Research Article

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Normalized solutions for a class of scalar field equations involving mixed fractional Laplacians

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Abstract: The purpose of this article is to establish sharp conditions for the existence of normalized solutions to a class of scalar field equations involving mixed fractional Laplacians with different orders. This study includes the case when one operator is local and the other one is non-local. This type of equation arises in various fields ranging from biophysics to population dynamics. Due to the importance of these applications, this topic has very recently received an increasing interest. In this article, we provide a complete description of the existence/non-existence of ground state solutions using constrained variational approaches. This study addresses the mass subcritical, critical and supercritical cases. Our model presents some difficulties due to the "conflict" between the different orders and requires a novel analysis, especially in the mass supercritical case. We believe that our results will open the door to other valuable contributions in this important field.

Keywords: variational method, normalized solutions, existence/non-existence, mixed fractional Laplacians

MSC 2020: 35J50, 35Q41, 35Q55, 37K45

1 Introduction

Fractional Schrödinger equations have attracted many scientists, working in different fields, during the past few decades. They have become an entire branch of non-linear analysis. Motivated by numerous and various applications, a lot of significant advances have been made. See, e.g., [28–30] for the existence and symmetry of minimizers, [17,18] for the Cauchy problem, and [31] for some fractional functional inequalities, to only cite a few. The main class of Schrödinger equations that have been studied until very recently is given by:

$$\begin{cases} i\partial_t\psi(t,x) + (-\Delta)^s\psi(t,x) - f(|\psi(t,x)|)\psi(t,x) = 0, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d, \\ \psi(0,x) = \psi_0(x), & x \in \mathbb{R}^d, \end{cases} \tag{1.1}$$

where $(-\Delta)^s$ is the fractional Laplacian, that will be formally defined below, and f is a non-negative function defined on $\mathbb{R}^+$.

Motivated by various applications, we are concerned with

$$\begin{cases} i\partial_t\psi(t,x) + (-\Delta)^{s_1}\psi(t,x) + (-\Delta)^{s_2}\psi(t,x) - |\psi(t,x)|^p\psi(t,x) = 0, \\ \psi(0,x) = \psi_0(x), \end{cases} \tag{1.2}$$

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where $0 < s_1 < s_2 < 1$, $p > 0$, and $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^d$. Equation (1.2) naturally arises in the superposition of two stochastic processes with a different random walk and a Lévy flight. This is the case when a particle can follow either of these two processes according to a certain probability. The associated limit diffusion is described by a sum of two fractional Laplacians with different orders, see [6] and references therein for a more detailed account. Equation (1.2) plays a crucial role in chemical reaction design [1], plasma physics [23], and biophysics [45]. Very recently, [21] tackled another interesting problem related to equation (1.2). More precisely, it turns out that the mixed fractional Laplacians model the population dynamics, see [21, Appendix B] for a nice detailed account. Equation (1.2) also models some heart anomalies caused by arteries issues. Those heart problems can be modeled thanks to the superposition of two to five mixed fractional Laplacians since it is not necessarily the same anomaly in the five arteries, see [22,37,39,40].

Most of these models have been lately "discovered." This would explain why only a very few papers have addressed (1.2) so far. To the best of our knowledge, [2,6,12,16,19] and [21] count among the short list of contributions to the mixed fractional Laplacians-type equation (1.2). Many reasons lead us to believe that this field will soon receive an increasing interest.

In this article, we are interested in standing wave solutions of (1.2), which are solutions of the form $\psi(t, x) := e^{-i\lambda t}u(x)$, where $\lambda \in \mathbb{R}$ denotes the frequency. The function $u(x)$ then satisfies the following stationary scalar field equation:

$$(-\Delta)^{s_1}u(x) + (-\Delta)^{s_2}u(x) + \lambda u(x) - |u(x)|^p u(x) = 0, \quad x \in \mathbb{R}^d, \tag{1.3}$$

where $0 < s_1 < s_2 < 1$, $p > 0$, $\lambda \in \mathbb{R}$. The fractional Laplacian is given by

$$(-\Delta)^{s_i}u := C_{d,s_i}\int_{\mathbb{R}^d} \frac{u(x) - u(y)}{|x - y|^{d+2s_i}}\mathrm{d}y, \quad i = 1, 2,$$

with $C_{d,s_i} := 2^{2s_i}\pi^{-\frac{d}{2}}s_i\frac{\Gamma\left(\frac{d+2s_i}{2}\right)}{\Gamma(1-s_i)}$, where Γ is the Gamma function, see [42].

Since equation (1.3) has an underlying variational formulation, we will focus on its variational structure. First, we introduce the fractional Sobolev space

$$H^s(\mathbb{R}^d) := \left\{u : \mathbb{R}^d \to \mathbb{R} \quad | \quad u \in L^2(\mathbb{R}^d), \quad \text{and} \quad \int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|u(x) - u(y)|^2}{|x - y|^{d+2s}}\mathrm{d}x\mathrm{d}y < \infty\right\}, \quad s > 0,$$

and for $u \in H^s(\mathbb{R}^d)$, denote

$$\|\nabla_s u\|_2^2 := \int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|u(x) - u(y)|^2}{|x - y|^{d+2s}}\mathrm{d}x\mathrm{d}y.$$

For $0 < s_1 < s_2 < 1$, $d \geq 1$, we recall from [15, Remark 1.4.1] that $H^{s_2}(\mathbb{R}^d) \hookrightarrow H^{s_1}(\mathbb{R}^d)$, hence,

$$H^{s_1}(\mathbb{R}^d) \cap H^{s_2}(\mathbb{R}^d) = H^{s_2}(\mathbb{R}^d).$$

Moreover,

$$H^{s_2}(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d) \quad \text{continuously}, \quad \forall q \in \left[2, \frac{2d}{(d-2s_2)^+}\right),$$

where $\frac{2d}{(d-2s_2)^+} = \frac{2d}{d-2s_2}$ if $d > 2s_2$, and $\frac{2d}{(d-2s_2)^+} = +\infty$ if $d \leq 2s_2$. For more information about the fractional Sobolev spaces and the fractional Laplacian operator, we refer the readers to see, e.g., [15,41].

In Physics and other applications, researchers are very interested in solutions of (1.3) having a prescribed L^2-norm, called normalized solutions. More precisely, for a given $c > 0$, one looks at solutions (u, λ) of (1.3), with $\int_{\mathbb{R}^d}|u|^2\mathrm{d}x = c^2$. Normalized solutions can be obtained as critical points of an associated functional on the L^2-constraint. Therefore, the parameter $\lambda \in \mathbb{R}$ appears as a Lagrange multiplier and is not *a priori* given. This method is referred to as "the prescribed mass approach." It has attracted a great interest in recent years, thanks to the wide range of its applications, see, e.g., [4,9,10,33,34] and references therein.

This approach is particularly relevant from a physical point of view. Indeed, the L^2-norm is a preserved quantity of the evolution and the variational characterization of such solutions is often crucial to analyze their orbital stability/instability, see, e.g., [8,11,15,43,44].

It is standard to show that the associated energy functional for (1.3)

$$E^p(u) := \frac{1}{2}\|\nabla_{s_1}u\|_2^2 + \frac{1}{2}\|\nabla_{s_2}u\|_2^2 - \frac{1}{p+2}\int_{\mathbb{R}^d} |u|^{p+2}\mathrm{d}x, \tag{1.4}$$

is of class C^1 on $H^{s_2}(\mathbb{R}^d)$, and therefore on the L^2-constraint:

$$S_c := \left\{u \in H^{s_2}(\mathbb{R}^d) \mid \int_{\mathbb{R}^d} |u|^2\mathrm{d}x = c^2,\ c > 0\right\},$$

see, e.g., [7].

To find critical points of E^p on S_c, we first consider the following global minimization problem:

$$I_c^p := \inf_{u\in S_c} E^p(u), \quad c > 0. \qquad (GM_c)$$

Clearly, a minimizer of I_c^p is a ground state solution to (1.3), in the following sense:

Definition 1.1. We say that $u_c \in S_c$ is a ground state solution to (1.3), if it is a solution having minimal energy among all the solutions which belong to S_c. Namely,

$$E^p(u_c) = \inf\left\{E^p(u) \mid u \in S_c,\ \left(E^p|_{S_c}\right)'(u) = 0\right\}.$$

This definition was first introduced in [8] on a related problem, and it makes sense when the functional E^p is unbounded from below on S_c.

Denote by

$$\mathcal{M}_c := \left\{u \in H^{s_2}(\mathbb{R}^d) | u \text{ is a minimizer of } I_c^p\right\},$$

then our aim is to determine under which conditions on p, s_1, s_2 and c, $\mathcal{M}_c \neq \emptyset$ and $\mathcal{M}_c = \emptyset$.

Recall that from [7, Appendix B], if $0 < s < 1$, $d \geq 1$, $0 < p < \frac{4s}{(d-2s)^+}$, then the fractional Gagliardo-Nirenberg inequality

$$\|u\|_{p+2}^{p+2} \leq B(p,d,s)\|\nabla_s u\|_2^{\frac{\mathrm{d}p}{2s}}\|u\|_2^{p+2-\frac{\mathrm{d}p}{2s}}, \quad \forall u \in H^s(\mathbb{R}^d), \tag{1.5}$$

holds with the optimal constant $B(p,d,s) > 0$, given by

$$B(p,d,s) := \left(\frac{2s(p+2)-\mathrm{d}p}{\mathrm{d}p}\right)^{\frac{\mathrm{d}p}{4s}} \frac{2s(p+2)}{[2s(p+2)-\mathrm{d}p]\|Q_s\|_2^p}, \tag{1.6}$$

and Q_s is the ground state of

$$(-\Delta)^s u + u = |u|^p u, \quad x \in \mathbb{R}^d, \tag{1.7}$$

whose existence and uniqueness have been proved by [25,26]. In particular,

$$B(p,d,s) = \frac{2s+d}{d}\|Q_s\|_2^{-\frac{4s}{d}}, \quad \text{if } p = \frac{4s}{d}. \tag{1.8}$$

First of all, we prove the following properties of I_c^p, which are interesting by themselves and also important for the proof of the existence of the minimizers.

