Research Article

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Characterizations of Urysohn universal ultrametric spaces

https://doi.org/10.1515/agms-2025-0024
received September 18, 2024; accepted April 23, 2025

Abstract: In this article, using the existence of infinite equidistant subsets of closed balls, we first characterize the injectivity of ultrametric spaces for finite ultrametric spaces. This method also gives characterizations of the Urysohn universal ultrametric spaces. As an application, we find that the operations of the Cartesian product and the hyperspaces preserve the structures of the Urysohn universal ultrametric spaces. Namely, let (X, d) be the Urysohn universal ultrametric space. Then, we show that $(X \times X, d \times d)$ is isometric to (X, d), and show that the hyperspace consisting of all non-empty compact subsets of (X, d) and symmetric products of (X, d) are isometric to (X, d). We also establish that every complete ultrametric space injective for finite ultrametric space contains a subspace isometric to (X, d).

Keywords: Urysohn universal ultrametric spaces

MSC 2020: Primary 54E35, Secondary 51F99

1 Introduction

1.1 Backgrounds

For a class $\mathcal{C}$ of metric spaces, we say that a metric space (X, d) is *$\mathcal{C}$-injective* or *injective for $\mathcal{C}$* if it is non-empty and for every pair of metric spaces (Y, e) and (Z, h) in $\mathcal{C}$, and for every pair of isometric embeddings $\phi : Y \to Z$ and $f : Y \to X$, there exists an isometric embedding $F : Z \to X$ such that $F \circ \phi = f$. In other words, (X, d) is $\mathcal{C}$-injective if and only if for every metric space (Z, h) in $\mathcal{C}$, and for every metric subspace (Y, e) of (Z, h) with $(Y, e) \in \mathcal{C}$, every isometric embedding from (Y, e) into (X, d) can be extended to an isometric embedding from (Z, h) into (X, d). We denote by $\mathcal{F}$ the class of all finite metric spaces. Urysohn [29] constructed a separable complete $\mathcal{F}$-injective metric space and proved its uniqueness up to isometry. This space is nowadays called the *Urysohn universal (metric) space*, and it has been studied in geometry, topology, dynamics, and model theory. In what follows, let $(\mathbb{U}, \rho)$ denote the Urysohn universal metric space. Our main subjects of this article are non-Archimedean analogues of $(\mathbb{U}, \rho)$. For more information of $(\mathbb{U}, \rho)$ and related topics, we refer the readers to, for instance, [3,8,14,18,20,23–25,30,32].

To explain our main results and backgrounds, we prepare some notations and notions. A metric d on a set X is said to be an *ultrametric* if it satisfies the so-called strong triangle inequality $d(x, y) \leq d(x, z) \vee d(z, y)$ for all $x, y, z \in X$, where $\vee$ stands for the maximal operator on $\mathbb{R}$. We say that a set R is a *range set* if it is a subset of $[0, \infty)$ and $0 \in R$. For a range set R, an ultrametric space (X, d) is *R-valued* if $d(x, y) \in R$ for all $x, y \in R$. For a range set R, we denote by $\mathcal{N}(R)$ the class of all finite R-valued ultrametric spaces.

For a finite or countable range set R, we say that an ultrametric (X, d) is the *R-Urysohn universal ultrametric space* if it is a separable complete $\mathcal{N}(R)$-injective R-valued ultrametric space. Similar to $(\mathbb{U}, \rho)$, in the

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case where R is finite or countable, using the so-called back-and-forth argument, the uniqueness up to isometry of the R-Urysohn universal ultrametric space has been proven in [11] and [3].

Gao and Shao [11] provided various constructions of the R-Urysohn universal ultrametric spaces for every finite or countable range set R. Wan [32] established that the space of all non-empty compact R-valued ultrametric spaces equipped with the Gromov-Hausdorff ultrametric is the R-Urysohn universal ultrametric space for every finite or countable range set R. Ishiki [15] gave other constructions of Urysohn universal ultrametric spaces using mapping spaces and spaces of continuous pseudo-ultrametrics.

Since for every uncountable range set R all $\mathcal{N}(R)$-injective ultrametric spaces are non-separable (this is a simple corollary of [7, (12) in Theorem 1.6]), mathematicians often treat the R-Urysohn universal ultrametric space only in the case where R is finite or countable to guarantee the separability. In [16], to expand the theory of Urysohn universal space into the non-separable spaces, for every uncountable range set R, the author introduced the *R-petaloid ultrametric space*, which is intended to be a standard class of non-separable Urysohn universal ultrametric spaces. The author also proved its uniqueness up to isometry using the back-and-forth argument together with transfinite induction ([16, Theorem 1.1] and Theorem 2.2 in the present article). The definition and basic properties of the petaloid spaces will be presented in Section 2 in this article.

