contains repeated 3.

(*b*) *The graph G realizes* $S_{\{1,3\}_n^{n-1}}$ *if and only if* $n \equiv 2$ *or* $3 \bmod 4$, *and G is formed in one of the following two ways*:

(*i*) $G = (K_1 \cup K_1) \vee (K_1 \cup H)$, *where the graph H realizes* $S_{\{1,n-4\}_{n-3}^{n-3}}$;

(*ii*) $G = K_1 \vee F$, *where the graph F realizes* $S_{\{2,n-1\}_{n-1}^{n-2}}$.

6 $S_{\{i,n\}_n^m}$-conjecture

This section is devoted to describing necessary conditions on the multiset $S_{\{i,n\}_n^m}$ to be Laplacian realizable. Since the results below do not resolve the question of the realizability of $S_{\{i,n\}_n^m}$, we pose Conjecture 6.4 at the end of this section.

First, we note that by Theorem 2.6, any graph realizing $S_{\{i,n\}_n^m}$ is not a join of two graphs. However, it can be a Cartesian product of two graphs. Indeed, the ladder graph on six vertices, $P_2 \times P_3$, realizes the multiset $S_{\{4,6\}_6^3}$ (Table A3). Moreover, from Theorem 2.4, it immediately follows that if G realizes a multiset $S_{\{i,n\}_n^m}$, then its complement realizes a multiset of the same kind, as well.

Proposition 6.1. *If G realizes* $S_{\{i,n\}_n^m}$, *then* $\overline{G}$ *is a connected graph that realizes* $S_{\{n-i,n\}_n^{n-m}}$.

Thus, together with $S_{\{4,6\}_6^3}$, we obtain that $S_{\{2,6\}_6^3}$ is Laplacian realizable (Figure 2 and Table A3).

Note that since the equalities $n - i = i$ and $n - m = m$ cannot hold simultaneously due to $i \neq m$, the graphs realizing $S_{\{i,n\}_n^m}$ are not self-complementary, so the number of graphs realizing multisets $S_{\{i,n\}_n^m}$ is even (if finite).

According to the study by Cvetković et al. [5, p. 286–289] and Tables A1–A3 in Appendix A, there are no graphs of order less than 6 realizing $S_{\{i,n\}_n^m}$ and exactly two such graphs of order 6. It can be shown that the multisets $S_{\{i,n\}_n^m}$ are not Laplacian realizable for $n = 7$ and for any prime n.

Proposition 6.2. *If* $n \geqslant 7$ *is a prime number, then* $S_{\{i,n\}_n^m}$ *is not Laplacian realizable.*

Proof. Let G realizes a multiset $S_{\{i,n\}_n^m}$, and let the eigenvalues of $L(G)$ be given by $0, \mu_2, \ldots, \mu_n$. According to the matrix tree theorem, see, for instance, [5, p. 190], the number of spanning trees of G is equal to

$$\tau(G) = \frac{\lambda_2 \cdots \lambda_n}{n} = \frac{(n-1)! \cdot m}{n \cdot i}.$$

As the number of spanning trees is an integer, n must divide $\frac{(n-1)! \cdot m}{i}$, and so, in particular, n cannot be a prime number, a contradiction. □

From Lemma 3.3, we obtain another simple observation.

Proposition 6.3. *Let n be a positive integer,* $n \geqslant 6$. *If* $n \equiv 0$ *or* $1 \bmod 4$ *and* $i - m$ *is odd, then* $S_{\{i,n\}_n^m}$ *is not Laplacian realizable.*

As we noted above, graphs realizing $S_{\{i,n\}_n^m}$ are not joins but may be Cartesian products. However, by Theorem 3.6 such graphs can be of order less than 9. As we mentioned in Remark 3.7, there are only three Cartesian products of order less than 9, two of which realize multisets of kind $S_{\{i,n\}_n^m}$. One of them is $S_{\{4,6\}_6^3}$ mentioned above in this section. Another one is $S_{\{7,8\}_8^3}$. The graph realizing this multiset is the Cartesian product of K_2 and $K_1 \vee \overline{P_3} = K_1 \vee (K_1 \cup K_2)$, whose Laplacian spectrum is $S_{2,4}$. By Theorem 6.1, the complement of $G = K_2 \times [K_1 \vee (K_1 \cup K_2)]$ realizes a multisets of kind $S_{\{i,n\}_n^m}$ as well. Namely, $\sigma_L(\overline{G}) = S_{\{1,8\}_8^5}$. Both graphs are depicted in Figure 3.

Thus, for $n \leqslant 8$, we found four Laplacian realizable multiset $S_{\{i,n\}_n^m}$ and two Cartesian products and their complements. This fact together with Theorem 3.6 inspired us to make the following conjecture.

Conjecture 6.4. *For $n \geqslant 4$, the only Laplacian realizable multiset of kind $S_{\{i,n\}_n^m}$ are $S_{\{4,6\}_6^3}$, $S_{\{2,6\}_6^3}$, $S_{\{7,8\}_8^3}$, and $S_{\{1,8\}_8^5}$.*

In light of this conjecture, we have the following.