Theorem 1.1. *Assume that* $0 < s_1 < s_2 < 1$, $d \geq 1$, *and* $0 < p \leq \frac{4s_2}{d}$, *then*

$$I_c^p \leq 0, \quad \forall c > 0,$$

and furthermore,

(1) *If* $0 < p < \frac{4s_1}{d}$, *then* $-\infty < I_c^p < 0, \quad \forall c > 0$;

(2) *If* $\frac{4s_1}{d} \le p < \frac{4s_2}{d}$, *denoting*

$$c_0 := \sup\{c > 0 \mid I_c^p = 0\}, \tag{1.9}$$

then $0 < c_0 < +\infty$, *and*

$$\begin{cases} I_c^p = 0, & 0 < c \le c_0; \\ -\infty < I_c^p < 0, & c > c_0. \end{cases}$$

(3) *If* $p = \frac{4s_2}{d}$, *setting* $c_* := \|Q_{s_2}\|_2$, *where* Q_{s_2} *is the ground state solution of* (1.7) *with* $s = s_2$, *then*

$$\begin{cases} I_c^p = 0, & 0 < c \le c_*; \\ I_c^p = -\infty, & c > c_*. \end{cases}$$

(4) *The mapping*: $c \mapsto I_c^p$ *is continuous, concave, and non-increasing on* $(0, \infty)$.

Remark 1.1. When $p = \frac{4s_1}{d}$, c_0 in (1.9) can be given explicitly as:

$$c_0 = \|Q_{s_1}\|_2,$$

where Q_{s_1} is the ground state of (1.7) with $s = s_1$. For more details, see (3.16) and the proof of Theorem 1.2. Figure 1 shows the properties of I_c^p:

Our first main result reads as follows.

Theorem 1.2. *Let* $0 < s_1 < s_2 < 1$, $d \ge 1$, *and* $0 < p \le \frac{4s_2}{d}$.

(1) *When* $0 < p < \frac{4s_1}{d}$, *then for all* $c > 0$, I_c^p *admits at least one minimizer (namely* $\mathcal{M}_c \ne \emptyset$*)*;

(2) *When* $\frac{4s_1}{d} < p < \frac{4s_2}{d}$, *then* I_c^p *has a minimizer if and only if* $c \ge c_0$, *where* c_0 *is given by* (1.9);

(3) *When* $p = \frac{4s_1}{d}$, *then* I_c^p *has a minimizer if and only if* $c > c_0$;

(4) *When* $p = \frac{4s_2}{d}$, *then for all* $c > 0$, I_c^p *has no minimizers.*

Remark 1.2. This theorem gives a complete description of the existence/non-existence of minimizers of I_c^p. Figure 2 shows the range of c for the existence and non-existence of minimizers of I_c^p:

Remark 1.3. In the existence part of our theorem, except for the case that $\frac{4s_1}{d} < p < \frac{4s_2}{d}$, and $c = c_0$, we will prove that any minimizing sequence of I_c^p is relatively compact modulo translations, see Proposition 3.1. This guarantees the orbital stability of the standing waves if we have the local well-posedness of the Cauchy problem (1.2), see [20], or [32]. To prove this theorem, we mainly apply the concentration compactness principle of Lions [35]. However, compared with its classical form, our arguments are more simple.

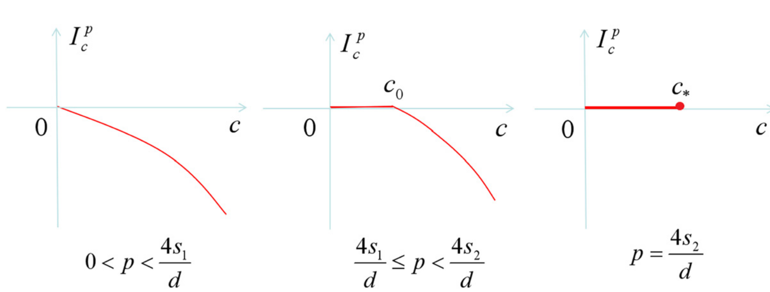

Figure 1: The properties of the minimization I_c^p.

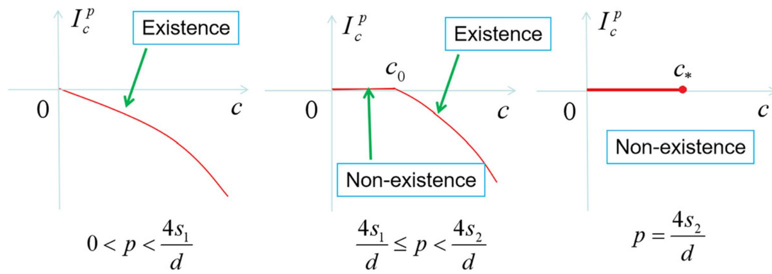

Figure 2: Sharp existence and non-existence of minimizers of I_c^p.

By the Lagrange multiplier theory, we have the following conclusion.

Theorem 1.3. *Let $u_c \in S_c$ be a minimizer of I_c^p, whose existence has been proved in* Theorem 1.2. *Then, there exists a Lagrange multiplier $\lambda_c > 0$, such that u_c is a ground state solution of* (1.3) *with $\lambda = \lambda_c$.*

Remark 1.4. We point out that when $s_1 = 1, s_2 = 2$, similar results to Theorems 1.1 and 1.2 have been observed in the references such as [5,13,36]. But since our equation involves two non-local terms with different orders, the analyses shall be more delicate.

In what follows, we focus on the mass critical and supercritical cases, namely $\frac{4s_2}{d} \le p < \frac{4s_2}{(d-2s_2)^+}$. In Lemma 2.1, we will show that if $\frac{4s_2}{d} < p < \frac{4s_2}{(d-2s_2)^+}$ and $c > 0$, or $p = \frac{4s_2}{d}$ and $c > c_*$, then the functional E^p is unbounded from below on S_c. Therefore, one cannot hope to obtain a solution using the global minimization (GM_c). Inspired by the work of Jeanjean [33], whose idea has been significantly carried out in the analyses of other equations, see, e.g., [4,8,11,24,43,44], we consider the following local type of minimization problem,

$$J_c^p := \inf_{u \in V_c} E^p(u), \qquad (LM_c),$$

where V_c is defined by

$$V_c := \{u \in S_c \mid Q^p(u) = 0\}, \tag{1.10}$$

with $Q^p(u)$ given by

$$Q^p(u) := s_1\|\nabla_{s_1} u\|_2^2 + s_2\|\nabla_{s_2} u\|_2^2 - \frac{dp}{2(p+2)}\|u\|_{p+2}^{p+2}. \tag{1.11}$$

Note that $Q^p(u) = 0$ is indeed a variant of Pohozaev identity. Namely, if $u_0 \in S_c$ is a critical point of E^p on S_c, then necessarily $Q^p(u_0) = 0$, see Lemma 2.2 for details.

Our second main result reads as follows.

Theorem 1.4. *Assume that $0 < s_1 < s_2 < 1$, and $\frac{4s_2}{d} \le p \le \frac{4s_1}{d-2s_1}$, $2s_1 < d \le \frac{2s_1 s_2}{s_2 - s_1}$. Then under either of the following conditions:*

(1) $\frac{4s_2}{d} < p \le \frac{4s_1}{d-2s_1}$ *and $c \in (0, +\infty)$;*

(2) $p = \frac{4s_2}{d}$ *and $c \in (c_*, +\infty)$, where c_* is given in* Theorem 1.1 (3),

we have that $J_c^p > 0$ and that J_c^p has a minimizer $u_c \in V_c$. In particular, there exists $\lambda_c > 0$ such that u_c is a ground state solution to

$$(-\Delta)^{s_1}u + (-\Delta)^{s_2}u + \lambda u - |u|^p u = 0, \quad x \in \mathbb{R}^d, \tag{1.12}$$

with $\lambda = \lambda_c$. Moreover, when $\frac{4s_2}{d} < p \le \frac{4s_1}{d-2s_1}$, we have

$$\|\nabla_{s_1}u_c\|_2 \to +\infty,\ \|\nabla_{s_2}u_c\|_2 \to +\infty,\quad \lambda_c \to +\infty,\quad E^p(u_c) = J_c^p \to +\infty,$$

as $c \to 0^+$.

Remark 1.5. The condition "$p \le \frac{4s_1}{d-2s_1}$, $2s_1 < d \le \frac{2s_1s_2}{s_2-s_1}$" in this theorem, seems to be technical. It stems from Lemma 4.5 or Lemma 4.4. Indeed, if this condition can be extended to "$p < \frac{4s_2}{(d-2s_2)^+}$" in Lemma 4.5, then the conclusions of Theorem 1.4 hold also for "$p < \frac{4s_2}{(d-2s_2)^+}$." In addition, we will prove in Lemma 3.1(*ii*) that if $p = \frac{4s_2}{d}$ and $c \le c_*$, then E^p has no minimizers on S_c, hence Theorem 1.4 actually gives a sharp existence result of normalized solutions in the mass critical case $p = \frac{4s_2}{d}$.

Remark 1.6. To show this theorem, the main ingredient is to prove that a minimizing sequence of J_c^p is relatively compact modulo translations. However, compared with the case of I_c^p in Theorem 1.2, the situation is more complicated. Since two constraints are involved, then two Lagrange multipliers appear. Therefore, one needs to prove that one of them is indeed zero. To this aim, we apply some arguments from [11] and [24]. To show that the limit function of the minimizing sequence is in the L^2 constraint, we make use of the strict monotonicity of $c :\mapsto J_c^p$, where we need to restrict the range of p as the assumption in Theorem 1.4, see Lemmas 4.4 and 4.5 for more details.

Remark 1.7. Finally, we point out that, throughout this article, we assume that $0 < s_1 < s_2 < 1$; however, keeping other conditions invariant, all results obtained in this article remain true when $s_2 = 1$.

This article is organized as follows. In Section 2, we give two important preliminary results. In Section 3, we treat the mass subcritical and critical cases. In particular, we prove Theorems 1.1 and 1.2. Section 4 is dedicated to the proof of Theorem 1.4, which deals with the existence and asymptotic behavior of normalized solutions in the mass critical and supercritical cases. Finally, in the appendix, we give a technical lemma on the decomposition of the norm $\|\nabla_s u\|_2^2$.

Notation: Throughout the article, we denote by $\|\cdot\|_p$ the standard norm on $L^p(\mathbb{R}^d)$.