Based on the notion of the petaloid ultrametric space, even when R is uncountable, we introduce the R-Urysohn universal ultrametric space as the R-petaloid ultrametric space. Namely, in this article, we define the concept of the R-Urysohn universal ultrametric space as follows.

Definition 1.1. Let R be a range set. When R is finite or countable, the *R-Urysohn universal ultrametric space* means a (unique) separable complete $\mathcal{N}(R)$-injective R-valued ultrametric space. When R is uncountable, the *R-Urysohn universal ultrametric space* is the R-petaloid ultrametric space.

As a development of the theory of the Urysohn universal ultrametric spaces (and the petaloid spaces), in this article, we investigate the characterizations of injective spaces for finite ultrametric spaces, and we prove that the Cartesian product and the hyperspaces preserve the structures of the Urysohn universal ultrametric spaces. We also establish that every complete $\mathcal{N}(R)$-injective ultrametric spaces contains a subspace isometric to the R-Urysohn universal ultrametric space.

1.2 Main results

We now explain our main results. We prepare some definitions.

We denote by ω_0 the set of all non-negative integers according to a set-theoretical notation. Remark that $\omega_0 = \mathbb{Z}_{\geq 0}$ as a set and $\omega_0 = \aleph_0$ in set theory. We sometimes write $n < \omega_0$ if $n \in \omega_0$. Thus, the relation $\kappa \leq \omega_0$ means that $\kappa \in \omega_0$ or $\kappa = \omega_0$.

For a set E, we write $\mathrm{Card}(E)$ as the cardinality of E.

Definition 1.2. For a cardinal κ, and for a range set R, we denote by $\mathcal{N}(R, \kappa)$ the class of all R-valued ultrametric spaces (X, d) such that $\mathrm{Card}(X) < \kappa$. Remark that if $\kappa = \omega_0$, then $\mathcal{N}(R, \kappa)$ coincides with the class $\mathcal{N}(R)$. In what follows, we use $\mathcal{N}(R, \omega_0)$ rather than $\mathcal{N}(R)$.

For a metric space (X, d), for $a \in X$, and for $r \in (0, \infty)$, we denote by $\mathrm{B}(a, r; d)$ the closed ball centered at a with radius r. We often simply write $\mathrm{B}(a, r)$ instead of $\mathrm{B}(a, r; d)$.

For a metric space (X, d) and for $r \in (0, \infty)$, a subset A of X is called an *r-equidistant set* if $d(x, y) = r$ for all distinct $x, y \in A$.

Definition 1.3. Let $\kappa \leq \omega_0$ and R be a range set. An ultrametric space (X, d) is said to be (R, κ)-*haloed* if for every $a \in X$ and for every $r \in R \setminus \{0\}$, there exists an r-equidistant subset A of $\mathrm{B}(a, r)$ such that $\kappa \leq \mathrm{Card}(A)$ (Figure 1).

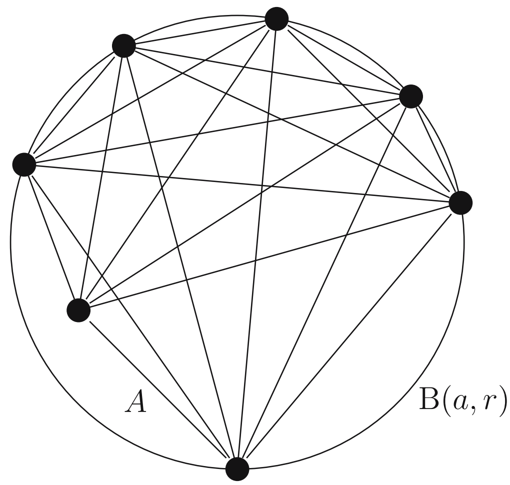

Figure 1: (R, κ)-haloed: each line has the same length r.

Definition 1.4. Let $\kappa \leq \omega_0$ and R be a range set. An ultrametric space (X, d) is (R, κ)-*avoidant* if for every $a \in X$, every $r \in R \setminus \{0\}$, and for every subset A of $\mathrm{B}(a, r)$ with $\mathrm{Card}(A) < \kappa$, there exists $p \in \mathrm{B}(a, r)$ such that $d(x, p) = r$ for all $x \in A$ (Figure 2).