Conjecture 6.5. *For $n \geqslant 5$ and for each admissible i and j, where $1 \leqslant i < j < n$, the spectra $S_{\{i,j\}_n^n}$ and $S_{\{i,j\}_n^{n-1}}$ are realized by unique graphs.*

Note that Conjecture 6.5 follows from Conjecture 6.4, the $S_{n,n}$-conjecture, and Theorems 4.3, 5.3, and 5.5.

7 Conclusion

Inspired by the work [9] by Fallat et al. on the sets $S_{i,n}$, we study the Laplacian realizability of the multisets $S_{\{i,j\}_n^m}$. We completely described the number of graphs realizing $S_{\{i,j\}_n^m}$ for $m = n$ and $m = n - 1$ and gave an algorithm for constructing those graphs. In addition, we also discussed some aspects of the Conjecture 6.4 about the realizability of the multiset $S_{\{i,n\}_n^m}$. This conjecture correlates with the conjecture on the realizability of sets $S_{n,n}$ introduced by Fallat et al. [9]. The next step we took was to consider the repeated eigenvalues $m = 1$ and $m = 2$. It turned out [15] that the graphs realizing multisets $S_{\{i,j\}_n^1}$ and $S_{\{i,j\}_n^2}$ are related but have a different construction than the graphs investigated in the present work.

There is another approach to study the multisets $S_{\{i,j\}_n^m}$. Instead of fixing the number m and seeking for admissible numbers i and j absent in Laplacian spectra of graphs as we did in this work and in [15], one can fix i and j and study the admissible m and n. For example, from Tables A1–A3 in Appendix A, it follows that the multiset $S_{\{1,3\}_4^2}$ and $S_{\{1,3\}_4^4}$, $S_{\{2,4\}_5^1}$ and $S_{\{2,4\}_5^3}$, and $S_{\{2,4\}_6^3}$ and $S_{\{2,4\}_6^5}$ are Laplacian realizable. Moreover, despite of Conjecture 6.5 for $m = n$ and $m = n - 1$, there exist numbers m such that some multisets are realizable by more than one graph. For instance, the multiset $S_{\{1,3\}_7^4}$ is realized by two non-isomorphic graphs $2K_1 \vee (K_1 \cup C_4)$ and $K_1 \vee \overline{P_2 \times P_3}$ (Table A4). Thus, for the future research, we pose the following problem.

Problem 1. Given a multiset $S_{\{i,j\}_n^m}$ for fixed n and $i < j$, find all admissible m such that $S_{\{i,j\}_n^m}$ is Laplacian realizable. Find all m admitting realizability of $S_{\{i,j\}_n^m}$ by a few non-isomorphic graphs.

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Appendix

A List of Laplacian integral graphs realizing $S_{\{i,j\}_n^m}$ up to order 7

From the studies by Cvetković and Petrić [4] and Cvetković et al. [5, p. 286], it follows that there are totally six connected graphs of order 4, 21 connected graphs of order 5, and 112 ones of order 6. In [5, p. 301–304], the authors depicted graphs of the order less than 6 and found their Laplacian spectra. Thus, it is known that there are exactly five Laplacian integral graphs of order 4 and 12 Laplacian graphs of order 5. Cvetković and Petrić [4] depicted all these graphs of order 6 without calculating their Laplacian spectra. In the study by Grone and Merris [12], it was mentioned that there are exactly 37 Laplacian integral graphs. Here, we list all graphs realizing multisets $S_{\{i,j\}_n^m}$ up to order 6. We also provide some graphs of order 7 realizing those multisets. However, finding all the graphs of order 7 realizing $S_{\{i,j\}_n^m}$ is an open problem.

Below and throughout the text of the article, we use the following notations: $nK_1 = \bigcup_{i=1}^{n} K_1$, S_n means the star graph on n vertices, F_n is the friendship graph on $2n+1$ vertices, C_n is the cycle on n vertices, P_n is the path graph on n vertices, and $K_{p_1,\dots,p_k}$ is the complete k-partite graph on $p_1 + \cdots + p_k$ vertices.

Table A1: Laplacian integral graphs realizing $S_{\{i,j\}_n^m}$ for $n = 4$.