2 Preliminary results

By the Gagliardo-Nirenberg inequality (1.5) and scaling arguments, we will show the following lemma. We point out that similar results have been proved for the case with the Laplacian operator only, see for example, [3,27,38]. The ideas of our proofs somehow are similar to them, but since there are two non-local terms with different orders, we need to control carefully the two terms.

Lemma 2.1. *Assume that* $0 < s_1 < s_2 < 1$, $d \ge 1$, *and* $0 < p < \frac{4s_2}{(d-2s_2)^+}$, *and let* $c_* := \|Q_{s_2}\|_2$, *with* Q_{s_2} *being the ground state solution of* (1.7) *with* $s = s_2$. *Then under either of the following conditions*:

(1) $0 < p < \frac{4s_2}{d}$ *and* $c \in (0, +\infty)$;

(2) $p = \frac{4s_2}{d}$ *and* $c \in (0, c_*]$,

the functional E^p *is bounded from below, namely* $I_c^p > -\infty$.

Moreover, under either of the following conditions:

(3) $p = \frac{4s_2}{d}$ *and* $c \in (c_*, +\infty)$;

(4) $\frac{4s_2}{d} < p < \frac{4s_2}{(d-2s_2)^+}$ *and* $c \in (0, +\infty)$,

the functional E^p *is unbounded from below, namely* $I_c^p = -\infty$.

Proof. By the Gagliardo-Nirenberg inequality (1.5) with $s = s_2$, for any $u \in S_c$,

$$\begin{aligned} E^p(u) &\geq \frac{1}{2}\|\nabla_{s_1}u\|_2^2 + \frac{1}{2}\|\nabla_{s_2}u\|_2^2 - \frac{B(p,d,s_2)}{p+2}\|\nabla_{s_2}u\|_2^{\frac{dp}{2s_2}} \cdot c^{p+2-\frac{dp}{2s_2}} \\ &\geq \frac{1}{2}\|\nabla_{s_2}u\|_2^2 - \frac{B(p,d,s_2)}{p+2}\|\nabla_{s_2}u\|_2^{\frac{dp}{2s_2}} \cdot c^{p+2-\frac{dp}{2s_2}}. \end{aligned} \tag{2.1}$$

Note that $\frac{dp}{2s_2} < 2$ if $0 < p < \frac{4s_2}{d}$, thus we deduce from (2.1) that $I_c^p > -\infty$ for all $c > 0$.

If $p = \frac{4s_2}{d}$, (2.1) is reduced to

$$E^p(u) \geq \frac{1}{2}\|\nabla_{s_2}u\|_2^2\left[1 - \left(\frac{c}{c_*}\right)^{\frac{4s_2}{d}}\right], \quad \forall u \in S_c, \tag{2.2}$$

which also implies that $I_c^p > -\infty$ for all $c \in (0, c_*]$.

To show that $I_c^p = -\infty$ in cases (3) and (4), for any $c > 0$, given $u \in S_c$, we consider the scaling

$$u^t(x) := t^{\frac{d}{2}}u(tx), \quad t > 0.$$

Then $u^t \in S_c, \ \forall t > 0$, and

$$E^p(u^t) = \frac{t^{2s_1}}{2}\|\nabla_{s_1}u\|_2^2 + \frac{t^{2s_2}}{2}\|\nabla_{s_2}u\|_2^2 - \frac{t^{\frac{dp}{2}}}{p+2}\|u\|_{p+2}^{p+2}. \tag{2.3}$$

Clearly, if $\frac{4s_2}{d} < p < \frac{4s_2}{(d-2s_2)^+}$, then $\frac{dp}{2} > 2s_2$, thus $\lim_{t\to\infty}E^p(u^t) = -\infty$. This implies that $I_c^p = -\infty$ for all $c > 0$.

When $p = \frac{4s_2}{d}$, we replace u in (2.3) by $v := \frac{c}{c_*}Q_{s_2}$, then $v \in S_c$ and

$$\begin{aligned} E^p(v^t) &= \left(\frac{c}{c_*}\right)^2\frac{t^{2s_1}}{2}\|\nabla_{s_1}Q_{s_2}\|_2^2 + \left(\frac{c}{c_*}\right)^2\frac{t^{2s_2}}{2}\|\nabla_{s_2}Q_{s_2}\|_2^2 - \left(\frac{c}{c_*}\right)^{p+2}\frac{t^{2s_2}}{p+2}\|Q_{s_2}\|_{p+2}^{p+2} \\ &= \left(\frac{c}{c_*}\right)^2\left[\frac{t^{2s_1}}{2}\|\nabla_{s_1}Q_{s_2}\|_2^2 + \frac{t^{2s_2}}{2}\|\nabla_{s_2}Q_{s_2}\|_2^2 - \left(\frac{c}{c_*}\right)^p\frac{t^{2s_2}}{2}\|\nabla_{s_2}Q_{s_2}\|_2^2\right] \\ &= \left(\frac{c}{c_*}\right)^2\left[\frac{t^{2s_1}}{2}\|\nabla_{s_1}Q_{s_2}\|_2^2 + \frac{t^{2s_2}}{2}\left(1 - \left(\frac{c}{c_*}\right)^p\right)\|\nabla_{s_2}Q_{s_2}\|_2^2\right], \end{aligned} \tag{2.4}$$

from which, we see that $\lim_{t\to\infty}E^p(v^t) = -\infty$ as $c > c_*$. Hence, $I_c^p = -\infty$ for all $c > c_*$, which completes the proof. □

Now we prove the following Pohozaev-type identity result.

Lemma 2.2. *Let $u_0 \in S_c$ be a critical point of E^p on S_c, then necessarily $Q^p(u_0) = 0$, where $Q^p(u)$ is given by* (1.11).

Proof. If $u_0 \in S_c$ is a critical point of E^p on S_c, then in the weak sense, there exists $\lambda_0 \in \mathbb{R}$, such that (u_0, λ_0) solves the equation:

$$(-\Delta)^{s_1}u_0 + (-\Delta)^{s_2}u_0 - |u_0|^p u_0 + \lambda_0 u_0 = 0, \quad x \in \mathbb{R}^d. \tag{2.5}$$

Multiplying (2.5) by u_0 and $x \cdot \nabla\bar{u}_0$, respectively, and then integrating by parts, we have

$$\|\nabla_{s_1}u_0\|_2^2 + \|\nabla_{s_2}u_0\|_2^2 - \|u_0\|_{p+2}^{p+2} + \lambda_0\|u\|_2^2 = 0 \tag{2.6}$$

and

$$\frac{2s_1 - d}{2}\|\nabla_{s_1}u_0\|_2^2 + \frac{2s_2 - d}{2}\|\nabla_{s_2}u_0\|_2^2 - \frac{-d}{p+2}\|u_0\|_{p+2}^{p+2} - \frac{d\lambda_0}{2}\|u_0\|_2^2 = 0, \tag{2.7}$$

where in the second identity, we used that $\langle(-\Delta)^s u, x\cdot\nabla\bar{u}\rangle_{L^2} = \frac{2s-d}{2}\|\nabla_s u\|_2^2$, $\langle|u|^p u, x\cdot\nabla\bar{u}\rangle_{L^2} = \frac{-d}{p+2}\|u\|_{p+2}^{p+2}$, see [7, Proposition B1].

From (2.6) and (2.7), we have

$$s_1\|\nabla_{s_1}u_0\|_2^2 + s_2\|\nabla_{s_2}u_0\|_2^2 - \frac{dp}{2(p+2)}\|u_0\|_{p+2}^{p+2} = 0, \tag{2.8}$$

which is exactly $Q^p(u_0) = 0$. □

3 Mass subcritical and critical cases

To prove Theorem 1.2, we first show the following proposition.

Proposition 3.1. *Let* $0 < p \le \frac{4s_2}{d}$ *and* $c > 0$ *be such that*

$$-\infty < I_c^p < 0,$$

then any minimizing sequence of I_c^p *is relatively compact modulo translations. In particular,* I_c^p *admits a minimizer.*

Proof. Let $\{u_n\} \subset S_c$ be an arbitrary minimizing sequence of I_c^p, namely,

$$\|u_n\|_2 = c \quad \text{and} \quad E^p(u_n) \to I_c^p, \quad n \to \infty.$$

Then by (2.1), one may observe that $\{u_n\}$ is bounded in H^{s_2}. Now we claim that

$$\int_{\mathbb{R}^d} |u_n|^{p+2}\mathrm{d}x \nrightarrow 0, \quad \text{as } n \to \infty. \tag{3.1}$$

Indeed, if $\int_{\mathbb{R}^d}|u_n|^{p+2}\mathrm{d}x \to 0$, then

$$I_c^p = E^p(u_n) + o_n(1) = \frac{1}{2}\|\nabla_{s_1}u_n\|_2^2 + \frac{1}{2}\|\nabla_{s_2}u_n\|_2^2 + o_n(1),$$

which contradicts the fact that $I_c^p < 0$. Thus, by (3.1) and the Lions vanishing lemma [35, Lemma I.1], there exist a constant $\delta > 0$ and a sequence $\{x_n\} \subset \mathbb{R}^d$ such that

$$\int_{B(x_n,1)} |u_n|^2\mathrm{d}x \ge \delta > 0,$$

or equivalently

$$\int_{B(0,1)} |u_n(\cdot + x_n)|^2\mathrm{d}x \ge \delta > 0. \tag{3.2}$$

Here $B(0,1)$ denotes the unit ball centered at 0. Now let $v_n(\cdot) := u_n(\cdot + x_n)$, then obviously $\{v_n\}$ is bounded in H^{s_2}, and thus up to a subsequence (still denoted it by $\{v_n\}$), there exists $u \in H^{s_2}$ such that

$$v_n \rightharpoonup u \quad \text{weakly in } H^{s_2} \quad \text{and } v_n \to u \quad \text{in } L^2_{\mathrm{loc}}(\mathbb{R}^d).$$

We note that $u \neq 0$, since by (3.2),

$$0 < \delta \le \lim_{n\to\infty} \int_{B(0,1)} |v_n|^2\mathrm{d}x = \int_{B(0,1)} |u|^2\mathrm{d}x.$$