The following theorem is our first main result.

Theorem 1.1. *Let R be a range set. Let (X, d) be an R-valued ultrametric space. Then, for every fixed integer $n \in \mathbb{Z}_{\geq 1}$, the following statements are equivalent*:

(A1) *The space (X, d) is (R, n)-haloed.*
(A2) *The space (X, d) is (R, n)-avoidant.*
(A3) *The space (X, d) is $\mathcal{N}(R, n + 1)$-injective.*

As a consequence, the following statements are equivalent:

(B1) *The space (X, d) is (R, ω_0)-haloed.*
(B2) *The space (X, d) is (R, ω_0)-avoidant.*
(B3) *The space (X, d) is $\mathcal{N}(R, \omega_0)$-injective.*

Remark 1.1. Some of the equivalences in Theorem 1.1 have been discovered. The equivalence between the conditions (B2) and (B3) is implicitly proven in [15, Lemma 2.48]. The proof of key lemma in [27] indicates the

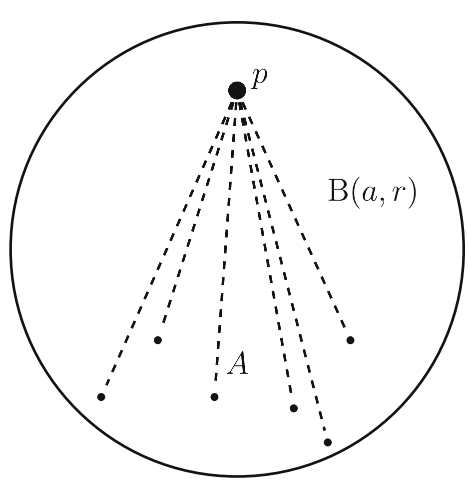

Figure 2: (R, κ)-avoidant: each dashed line has the same length r.

equivalence between the conditions (B1) and (B3). In [17, Lemma 1 in §2], it is proven that a metric space (X, d) is $\mathcal{N}(R, \omega_0)$-injective if and only if for every $r \in R$, for every finite r-equidistant subset A of X, there exists a point $y \in X$ with $y \notin A$ such that $A \sqcup \{y\}$ is r-equidistant. This means that the condition (B3) is equivalent to the condition that (X, d) is (R, n)-haloed for all $n \in \mathbb{Z}_{\geq 0}$ (Lemma 3.1). This article first clearly summarizes these individual implicit results as Theorem 1.1, which enables us to highlight new aspects of isometry classes of the Urysohn universal ultrametric spaces appearing in the following (for instance, see Theorems 1.2 and 1.4, and Lemma 3.6, and Proposition 4.5). The author also expects that Theorem 1.1 and its proof method can be extended to the study of $\mathcal{N}(R, \kappa)$-injective spaces for a finite or uncountable κ.

We next explain the applications of Theorem 1.1. For metric spaces (X, d) and (Y, e), we define the *ℓ^∞-product metric $d \times_\infty e$ on $X \times Y$* by $(d \times_\infty e)((x, y), (u, v)) = d(x, u) \vee e(y, v)$. For $p \in [1, \infty)$, we define the *ℓ^p-product metric $d \times_p e$* by $(d \times_p e)((x, y), (u, v)) = (d(x, u)^p + e(y, v)^p)^{1/p}$.

First, we prove that the Cartesian product preserves the structure of the Urysohn universal ultrametric spaces.

Theorem 1.2. *For every pair of range sets R_0 and R_1, let (X, d) and (Y, e) denote the R_0-Urysohn and the R_1-Urysohn universal ultrametric spaces, respectively. Then, the product $(X \times Y, d \times_\infty e)$ is the $(R_0 \cup R_1)$-Urysohn universal ultrametric space. In particular, for a range set R, if (X, d) is the R-Urysohn universal ultrametric space, then so is $(X \times X, d \times_\infty d)$. Namely, the space $(X \times X, d \times_\infty d)$ is isometric to (X, d).*

In contrast, we observe that the Cartesian product of the ordinary Urysohn universal metric space $(\mathbb{U}, \rho)$ is not isometric to $(\mathbb{U}, \rho)$. Recall that $\mathcal{F}$ stands for the class of all finite metric spaces, and $(\mathbb{U}, \rho)$ is defined as the complete separable $\mathcal{F}$-injective metric space, whereas the R-Urysohn universal ultrametric space is characterized by $\mathcal{N}(R)$-injectivity.