Construction	Laplacian spectrum	$S_{\{i,j\}_n^m}$
$S_4 \cong K_{3,1} \cong K_1 \vee 3K_1$	$\{0,1,1,4\}$	$S_{\{2,3\}_4^1}$
$C_4 \cong K_{2,2} \cong 2K_1 \vee 2K_1$	$\{0,2,2,4\}$	$S_{\{1,3\}_4^2}$
$K_{1,1,2} \cong K_2 \vee 2K_1$, the diamond graph	$\{0,2,4,4\}$	$S_{\{1,3\}_4^4}$

Table A2: Laplacian integral graphs realizing $S_{\{i,j\}_n^m}$ for $n = 5$

Construction	Laplacian spectrum	$S_{\{i,j\}_n^m}$
$K_1 \vee \overline{K_{1,1,2}}$	$\{0,1,1,3,5\}$	$S_{\{2,4\}_5^1}$
$K_{3,2}$	$\{0,2,2,3,5\}$	$S_{\{1,4\}_5^2}$
$F_2 \cong K_1 \vee 2K_2$, the butterfly graph	$\{0,1,3,3,5\}$	$S_{\{2,4\}_5^3}$
$K_1 \vee \overline{S_4}$	$\{0,1,4,4,5\}$	$S_{\{2,3\}_5^4}$
$K_2 \vee \overline{P_3}$	$\{0,2,4,5,5\}$	$S_{\{1,3\}_5^5}$

Table A3: Laplacian integral graphs that realizing $S_{\{i,j\}_n^m}$ for $n = 6$

Graph	Lapl. Spectrum	$S_{\{i,j\}_n^m}$	Graph	Lapl. Spectrum	$S_{\{i,j\}_n^m}$
$K_1 \vee (2K_1 \cup P_3)$	$\{0,1,1,2,4,6\}$	$S_{\{3,5\}_6^1}$	$K_1 \vee \overline{K_{2,3}}$	$\{0,1,3,4,4,6\}$	$S_{\{2,5\}_6^4}$
$K_1 \vee (K_1 \cup S_4)$	$\{0,1,2,2,5,6\}$	$S_{\{3,4\}_6^2}$	$2K_1 \vee \overline{K_{1,3}}$	$\{0,2,4,5,5,6\}$	$S_{\{1,3\}_6^5}$
$K_1 \vee (K_1 \cup C_4)$	$\{0,1,3,3,5,6\}$	$S_{\{2,4\}_6^3}$	$K_1 \vee (K_1 \cup K_{1,1,2})$	$\{0,1,3,5,5,6\}$	$S_{\{2,4\}_6^5}$
$\overline{P_2 \times P_3}$	$\{0,1,3,3,4,5\}$	$S_{\{2,6\}_6^3}$	$P_3 \vee \overline{P_3}$	$\{0,3,4,5,6,6\}$	$S_{\{1,2\}_6^6}$
$P_2 \times P_3$	$\{0,1,2,3,3,5\}$	$S_{\{4,6\}_6^3}$	$K_2 \vee (K_1 \cup P_3)$	$\{0,2,3,5,6,6\}$	$S_{\{1,4\}_6^6}$

Table A4: Some Laplacian integral graphs that realizing $S_{\{i,j\}_n^m}$ for $n = 7$

Construction	Laplacian spectrum	$S_{\{i,j\}_n^m}$
$K_1 \vee (P_3 \cup \overline{P_3})$	$\{0, 1, 1, 2, 3, 4, 7\}$	$S_{\{5,6\}_7^1}$
$K_1 \vee [2K_1 \cup (K_1 \vee \overline{P_3})]$	$\{0, 1, 1, 2, 4, 5, 7\}$	$S_{\{3,6\}_7^1}$
$2K_1 \vee (K_1 \cup S_4)$	$\{0, 2, 3, 3, 5, 6, 7\}$	$S_{\{1,4\}_7^3}$
$2K_1 \vee (K_1 \cup C_4)$	$\{0, 2, 4, 4, 5, 6, 7\}$	$S_{\{1,3\}_7^4}$
$K_1 \vee \overline{P_2 \times P_3}$	$\{0, 2, 4, 4, 5, 6, 7\}$	$S_{\{1,3\}_7^4}$
$K_1 \vee (P_2 \times P_3)$	$\{0, 2, 3, 4, 4, 6, 7\}$	$S_{\{1,5\}_7^4}$
$2K_1 \vee (K_2 \cup P_3)$	$\{0, 2, 3, 4, 5, 5, 7\}$	$S_{\{1,5\}_7^5}$
$K_1 \vee [K_1 \cup (K_1 \vee \overline{S_4})]$	$\{0, 1, 2, 5, 5, 6, 7\}$	$S_{\{3,4\}_7^5}$
$2K_1 \vee (K_1 \cup K_{1,1,2})$	$\{0, 2, 4, 5, 6, 6, 7\}$	$S_{\{1,3\}_7^6}$
$K_1 \vee [K_1 \cup (K_2 \vee \overline{P_3})]$	$\{0, 1, 3, 5, 6, 6, 7\}$	$S_{\{2,4\}_7^6}$
$K_1 \vee [2K_1 \vee (K_1 \cup P_3)]$	$\{0, 3, 4, 5, 6, 7, 7\}$	$S_{\{1,2\}_7^7}$
$K_2 \vee [K_1 \cup (K_1 \vee \overline{P_3})]$	$\{0, 2, 3, 5, 6, 7, 7\}$	$S_{\{1,4\}_7^7}$
$K_2 \vee (K_2 \cup P_3)$	$\{0, 2, 3, 4, 5, 7, 7\}$	$S_{\{1,6\}_7^7}$