Next, we prove that $u \in S_c$ is a minimizer of I_c^p. Since $u \neq 0$, by the Brézis-Lieb lemma and Lemma A.1, we have

$$c^2 = \|v_n\|_2^2 = \|v_n - u\|_2^2 + \|u\|_2^2 + o_n(1). \tag{3.3}$$

$$E^p(v_n) = E^p(v_n - u) + E^p(u) + o_n(1). \tag{3.4}$$

If $\|u\|_2^2 < c^2$, we denote $a := \frac{c}{\|u\|_2}$, then $a > 1$, $au \in S_c$, and

$$E^p(au) = \frac{a^2}{2}\|\nabla_{s_1} u\|_2^2 + \frac{a^2}{2}\|\nabla_{s_2} u\|_2^2 - \frac{a^{p+2}}{p+2}\|u\|_{p+2}^{p+2}, \tag{3.5}$$

which implies that

$$E^p(u) = \frac{1}{a^2}E^p(au) + \frac{(a^p - 1)}{p+2}\|u\|_{p+2}^{p+2}. \tag{3.6}$$

Similarly, let $a_n := \frac{c}{\|v_n - u\|_2}$, then $a_n > 1$, $a_n(v_n - u) \in S_c$, and

$$E^p(v_n - u) = \frac{1}{a_n^2}E^p(a_n(v_n - u)) + \frac{(a_n^p - 1)}{p+2}\|v_n - u\|_{p+2}^{p+2}. \tag{3.7}$$

Therefore, by (3.3), (3.4), (3.6), and (3.7), we have

$$\begin{aligned}
I_c^p &= E^p(v_n) + o_n(1) = E^p(v_n - u) + E^p(u) + o_n(1)\\
&= \frac{1}{a^2}E^p(au) + \frac{1}{a_n^2}E^p(a_n(v_n - u)) + \frac{(a^p - 1)}{p+2}\|u\|_{p+2}^{p+2} + \frac{(a_n^p - 1)}{p+2}\|v_n - u\|_{p+2}^{p+2} + o_n(1)\\
&\geq \left(\frac{1}{a^2} + \frac{1}{a_n^2}\right)I_c^p + \frac{(a^p - 1)}{p+2}\|u\|_{p+2}^{p+2} + \frac{(a_n^p - 1)}{p+2}\|v_n - u\|_{p+2}^{p+2} + o_n(1)\\
&= I_c^p + \frac{(a^p - 1)}{p+2}\|u\|_{p+2}^{p+2} + \frac{(a_n^p - 1)}{p+2}\|v_n - u\|_{p+2}^{p+2} + o_n(1),
\end{aligned}$$

which is impossible, since $p > 0$ and $a > 1$, $a_n > 1$. Hence $\|u\|_2^2 = c^2$, thus $v_n \to u$ in $L^2(\mathbb{R}^d)$, and by interpolation, $v_n \to u$ in $L^{p+2}(\mathbb{R}^d)$. Furthermore, since $u \in S_c$, then

$$I_c^p \leq E^p(u) \leq \lim_{n\to\infty} E^p(u_n) = I_c^p.$$

This implies that $E^p(u) = I_c^p$, and then $u \in S_c$ is a minimizer of I_c^p. □

By Lemma 2.2, we can establish the following non-existence results.

Lemma 3.1. *Let* $\frac{4s_1}{d} \leq p \leq \frac{4s_2}{d}$, *then,*

(*i*) *When* $\frac{4s_1}{d} \leq p < \frac{4s_2}{d}$, *there exists* $\hat{c} > 0$ *such that, for all* $c < \hat{c}$, E^p *has no critical points on* S_c;

(*ii*) *When* $p = \frac{4s_2}{d}$, *then for all* $c \leq c_*$, E^p *has no critical points on* S_c. *In particular, for all* $c > 0$, I_c^p *has no minimizers.*

Proof. We argue by contradiction to prove the non-existence. Indeed, when $\frac{4s_1}{d} \leq p < \frac{4s_2}{d}$, we assume that there exists a sequence $\{c_n\}$ with $c_n \to 0$ as $n \to \infty$, and $u_n \in S_{c_n}$ being a critical point of E^p on S_{c_n}. Then by Lemma 2.2, $Q^p(u_n) = 0$, $\forall n \in \mathbb{N}^+$. Since $\|u_n\|_2 = c_n \to 0$, then using $Q^p(u_n) = 0$ and interpolation, we have

$$\|u_n\|_{p+2} \to 0, \quad \|\nabla_{s_1} u_n\|_2 \to 0, \quad \|\nabla_{s_2} u_n\|_2 \to 0. \tag{3.8}$$

Note that, since $\frac{4s_1}{d} \leq p < \frac{4s_2}{d}$, clearly, for each $n \in \mathbb{N}^+$, we have

$$\int |u_n|^{p+2}\mathrm{d}x \leq \int |u_n|^{\frac{4s_1}{d}+2}\mathrm{d}x + \int |u_n|^{\frac{4s_2}{d}+2}\mathrm{d}x, \tag{3.9}$$

thus by the Gagliardo-Nirenberg inequality (1.5) with $s = s_1$ and $s = s_2$ respectively,

$$\int |u_n|^{p+2}\mathrm{d}x \leq B\left(\frac{4s_1}{d}, d, s_1\right)\|\nabla_{s_1} u_n\|_2^2 c_n^{\frac{4s_1}{d}} + B\left(\frac{4s_2}{d}, d, s_2\right)\|\nabla_{s_2} u_n\|_2^2 c_n^{\frac{4s_2}{d}}. \tag{3.10}$$

Therefore, we obtain

$$\begin{aligned} Q^p(u_n) &= s_1\|\nabla_{s_1}u_n\|_2^2 + s_2\|\nabla_{s_2}u_n\|_2^2 - \frac{dp}{2(p+2)}\|u_n\|_{p+2}^{p+2} \\ &\geq s_1\|\nabla_{s_1}u_n\|_2^2 + s_2\|\nabla_{s_2}u_n\|_2^2 - \sum_{i=1}^{2}\frac{dp\cdot B\left(\frac{4s_i}{d},d,s_i\right)}{2(p+2)}\|\nabla_{s_i}u_n\|_2^2 c_n^{\frac{4s_i}{d}} \\ &= \sum_{i=1}^{2}\|\nabla_{s_i}u_n\|_2^2\left[s_i - \frac{dp\cdot B\left(\frac{4s_i}{d},d,s_i\right)}{2(p+2)}c_n^{\frac{4s_i}{d}}\right]. \end{aligned} \tag{3.11}$$

Then from (3.8) and (3.11), we deduce that for $n \in \mathbb{N}^+$ sufficiently large, $Q^p(u_n) > 0$, which is a contradiction with the fact that $Q^p(u_n) = 0,\ \forall n \in \mathbb{N}^+$. Thus, we have proved (i).

When $p = \frac{4s_2}{d}$, if there exist $c_1 \leq c_*$ and $u_1 \in S_{c_1}$ a critical point of E^p on S_{c_1}, then $Q^p(u_1) = 0$. However, by the Gagliardo-Nirenberg inequality (1.5) with $s = s_2$,

$$Q^p(u_1) \geq s_1\|\nabla_{s_1}u_1\|_2^2 + s_2\|\nabla_{s_2}u_1\|_2^2\left[1 - \left(\frac{c_1}{c_*}\right)^{\frac{4s_2}{d}}\right], \tag{3.12}$$

this implies that $Q^p(u_1) > 0$, which is a contradiction. Hence for any $c \leq c_*$, E^p has no critical points on S_c. Moreover, by the fact that $I_c^p = -\infty$ if $c > c_*$, we conclude that for all $c > 0$, I_c^p is not attained. □

To prove Theorem 1.2, we need to treat independently the case where $c = c_0$.

Lemma 3.2. *Assume that $\frac{4s_1}{d} < p < \frac{4s_2}{d}$, then $I_{c_0}^p$ admits at least one minimizer.*

Proof. To show this lemma, we will use some arguments from [34]. Let $c_n := c_0 + \frac{1}{n}$, then $c_n \to c_0$ as $n \to \infty$. By Theorem 1.1 (2) and (4), for all $n \in \mathbb{N}^+$, $I_{c_n}^p < 0$, and $I_{c_n}^p \to I_{c_0}^p = 0$ as $n \to \infty$. Furthermore, by Proposition 3.1, $I_{c_n}^p$ admits a minimizer $u_n \in S_{c_n}$, namely $E^p(u_n) = I_{c_n}^p$. Now we claim that $\{u_n\}$ is bounded in H^{s_2}. Indeed, since $E^p(u_n) = I_{c_n}^p \to I_{c_0}^p = 0$, then by (2.1), $\{\|\nabla_{s_i}u_n\|_2^2\}$ is bounded, where $i = 1, 2$. Additionally, $\|u_n\|_2 = c_n \to c_0 > 0$, then we conclude that $\{u_n\}$ is bounded in H^{s_2}. Now we prove that

$$\int_{\mathbb{R}^d}|u_n|^{p+2}dx \nrightarrow 0, \quad \text{as } n \to \infty. \tag{3.13}$$

Indeed, if $\int_{\mathbb{R}^d}|u_n|^{p+2}dx \to 0$, then by $E^p(u_n) \to 0$, we have

$$\|\nabla_{s_1}u_n\|_2^2 \to 0, \quad \|\nabla_{s_2}u_n\|_2^2 \to 0, \quad \text{as } n \to \infty. \tag{3.14}$$

On the other hand, by the Gagliardo-Nirenberg inequality (1.5) with $s = s_1$,

$$\begin{aligned} E^p(u_n) &\geq \frac{1}{2}\|\nabla_{s_1}u_n\|_2^2 + \frac{1}{2}\|\nabla_{s_2}u_n\|_2^2 - \frac{B(p,d,s_1)}{p+2}\|\nabla_{s_1}u_n\|_2^{\frac{dp}{2s_1}}\cdot c_n^{p+2-\frac{dp}{2s_1}} \\ &\geq \|\nabla_{s_1}u_n\|_2^2\left(\frac{1}{2} - \frac{B(p,d,s_1)}{p+2}\|\nabla_{s_1}u_n\|_2^{\frac{dp}{2s_1}-2}\cdot c_n^{p+2-\frac{dp}{2s_1}}\right). \end{aligned} \tag{3.15}$$

Note that $\frac{dp}{2s_1} - 2 > 0$ if $\frac{4s_1}{d} < p < \frac{4s_2}{d}$, then by (3.14) and (3.15), we have that $E^p(u_n) \geq 0$ for n large enough. This contradicts the fact that $E^p(u_n) = I_{c_n}^p < 0$ for all $n \in \mathbb{N}^+$. Then the claim is verified. Thus, up to a subsequence of $\{u_n\}$ (we still denote it by $\{u_n\}$), there exist a sequence $\{x_n\}$ and a function $u \in H^{s_2}\backslash\{0\}$, such that

$$u_n(\cdot + x_n) \rightharpoonup u, \quad \text{weakly in } H^{s_2}.$$

Let $v_n = u_n(\cdot + x_n)$, then following the same argument as in the proof of Proposition 3.1, we conclude that

$$v_n \to u, \quad \text{strongly in } H^{s_2}.$$

Hence, $u \in S_{c_0}$ is a minimizer of $I^p_{c_0}$, and this ends the proof. □

Proof of Theorem 1.1. For any $c > 0$, from (2.3) we observe that $I^p_c \le 0$. Moreover, if $0 < p < \frac{4s_1}{d}$, then $\frac{dp}{2} < 2s_1 < 2s_2$. Letting $t > 0$ small enough, we have $E^p(u^t) < 0$. This together with Lemma 2.1 implies that $-\infty < I^p_c < 0$. Then point (1) follows.