Theorem 1.3. *For any $p \in [1, \infty]$, the product space $(\mathbb{U} \times \mathbb{U}, \rho \times_p \rho)$ is not $\mathcal{F}$-injective. In particular, it is not isometric to $(\mathbb{U}, \rho)$.*

For a metric space (X, d) and for a subset of X, we write $\mathrm{diam}(A)$ as the diameter of A. We denote by $\mathcal{K}(X)$ the set of all non-empty compact subsets of X. For $m \in \mathbb{Z}_{\geq 2} \sqcup \{\infty\}$ and $l \in (0, \infty]$, we also denote by $\mathcal{K}_{m,l}(X)$ the space of all $E \in \mathcal{K}(X)$ such that $\mathrm{Card}(E) \leq m$ and $\mathrm{diam}(E) \leq l$, where in the case of $m = l = \infty$, the inequalities $\mathrm{Card}(E) \leq \infty$ and $\mathrm{diam}(E) \leq \infty$ mean that there are no restrictions on $\mathrm{Card}(E)$ and $\mathrm{diam}(E)$. In particular, $\mathcal{K}_{\infty,\infty}(X) = \mathcal{K}(X)$. For $E, F \in \mathcal{K}(X)$, we define the Hausdorff distance $\mathcal{HD}_d(E, F)$ by

$$\mathcal{HD}_d(E, F) = \max\left\{\sup_{a \in E} d(a, F), \sup_{b \in F} d(b, E)\right\}.$$

Note that $\mathcal{HD}_d$ is actually a metric on $\mathcal{K}(X)$. We represent the restricted metric $\mathcal{HD}_d$ on $\mathcal{K}_{m,l}(X)$ as the same symbol $\mathcal{HD}_d$ to the ambient metric. Remark that the space $(\mathcal{K}_{m,\infty}(X), \mathcal{HD}_d)$ is sometimes called the *mth symmetric product of* (X, d). In this article, the word "hyperspace" means a "space of subspace" of a given metric space. Throughout this article, the readers may regard "hyperspaces" as spaces in the form $(\mathcal{K}_{m,l}(X), \mathcal{HD}_d)$.

We next show that the hyperspace keeps the structure of the Urysohn universal ultrametric spaces.

Theorem 1.4. *Let $m \in \mathbb{Z}_{\geq 2} \sqcup \{\infty\}$, R be a range set, $l \in (0, \infty]$, and let (X, d) be the R-Urysohn universal ultrametric space. Then, $(\mathcal{K}_{m,l}(X), \mathcal{HD}_d)$ is the R-Urysohn universal ultrametric space. In particular, the space $(\mathcal{K}_{m,l}(X), \mathcal{HD}_d)$ is isometric to (X, d).*

We also establish that for every range set R, every complete $\mathcal{N}(R)$-injective ultrametric space contains a subspace isometric to the R-Urysohn universal ultrametric space.

Theorem 1.5. *Let R be an uncountable range set. If (X, d) is a complete $\mathcal{N}(R, \omega_0)$-injective ultrametric space, then (X, d) contains a subspace F isometric to the R-Urysohn universal ultrametric space. Moreover, if K is a compact subset of X satisfying $d(K^2) \subseteq R$, then we can choose F so that $K \subseteq F$.*

Remark 1.2. Theorem 1.5 in the case where R is finite or countable and an analogue of Theorem 1.5 for $(\mathbb{U}, \rho)$ can be proven by the back-and-forth argument.

This article is organized as follows. In Section 2, we prepare basic notations and notions. We will explain the petaloid ultrametric space. In Section 3, we characterize the $\mathcal{N}(R)$-injectivity using the existence of infinite equidistant subsets of closed balls (Theorem 1.1). We divide the proof of Theorem 1.1 into three parts (Lemmas 3.2, 3.3, and 3.4). As a consequence, we prove Theorem 1.2 on the product of Urysohn universal ultrametric spaces. In Section 4, we clarify the metric structures of the hyperspace of compact subsets and symmetric products of the R-Urysohn universal ultrametric space (Theorem 1.4). Section 5 is devoted to proving that all complete $\mathcal{N}(R)$-injective ultrametric spaces contain the R-Urysohn universal ultrametric space (Theorem 1.5). Section 6 presents some questions on the isometry problems of the product and hyperspaces of universal metric spaces such as the Urysohn universal metric space and the random graph.

2 Preliminaries

2.1 Generalities

The proof of the next lemma is presented in [28, Propositions 18.2, 18.4, and 18.5].