Now we prove point (4). Indeed, letting $\{u_n\} \subset S_c$ being such that $I^p_c = E^p(u_n) + o_n(1)$, then for any $t > 1$, $I^p_{tc} \le E^p(tu_n) \le t^2 E^p(u_n)$, which leads to

$$I^p_{tc} \le I^p_c, \quad \forall t > 1,$$

since $I^p_c \le 0$. Thus for any $0 < c_1 < c_2$,

$$I^p_{c_2} = I^p_{\frac{c_2}{c_1}c_1} \le I^p_{c_1}.$$

Namely I^p_c is non-increasing with respect to c on $(0, \infty)$. To show the continuity, we first let $c_n \to c^-$, then by the definition of I^p_c, for any $\varepsilon > 0$, there exists $u \in S_c$, such that $E^p(u) < I^p_c + \varepsilon$. Denote $b_n := \frac{c_n}{c}$, then $b_n \to 1$ and

$$\begin{aligned} 0 \le I^p_{c_n} - I^p_c &\le E^p(b_n u) - E^p(u) + \varepsilon \\ &= \frac{b_n^2 - 1}{2}(\|\nabla_{s_1} u\|_2^2 + \|\nabla_{s_2} u\|_2^2) - \frac{b_n^{p+2} - 1}{p+2}\|u\|_{p+2}^{p+2} + \varepsilon \\ &= \varepsilon + o_n(1), \end{aligned}$$

by letting $\varepsilon \to 0$, then $\lim_{c_n \to c^-} I^p_{c_n} = I^p_c$. Similarly, we can prove that $\lim_{c_n \to c^+} I^p_{c_n} = I^p_c$. Hence, the continuity is verified. In addition, we observe that

$$I^p_c = \inf_{v \in S_1} E^p(cv) = \inf_{v \in S_1} \left\{ c^2\left(\frac{1}{2}\|\nabla_{s_1} v\|_2^2 + \frac{1}{2}\|\nabla_{s_2} v\|_2^2\right) - \frac{c^{p+2}}{p+2}\|v\|_{p+2}^{p+2} \right\}.$$

On the other hand, for any $a > 0$, $b > 0$, $p > 0$, the function $x : \mapsto ax^2 - bx^{p+2}$ is concave on $(0, \infty)$, and the minimum of a family concave functions is also concave; therefore, the mapping $c : \mapsto I^p_c$ is concave on $(0, \infty)$. Thus, point (4) is verified.

Due to point (4) and the definition of c_0, to show point (2), we only need to prove that $0 < c_0 < +\infty$, the remaining cases will follow obviously. Note that, given $v_0 \in S_1$, we have

$$E^p(tv_0) = \frac{t^2}{2}(\|\nabla_{s_1} v_0\|_2^2 + \|\nabla_{s_2} v_0\|_2^2) - \frac{t^{p+2}}{p+2}\|v_0\|_{p+2}^{p+2}, \quad t > 0,$$

which implies that $E^p(tv_0) < 0$ as $t > 0$ large enough. Hence there exists $\bar{c} > 0$ large enough, such that $I^p_{\bar{c}} < 0$, then $c_0 < +\infty$. In addition, from Proposition 3.1 and Lemma 3.1, one sees that $I^p_c = 0$ as $c > 0$ small. Then by the definition of c_0 (1.9), we conclude that $c_0 > 0$. Hence, we have proved that $0 < c_0 < +\infty$.

Now we prove point (3). Indeed, if $p = \frac{4s_2}{d}$, by (2.2),

$$E^p(u) \ge \frac{1}{2}\|\nabla_{s_2} u\|_2^2 \left[1 - \left(\frac{c}{c_*}\right)^{\frac{4s_2}{d}}\right], \quad \forall u \in S_c,$$

we see that $I^p_c \ge 0$ for all $c \le c_*$. In view that $I^p_c \le 0$ for all $c > 0$, hence $I^p_c = 0$ for all $0 < c \le c_*$. As for the case $c > c_*$, it has been proved in Lemma 2.1 that $I^p_c = -\infty$ as $c > c_*$. Then the proof is complete. □

Proof of Theorem 1.2. First, thanks to Theorem 1.1, Proposition 3.1, and Lemma 3.1(*ii*), points (1) and (4) follow immediately. Additionally, by Lemma 3.2, we conclude that when $\frac{4s_1}{d} < p < \frac{4s_2}{d}$, $c \ge c_0$, or $p = \frac{4s_1}{d}$, $c > c_0$, I^p_c admits a minimizer. Hence, to prove points (2) and (3), it is enough to verify that when $\frac{4s_1}{d} < p < \frac{4s_2}{d}$, $c < c_0$, or $p = \frac{4s_1}{d}$, $c \le c_0$, I_c^p has no minimizers. Now, when $\frac{4s_1}{d} \le p < \frac{4s_2}{d}$, if there exists $c_2 < c_0$, such that $I_{c_2}^p$ admits a minimizer $u_2 \in S_{c_2}$, then by Theorem 1.1(2), $E^p(u_2) = I_{c_2}^p = 0$, thus for $c \in (c_2, c_0)$,

$$I_c^p \le E^p\left(\frac{c}{c_2}u_2\right) = \left(\frac{c}{c_2}\right)^2\left[E^p(u_2) - \frac{\left(\frac{c}{c_2}\right)^p - 1}{p+2}\|u_2\|_{p+2}^{p+2}\right] = -\frac{\left(\frac{c}{c_2}\right)^2\left[\left(\frac{c}{c_2}\right)^p - 1\right]}{p+2}\|u_2\|_{p+2}^{p+2} < 0.$$

This contradicts Theorem 1.1(2). Therefore, when $\frac{4s_1}{d} \le p < \frac{4s_2}{d}$, for any $c < c_0$, I_c^p has no minimizers, then point (2) is verified.

To end the proof, we only need to show that when $p = \frac{4s_1}{d}$ and $c = c_0$, I_c^p has no minimizers. To this aim, we first claim that if $p = \frac{4s_1}{d}$, then

$$c_0 = \|Q_{s_1}\|_2, \tag{3.16}$$

where $c_0 = \sup\{c > 0 | I_c^p = 0\}$ given as (1.9), and Q_{s_1} is the ground state of (1.7) with $s = s_1$. Indeed, when $p = \frac{4s_1}{d}$, as (2.2), we have

$$E^p(u) \ge \frac{1}{2}\|\nabla_{s_1}u\|_2^2\left[1 - \left(\frac{c}{\|Q_{s_1}\|_2}\right)^{\frac{4s_1}{d}}\right], \quad \forall u \in S_c.$$

This implies that if $c \le \|Q_{s_1}\|_2$, then $E^p(u) \ge 0$, $\forall u \in S_c$, from which we deduce that $I_c^p = 0$, for all $c \le \|Q_{s_1}\|_2$. Consequently, by the definition of c_0, we can conclude that $c_0 = \|Q_{s_1}\|_2$ after showing that $I_c^p < 0$ as $c > \|Q_{s_1}\|_2$. To prove that, let $v := \frac{c}{\|Q_{s_1}\|_2}Q_{s_1}$, then $v \in S_c$ and

$$\begin{aligned} E^p(v^t) &= \left(\frac{c}{\|Q_{s_1}\|_2}\right)^2\frac{t^{2s_2}}{2}\|\nabla_{s_2}Q_{s_1}\|_2^2 + \left(\frac{c}{\|Q_{s_1}\|_2}\right)^2\frac{t^{2s_1}}{2}\|\nabla_{s_1}Q_{s_1}\|_2^2 - \left(\frac{c}{\|Q_{s_1}\|_2}\right)^{p+2}\frac{t^{2s_1}}{p+2}\|Q_{s_1}\|_{p+2}^{p+2} \\ &= \left(\frac{c}{\|Q_{s_1}\|_2}\right)^2\left[\frac{t^{2s_2}}{2}\|\nabla_{s_2}Q_{s_1}\|_2^2 + \frac{t^{2s_1}}{2}\|\nabla_{s_1}Q_{s_1}\|_2^2 - \left(\frac{c}{\|Q_{s_1}\|_2}\right)^p\frac{t^{2s_1}}{2}\|\nabla_{s_1}Q_{s_1}\|_2^2\right] \\ &= \left(\frac{c}{\|Q_{s_1}\|_2}\right)^2\left[\frac{t^{2s_2}}{2}\|\nabla_{s_2}Q_{s_1}\|_2^2 + \frac{t^{2s_1}}{2}\left(1 - \left(\frac{c}{\|Q_{s_1}\|_2}\right)^p\right)\|\nabla_{s_1}Q_{s_1}\|_2^2\right], \end{aligned} \tag{3.17}$$

which implies that if $c > \|Q_{s_1}\|_2$, then there exists $t_0 > 0$ small enough, such that $E^p(v^{t_0}) < 0$. Thus, it follows that $I_c^p < 0$ as $c > \|Q_{s_1}\|_2$. Therefore, the claim is proved.

Now we argue by contradiction to assume that when $p = \frac{4s_1}{d}$, $I_{c_0}^p$ has a minimizer $u_3 \in S_{c_0}$. Then by Lemma 2.2, $Q^p(u_3) = 0$. However, as (3.12), by the Gagliardo-Nirenberg inequality (1.5) with $s = s_1$,

$$Q^p(u_3) \ge s_2\|\nabla_{s_2}u_3\|_2^2 + s_1\|\nabla_{s_1}u_3\|_2^2\left[1 - \left(\frac{c_0}{\|Q_{s_1}\|_2}\right)^{\frac{4s_1}{d}}\right]. \tag{3.18}$$

Since $u_3 \in S_c$ and $c_0 = \|Q_{s_1}\|_2$ in (3.16), then (3.18) implies that $Q^p(u_3) > 0$, which is a contradiction. This completes the proof. □

Proof of Theorem 1.3. Let $u_c \in S_c$ be a minimizer of I_c^p, then clearly u_c is a critical point of E^p on the sphere S_c, thus there exists a Lagrange multiplier $\lambda_c \in \mathbb{R}$, such that the couple (u_c, λ_c) solves weakly equation (2.5), and that (2.6) holds for (u_c, λ_c). Then we derive from (2.6) that

$$\begin{aligned} \lambda_c\|u_c\|_2^2 &= \|u_c\|_{p+2}^{p+2} - \|\nabla_{s_1}u_c\|_2^2 - \|\nabla_{s_2}u_c\|_2^2, \\ &= \frac{p}{p+2}\|u_c\|_{p+2}^{p+2} - 2E^p(u_c). \end{aligned} \tag{3.19}$$

Note that $E^p(u_c) = I_c^p < 0$, see Theorem 1.2, then we obtain from (3.19) that $\lambda_c > 0$. On the other hand, from the definition of I_c^p and Definition 1.1, it follows immediately that u_c is a ground state solution of (1.3) with $\lambda = \lambda_c$. □

4 Mass critical and supercritical cases

In this section, we consider the existence and properties of normalized solutions in the mass critical and super-critical cases, namely $\frac{4s_2}{d} \le p < \frac{4s_2}{(d-2s_2)^+}$. Note that in these cases, the functional is unbounded from below, thus we consider the following local minimization problem.

$$J_c^p := \inf_{u \in V_c} E^p(u), \tag{4.1}$$

where V_c is defined by

$$V_c := \{u \in S_c \mid Q^p(u) = 0\}. \tag{4.2}$$

We point out that, to prove the existence of minimizers of J_c^p, we use some ideas from [24], but the calculations here are more complicated.

Note that the manifold V_c is indeed a natural constraint for E^p on S_c, since we have

Lemma 4.1. *Assume that $\frac{4s_2}{d} \le p < \frac{4s_2}{(d-2s_2)^+}$. Then for $c > 0$, each critical point of $E^p|_{V_c}$ is a critical point of $E^p|_{S_c}$.*

Proof. Let u be a critical point of $E^p|_{V_c}$, then by [14, Corollary 4.1.2] we have that either (i) $(Q^p)'(u)$ and $(\|u\|_2^2)'$ are linearly dependent, or (ii) there exist $\lambda_1, \lambda_2 \in \mathbb{R}$ such that

$$(E^p)'(u) + \lambda_1 (Q^p)'(u) + \lambda_2 u = 0, \quad \text{in } \left(H^{s_2}\right)^*. \tag{4.3}$$

Indeed, (i) is impossible. Let us assume that it is true, then for some $\lambda_0 \in \mathbb{R}$, we have

$$(Q^p)'(u) + \lambda_0 (\|u\|_2^2)' = 0, \quad \text{in } \left(H^{s_2}\right)^*,$$

or equivalently

$$2s_1(-\Delta)^{s_1}u + 2s_2(-\Delta)^{s_2}u - \frac{\mathrm{d}p}{2}|u|^p u + 2\lambda_0 u = 0, \quad \text{in } \left(H^{s_2}\right)^*. \tag{4.4}$$

As in the proof of Lemma 2.2, multiplying (4.4) by $\bar{u}$ and $x \cdot \nabla \bar{u}$, respectively, and integrating by parts, we have

$$2s_1\|\nabla_{s_1}u\|_2^2 + 2s_2\|\nabla_{s_2}u\|_2^2 - \frac{\mathrm{d}p}{2}\|u\|_{p+2}^{p+2} + 2\lambda_0\|u\|_2^2 = 0 \tag{4.5}$$

and

$$s_1(2s_1 - d)\|\nabla_{s_1}u\|_2^2 + s_2(2s_2 - d)\|\nabla_{s_2}u\|_2^2 + \frac{\mathrm{d}^2 p}{2(p+2)}\|u\|_{p+2}^{p+2} - \lambda_0 d\|u\|_2^2 = 0. \tag{4.6}$$

From (4.5) and (4.6), we obtain that

$$4s_1^2\|\nabla_{s_1}u\|_2^2 + 4s_2^2\|\nabla_{s_2}u\|_2^2 - \frac{\mathrm{d}^2 p^2}{2(p+2)}\|u\|_{p+2}^{p+2} = 0. \tag{4.7}$$

Note that $u \in V_c$ and then $Q^p(u) = 0$. By this identity and (4.7), we conclude that

$$4s_1(s_1 - s_2)\|\nabla_{s_1}u\|_2^2 - \frac{\mathrm{d}p(\mathrm{d}p - 4s_2)}{2(p+2)}\|u\|_{p+2}^{p+2} = 0,$$

which is a contradiction, since $u \in S_c$, $s_1 < s_2$, and $p \geq \frac{4s_2}{d}$.

Now to end the proof, we only need to show that $\lambda_1 = 0$ in (4.3). As the above, multiplying (4.3) by $\bar{u}$ and $x \cdot \nabla\bar{u}$, respectively, and then integrating by parts, we have

$$(1 + 2\lambda_1 s_1)\|\nabla_{s_1}u\|_2^2 + (1 + 2\lambda_1 s_2)\|\nabla_{s_2}u\|_2^2 - \left(1 - \frac{\mathrm{d}p\lambda_1}{2}\right)\|u\|_{p+2}^{p+2} + \lambda_2\|u\|_2^2 = 0 \tag{4.8}$$

and

$$(1 + 2\lambda_1 s_1)\frac{2s_1 - d}{2}\|\nabla_{s_1}u\|_2^2 + (1 + 2\lambda_1 s_2)\frac{2s_2 - d}{2}\|\nabla_{s_2}u\|_2^2 - \left(1 - \frac{\mathrm{d}p\lambda_1}{2}\right)\frac{-d}{p+2}\|u\|_{p+2}^{p+2} - \frac{\mathrm{d}\lambda_2}{2}\|u\|_2^2 = 0. \tag{4.9}$$

From (4.8) and (4.9), we have

$$(1 + 2\lambda_1 s_1)s_1\|\nabla_{s_1}u\|_2^2 + (1 + 2\lambda_1 s_2)s_2\|\nabla_{s_2}u\|_2^2 - \left(1 - \frac{\mathrm{d}p\lambda_1}{2}\right)\frac{\mathrm{d}p}{2(p+2)}\|u\|_{p+2}^{p+2} = 0. \tag{4.10}$$

Using the fact that $Q^p(u) = 0$, we then deduce from (4.10) that

$$\lambda_1\left[s_1^2\|\nabla_{s_1}u\|_2^2 + s_2^2\|\nabla_{s_2}u\|_2^2 + \frac{\mathrm{d}^2p^2}{8(p+2)}\|u\|_{p+2}^{p+2}\right] = 0,$$

which implies that $\lambda_1 = 0$, this ends the proof. □

To show that J_c^p is attained for some $u \in V_c$, we consider the following equivalent minimization problem:

$$\tilde{J}_c^p := \inf\{\tilde{E}^p(v) \mid v \in S_c, \quad Q^p(v) \leq 0\}, \tag{4.11}$$

where

$$\tilde{E}^p(v) := E^p(v) - \frac{2}{\mathrm{d}p}Q^p(v) = \frac{\mathrm{d}p - 4s_1}{2\mathrm{d}p}\|\nabla_{s_1}v\|_2^2 + \frac{\mathrm{d}p - 4s_2}{2\mathrm{d}p}\|\nabla_{s_2}v\|_2^2. \tag{4.12}$$

We can prove that

Lemma 4.2. *Assume that* $\frac{4s_2}{d} \leq p < \frac{4s_2}{(d-2s_2)^+}$. *Then for any* $c > 0$, *there holds that*

$$\tilde{J}_c^p = \inf\{\tilde{E}^p(v) \mid v \in S_c,\ Q^p(v) = 0\} = J_c^p. \tag{4.13}$$

Proof. By the definition, obviously $J_c^p \geq \tilde{J}_c^p$. In addition, for any $u \in S_c$ with $Q^p(u) \leq 0$, by the continuity of $Q^p(u^t)$ about t, there exists $t_0 \in (0, 1]$, such that $Q^p(u^{t_0}) = 0$. Thus,

$$J_c^p \leq \tilde{E}^p\left(u^{t_0}\right) \leq \tilde{E}^p(u).$$

Taking the infimum, we have $J_c^p \leq \tilde{J}_c^p$, then (4.13) follows. □

Lemma 4.3. *Assume that* $\frac{4s_2}{d} \leq p < \frac{4s_2}{(d-2s_2)^+}$. *Then when* $\frac{4s_2}{d} < p < \frac{4s_2}{(d-2s_2)^+}$, *the function* $c \mapsto J_c^p$ *is non-increasing on* $(0, +\infty)$, *and when* $p = \frac{4s_2}{d}$, $c \mapsto J_c^p$ *is non-increasing on* $(c_*, +\infty)$.

Proof. To prove the monotonicity of J_c^p, it is essential to show that

$$J_c^p = \inf_{u \in S_c}\max_{t>0} E^p(u^t), \quad \text{if } \frac{4s_2}{d} < p < \frac{4s_2}{(d-2s_2)^+}, \tag{4.14}$$

and

$$J_c^p = \inf_{u \in \mathcal{K}_c} \max_{t>0} E^p(u^t), \quad \text{if } p = \frac{4s_2}{d}, \tag{4.15}$$

where $\mathcal{K}_c := \{u \in S_c : \|\nabla_{s_2} u\|_2^2 < \frac{\mathrm{d}}{d+2s_2}\|u\|_{\frac{4s_2}{d}+2}^{\frac{4s_2}{d}+2}\}$. To this aim, the key is to show that for any given $u \in S_c$ if $\frac{4s_2}{d} < p < \frac{4s_2}{(d-2s_2)^+}$ or $u \in \mathcal{K}_c$ if $p = \frac{4s_2}{d}$, there exists a unique $t_u > 0$ such that $Q^p(u^{t_u}) = 0$, and $E^p(u^{t_u}) = \max_{t>0} E^p(u^t)$. To prove that, we observe from (2.3) that, for any $u \in S_c$,

$$\frac{\mathrm{d}}{\mathrm{d}t}E^p(u^t) = s_1 t^{2s_1-1}\|\nabla_{s_1} u\|_2^2 + s_2 t^{2s_2-1}\|\nabla_{s_2} u\|_2^2 - \frac{\mathrm{d}pt^{\frac{\mathrm{d}p}{2}-1}}{2(p+2)}\|u\|_{p+2}^{p+2} = \frac{1}{t}Q^p(u^t). \tag{4.16}$$

For given $a, b, c > 0$, define

$$h(t) := at^{2s_1-1} + bt^{2s_2-1} - ct^{\frac{\mathrm{d}p}{2}-1}, \quad \forall t > 0,$$

then $h(t) = t^{2s_2-1}k(t)$, with

$$k(t) := at^{2(s_1-s_2)} + b - ct^{\frac{\mathrm{d}p}{2}-2s_2}, \ t > 0.$$

It is not difficult to check that under either of the following two conditions,

(1) $\frac{4s_2}{d} < p < \frac{4s_2}{(d-2s_2)^+}$;

(2) $p = \frac{4s_2}{d}$ and $b - c < 0$,

the function $k(t)$ is strictly decreasing on $(0, \infty)$, with a unique zero point $t_0 > 0$. Thus,

$$\begin{cases} h(t) > 0, & 0 < t < t_0, \\ h(t) = 0, & t = t_0; \\ h(t) < 0, & t > t_0. \end{cases}$$

Setting:

$$a = s_1\|\nabla_{s_1} u\|_2^2, \quad b = s_2\|\nabla_{s_2} u\|_2^2, \quad c = \frac{\mathrm{d}p}{2(p+2)}\|u\|_{p+2}^{p+2},$$

we conclude that under either of the conditions: (1) $\frac{4s_2}{d} < p < \frac{4s_2}{(d-2s_2)^+}$ and any given $u \in S_c$, or (2) $p = \frac{4s_2}{d}$ and any given $u \in \mathcal{K}_c$ (note that when $p = \frac{4s_2}{d}$, $b - c < 0 \Leftrightarrow \|\nabla_{s_2} u\|_2^2 < \frac{\mathrm{d}}{d+2s_2}\|u\|_{\frac{4s_2}{d}+2}^{\frac{4s_2}{d}+2}$), there exists a unique $t_u > 0$ such that $Q^p(u^{t_u}) = 0$, and

$$\frac{\mathrm{d}}{\mathrm{d}t}E^p(u^t) > 0 \text{ if } t \in (0, t_u), \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t}E^p(u^t) < 0 \text{ if } t \in (t_u, \infty), \tag{4.17}$$

from which we deduce that $E^p(u^{t_u}) = \max_{t>0} E^p(u^t)$.

Note particularly that when $p = \frac{4s_2}{d}$, for any $u \in \mathcal{K}_c$, by the Gagliardo-Nirenberg inequality (1.5) with $s = s_2$,

$$\|\nabla_{s_2} u\|_2^2 < \frac{d}{d+2s_2}\|u\|_{\frac{4s_2}{d}+2}^{\frac{4s_2}{d}+2} \Rightarrow \|\nabla_{s_2} u\|_2^2 < \|Q_{s_2}\|_2^{-\frac{4s_2}{d}}\|u\|_2^{\frac{4s_2}{d}}\|\nabla_{s_2} u\|_2^2 \Leftrightarrow c > \|Q_{s_2}\|_2 = c_*.$$

To end the proof, we clarify that since the other cases are standard as those in [11, Lemma 5.3], for simplicity, here we omit the details. □

Proposition 4.1. *Assume that* $\frac{4s_2}{d} \le p < \frac{4s_2}{(d-2s_2)^+}$, *then under either of the following conditions*:

(1) $\frac{4s_2}{d} < p < \frac{4s_2}{(d-2s_2)^+}$ *and* $c \in (0, +\infty)$;

(2) $p = \frac{4s_2}{d}$ *and* $c \in (c_*, +\infty)$,

we have $J_c^p > 0$. *Moreover, there exists* $u \in H^{s_2}\backslash\{0\}$, *such that* $\widetilde{E}^p(u) = J_c^p$ *and* $u \in V_{\|u\|_2}$ *with* $0 < \|u\|_2 \le c$.

Proof. We first show that $J_c^p > 0$. Indeed, for any $v \in S_c$ with $Q(v) \le 0$, by the Gagliardo-Nirenberg inequality (1.5) with $s = s_1$, we have

$$s_1\|\nabla_{s_1}v\|_2^2 + s_2\|\nabla_{s_2}v\|_2^2 \le \frac{dp}{2(p+2)}\|v\|_{p+2}^{p+2} \le \frac{dpB(p,d,s_1)}{2(p+2)}\|\nabla_{s_1}v\|_2^{\frac{dp}{2s_1}}\|v\|_2^{p+2-\frac{dp}{2s_1}} \tag{4.18}$$

which implies that

$$\frac{2s_1(p+2)}{dpB(p,d,s_1)}c^{\frac{dp}{2s_1}-p-2} \le \|\nabla_{s_1}v\|_2^{\frac{dp}{2s_1}-2}. \tag{4.19}$$

Therefore, in both cases, by (4.12), (4.19), and the definition of $\widetilde{J}_c^p$, we see that $\widetilde{J}_c^p > 0$. Thus by Lemma 4.2, $J_c^p > 0$ is verified.

Now let $\{v_n\}$ be an arbitrary minimizing sequence for (4.11), i.e., $\{v_n\} \subset S_c$, $Q^p(v_n) \le 0$, and $\widetilde{E}^p(v_n) \to \widetilde{J}_c^p = J_c^p$ as $n \to \infty$. We claim that $\{v_n\}$ is bounded in H^{s_2}. Indeed, since $v_n \in S_c$, it is enough to verify the boundedness of $\{\|\nabla_{s_i}v_n\|_2\}(i = 1, 2)$. When $\frac{4s_2}{d} < p < \frac{4s_2}{(d-2s_2)^+}$, by (4.12), $\{\|\nabla_{s_i}v_n\|_2\}(i = 1, 2)$ is surely bounded. When $p = \frac{4s_2}{d}$, by (4.12), $\{\|\nabla_{s_1}v_n\|_2\}$ is bounded, then by (4.18) we observe that $\{\|\nabla_{s_2}v_n\|_2\}$ is bounded. Thus, the claim is verified. In addition, by (4.18) and (4.19), there exists $C_0 > 0$ such that

$$\int_{\mathbb{R}^d} |v_n|^{p+2}\mathrm{d}x \ge C_0 > 0.$$

Using the same arguments as in the proof of Proposition 3.1, up to a translation and a subsequence (still denoted by $\{v_n\}$), there exists $u \in H^{s_2}\backslash\{0\}$ such that

$$v_n \rightharpoonup u \ne 0 \text{ in } H^{s_2}.$$

Moreover, from Lemma A.1 we deduce that

$$Q^p(v_n) - Q^p(v_n - u) - Q^p(u) \to 0, \tag{4.20}$$

$$\widetilde{E}^p(v_n) - \widetilde{E}^p(v_n - u) - \widetilde{E}^p(u) \to 0, \tag{4.21}$$

$$\|v_n\|_2^2 - \|v_n - u\|_2^2 - \|u\|_2^2 \to 0. \tag{4.22}$$

By (4.22), $0 < \|u\|_2 \le c$. We claim that $Q^p(u) \le 0$, which indeed can be proved by excluding the other possibilities:

(1) If $Q^p(u) > 0$ and $\|u\|_2 < c$, it follows from (4.20) and $Q^p(v_n) \le 0$ that $Q^p(v_n - u) \le 0$ for sufficiently large n. Set $c_1 = \sqrt{c^2 - \|u\|_2^2}$ and $w_n = c_1\|v_n - u\|_2^{-1}(v_n - u)$, then

$$\|v_n - u\|_2 \to c_1, \quad w_n \in S_{c_1}, \quad \text{and} \quad Q^p(w_n) \le 0.$$

Thus, by the definition of $\widetilde{J}_c^p$, it follows that

$$\widetilde{E}^p(w_n) \ge \widetilde{J}_{c_1}^p \quad \text{and} \quad \widetilde{E}^p(v_n - u) \ge \widetilde{J}_{c_1}^p.$$

From $Q^p(w_n) \le 0$ and (3.12), we see in particular that $c_1 > c_*$ as $p = \frac{4s_2}{d}$. Then applying Lemma 4.3, $\widetilde{J}_{c_1}^p = J_{c_1}^p \ge J_c^p$, and by (4.21) we obtain

$$\widetilde{E}^p(u) = \frac{dp - 4s_1}{2dp}\|\nabla_{s_1}u\|_2^2 + \frac{dp - 4s_2}{2dp}\|\nabla_{s_2}u\|_2^2 \le 0,$$

which is impossible, since due to $u \in S_c$ and $p \ge \frac{4s_2}{d} > \frac{4s_1}{d}$, we have $\widetilde{E}^p(u) > 0$.

(2) If $Q^p(u) > 0$ and $\|u\|_2 = c$, then $v_n \to u$ in L^2 as $n \to \infty$. This implies that $v_n \to u$ in L^{p+2} as $n \to \infty$. On the other hand, we deduce from $Q^p(u) > 0$ that $Q^p(v_n - u) \le 0$ for sufficiently large n. Thus, we can obtain $v_n \to u$ in H^{s_2} as $n \to \infty$. This yields $Q^p(v_n - u) \to 0$ as $n \to \infty$. Thus, it follows from (4.20) and $Q^p(u) > 0$ that $Q^p(v_n) > 0$ for sufficiently large n, which contradicts $Q^p(v_n) \le 0$.

Therefore, we have $Q^p(u) \le 0$ and $0 < \|u\|_2 \le c$. In particular, by (3.12), $c_* < \|u\|_2 \le c$ when $p = \frac{4s_2}{d}$. Thus, we deduce from the definition of $\widetilde{J}_c^p$, and Lemma 4.3 that

$$\widetilde{J}_c^p \le \widetilde{J}_{\|u\|_2}^p \le \widetilde{E}^p(u) \le \lim_{n\to\infty} \widetilde{E}^p(v_n) = \widetilde{J}_c^p,$$

which shows that $\widetilde{E}^p(u) = \widetilde{J}_c^p$. Using this equality, we can prove that $Q^p(u) = 0$. Indeed, if $Q^p(u) < 0$, then there exists $t_0 \in (0, 1)$ such that $Q^p(u^{t_0}) = 0$, thus by Lemma 4.3,

$$\widetilde{J}_c^p \le \widetilde{J}_{\|u\|_2}^p \le \widetilde{E}^p\big(u^{t_0}\big) < \widetilde{E}^p(u) = \widetilde{J}_c^p,$$

which is a contradiction. Therefore, by Lemma 4.2 we conclude that $\widetilde{E}^p(u) = J_c^p$ and $Q^p(u) = 0$. Then the proof is complete. □

To prove Theorem 1.4, we need the following two lemmas.

Lemma 4.4. *Assume that* $\frac{4s_2}{d} \le p \le \frac{4s_1}{d-2s_1}$, $2s_1 < d \le \frac{2s_1 s_2}{s_2 - s_1}$. *Let* $u \in H^{s_2}$ *be a non-trivial weak solution of the equation*

$$(-\Delta)^{s_1}u + (-\Delta)^{s_2}u + \lambda u - |u|^p u = 0, \quad x \in \mathbb{R}^d, \tag{4.23}$$

then necessarily $\lambda > 0$.

Proof. Since $u \in H^{s_2}(\mathbb{R}^d)$ is a non-trivial weak solution of (4.23), we deduce from (2.6) and (2.8) that

$$\begin{aligned}\lambda\|u\|_2^2 &= \|u\|_{p+2}^{p+2} - \|\nabla_{s_1}u\|_2^2 - \|\nabla_{s_2}u\|_2^2\\ &= \frac{4s_1 - (d-2s_1)p}{\mathrm{d}p}\|\nabla_{s_1}u\|_2^2 + \frac{4s_2 - (d-2s_2)p}{\mathrm{d}p}\|\nabla_{s_2}u\|_2^2.\end{aligned} \tag{4.24}$$

If $\frac{4s_2}{d} \le p \le \frac{4s_1}{d-2s_1}$, $2s_1 < d \le \frac{2s_1 s_2}{s_2 - s_1}$, and $u \ne 0$, then from (4.24), we observe that $\lambda > 0$. □

Lemma 4.5. *Assume that* $\frac{4s_2}{d} \le p \le \frac{4s_1}{d-2s_1}$, $2s_1 < d \le \frac{2s_1 s_2}{s_2 - s_1}$. *Then the mapping* $c \mapsto J_c^p$ *is strictly decreasing on the interval* $(0, +\infty)$.

Proof. By the fact that $c \mapsto J_c^p$ is non-increasing proved in Lemma 4.3, if we assume by contradiction that

$$J_c^p \equiv J_{c_3}^p, \quad \forall c \in (c_2, c_3),$$

for some $c_2, c_3 > 0$. Then by Proposition 4.1, for $c_3 > 0$, there exists $u_0 \in V_{\|u_0\|_2}$ such that

$$E^p(u_0) = J_{c_3}^p \quad \text{and} \quad 0 < \|u_0\|_2 < c_3.$$

Hence, u_0 is a free minimizer of E^p on the manifold $\mathcal{M} := \{u \in H^{s_2}\backslash\{0\} : \quad Q^p(u) = 0\}$. Thus, there exists a Lagrange multiplier $\lambda_3 \in \mathbb{R}$, such that

$$(E^p)'(u_0) + \lambda_3 (Q^p)'(u_0) = 0, \quad \text{in } \big(H^{s_2}\big)^*.$$

Using the same argument as in the proof of Lemma 4.1, we deduce that $\lambda_3 = 0$. Then

$$(E^p)'(u_0) = 0, \quad \text{in } \big(H^{s_2}\big)^*. \tag{4.25}$$

This means that u_0 is a non-trivial weak solution of (4.23) with $\lambda = 0$. However, by Lemma 4.4, this is impossible if $\frac{4s_2}{d} \le p \le \frac{4s_1}{d-2s_1}$, $2s_1 < d \le \frac{2s_1 s_2}{s_2 - s_1}$. Then the strict decreasing property of J_c^p about $c > 0$ follows. □

Proof of Theorem 1.4. By Proposition 4.1 and Lemma 4.5, we conclude immediately that when $\frac{4s_2}{d} \le p \le \frac{4s_1}{d-2s_1}$, $2s_1 < d \le \frac{2s_1 s_2}{s_2 - s_1}$, and $c > 0$, or when $p = \frac{4s_2}{d}$ and $c > c_*$, $J_c^p > 0$ and J_c^p admits at least one minimizer $u_c \in V_c$. In particular, by Lemma 4.1, u_c is a critical point of E^p on S_c. Thus, there exists $\lambda_c > 0$, such that (u_c, λ_c) solves weakly equation (1.12). To show the behavior of u_c as $c \to 0^+$, note that using $Q^p(u_c) = 0$ and the Gagliardo-Nirenberg inequality (1.5), we have as (4.19) that

$$\frac{2s_i(p+2)}{dpB(p,d,s_i)}c^{\frac{dp}{2s_i}-p-2} \le \|\nabla_{s_i}v\|_2^{\frac{dp}{2s_i}-2}, \quad i = 1, 2. \tag{4.26}$$

Note that $\frac{dp}{2s_i} - p - 2 < 0$, $\frac{dp}{2s_i} - 2 > 0$ if $\frac{4s_2}{d} < p < \frac{4s_2}{(d-2s_2)^+}$, then by (4.26),

$$\|\nabla_{s_1}u_c\|_2 \to +\infty, \quad \|\nabla_{s_2}u_c\|_2 \to +\infty, \quad \text{as} \quad c \to 0^+ \tag{4.27}$$

and thus $E^p(u_c) \to +\infty$ as $c \to 0^+$, by the following calculation ($Q^p(u_c) = 0$ is used)

$$E^p(u_c) = E^p(u_c) - \frac{2}{dp}Q^p(u_c) = \frac{dp - 4s_1}{2dp}\|\nabla_{s_1}u_c\|_2^2 + \frac{dp - 4s_2}{2dp}\|\nabla_{s_2}u_c\|_2^2. \tag{4.28}$$

Finally, the fact that $\lambda_c \to +\infty$ as $c \to 0^+$, follows from (4.24) and (4.27). Then the proof is complete. □

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Appendix

Lemma A.1. *Assume that* $\{u_n\} \subset H^s(\mathbb{R}^d)$ *and* $u \in H^s(\mathbb{R}^d)$, *being such that*

$$u_n \rightharpoonup u, \quad \text{in } H^s(\mathbb{R}^d),$$

then

$$\|\nabla_s u_n\|_2^2 - \|\nabla_s(u_n - u)\|_2^2 - \|\nabla_s u\|_2^2 = o_n(1), \quad \text{as } n \to +\infty. \tag{A1}$$

Proof. Let $v_n := u_n - u$, then $u_n = v_n + u$, and $v_n \rightharpoonup 0$, in $H^s(\mathbb{R}^d)$. Thus, it is not difficult to calculate that

$$\begin{aligned} &\|\nabla_s u_n\|_2^2 - \|\nabla_s(u_n - u)\|_2^2 - \|\nabla_s u\|_2^2 \\ &= 2 \int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{v_n(x)u(x) + v_n(y)u(y) - v_n(x)u(y) - v_n(y)u(x)}{|x-y|^{d+2s}} \mathrm{d}x\mathrm{d}y \\ &= 4 \int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{v_n(x)[u(x) - u(y)]}{|x-y|^{d+2s}} \mathrm{d}x\mathrm{d}y, \end{aligned} \tag{A2}$$

where in the last inequality we used the fact that

$$\begin{aligned} \int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{v_n(x)u(x)}{|x-y|^{d+2s}} \mathrm{d}x\mathrm{d}y &= \int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{v_n(y)u(y)}{|x-y|^{d+2s}} \mathrm{d}x\mathrm{d}y, \\ \int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{v_n(x)u(y)}{|x-y|^{d+2s}} \mathrm{d}x\mathrm{d}y &= \int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{v_n(y)u(x)}{|x-y|^{d+2s}} \mathrm{d}x\mathrm{d}y. \end{aligned}$$

Note that by changing the variable of the integrals, we have

$$\begin{aligned} \int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{v_n(x)[u(x) - u(y)]}{|x-y|^{d+2s}} \mathrm{d}x\mathrm{d}y &= \frac{1}{2} \int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{v_n(x)[u(x) - u(y)] + v_n(y)[u(y) - u(x)]}{|x-y|^{d+2s}} \mathrm{d}x\mathrm{d}y \\ &= \frac{1}{2} \int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{[v_n(x) - v_n(y)][u(x) - u(y)]}{|x-y|^{d+2s}} \mathrm{d}x\mathrm{d}y \\ &= \int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{[v_n(x) - v_n(y)]u(x)}{|x-y|^{d+2s}} \mathrm{d}x\mathrm{d}y. \end{aligned} \tag{A3}$$

Since $v_n \rightharpoonup 0$, in $H^s(\mathbb{R}^d)$, then

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{[v_n(x) - v_n(y)]u(x)}{|x-y|^{d+2s}} \mathrm{d}x\mathrm{d}y \to 0, \quad \text{as } n \to +\infty,$$

thus by (A2) and (A3) we obtain (A1). This completes the proof. □