Lemma 2.1. *Let (X, d) be an ultrametric space. Then, the following are true*:
(1) *For every triple $x, y, z \in X$, the inequality $d(x, z) < d(z, y)$ implies $d(z, y) = d(x, y)$.*
(2) *For every $a \in X$, for every $r \in (0, \infty)$, and for every $q \in B(a, r)$ we have $\mathrm{B}(a, r) = \mathrm{B}(q, r)$.*
(3) *For every pair $a, b \in X$, and for every pair $r, l \in (0, \infty)$, if $\mathrm{B}(a, r) \cap \mathrm{B}(b, l) \neq \emptyset$, then we have $\mathrm{B}(a, r) \subseteq \mathrm{B}(b, l)$ or $\mathrm{B}(b, l) \subseteq \mathrm{B}(a, r)$.*

2.2 Urysohn universal ultrametric spaces

A subset E of $[0, \infty)$ is said to be *semi-sporadic* if there exists a strictly decreasing sequence $\{a_i\}_{i \in \mathbb{Z}_{\geq 0}}$ in $(0, \infty)$ such that $\lim_{i \to \infty} a_i = 0$ and $E = \{0\} \cup \{a_i | i \in \mathbb{Z}_{\geq 0}\}$. A subset of $[0, \infty)$ is said to be *tenuous* if it is finite or semi-sporadic [15]. For a range set R, we denote by $\mathbf{TEN}(R)$ the set of all tenuous range subsets of R. For a metric space (X, d), and for a subset A of X, we write $d(x, A) = \inf_{a \in A} d(x, a)$. In this article, we often represent a restricted metric as the same symbol to the ambient one. We now provide the definition of the petaloid ultrametric space that is introduced in [16] by the author.

Definition 2.1. Let R be an uncountable range set. We say that a metric space (X, d) is *R-petaloid* if it is an R-valued ultrametric space and there exists a family $\{\Pi(X, S)\}_{S \in \mathbf{TEN}(R)}$ of subspaces of X satisfying the following properties:
(P1) If $S \in \mathbf{TEN}(R)$ satisfies $S \neq \{0\}$, then $(\Pi(X, S), d)$ is isometric to the S-Urysohn universal ultrametric space. Namely, $(\Pi(X, S), d)$ is a separable complete $\mathcal{N}(S)$-injective S-valued ultrametric space.
(P2) We have $\bigcup_{S \in \mathbf{TEN}(R)} \Pi(X, S) = X$.
(P3) If $S, T \in \mathbf{TEN}(R)$, then $\Pi(X, S) \cap \Pi(X, T) = \Pi(X, S \cap T)$.
(P4) If $S, T \in \mathbf{TEN}(R)$ and $x \in \Pi(X, T)$, then $d(x, \Pi(X, S))$ belongs to $\{0\} \cup (T \setminus S)$.

We call the family $\{\Pi(X, S)\}_{S \in \mathbf{TEN}(R)}$ an *R-petal of X* and call $\Pi(X, S)$ the S-piece of the R-petal $\{\Pi(X, S)\}_{S \in \mathbf{TEN}(R)}$. We simply write $\Pi(S) = \Pi(X, S)$ when the whole space is clear by the context.

Remark 2.1. For every uncountable range set R, and for every R-petaloid space (X, d), the petal $\Pi(X, \{0\})$ is a singleton.

Theorem 2.2. *Let R be an uncountable range set. The following statements hold*:
(1) *There exists an R-petaloid ultrametric space and it is unique up to isometry.*
(2) *The R-petaloid ultrametric space is complete, non-separable, and $\mathcal{F}(R)$-injective.*
(3) *Every separable R-valued ultrametric space can be isometrically embedded into the R-petaloid ultrametric space.*

Proof. The first statement is due to the author's result [16, Theorem 1.1]. In the second statement, the $\mathcal{N}(R)$-injectivity and the completeness are deduced from [16, Propositions 2.9 and 2.10]. The non-separability follows from [7, (12) in Theorem 1.6] and the fact that if (X, d) is $\mathcal{N}(R)$-injective, then $d(X \times X) = R$ (Section 1). All complete $\mathcal{N}(R)$-injective ultrametric spaces have the property stated in the third statement, which can be shown by induction. □

Lemma 2.3. *Let R be a range set and (X, d) be the R-petaloid ultrametric space. Then, for every $S \in \mathbf{TEN}(R)$, and for every $x \in X$, there exists $p \in \Pi(X, S)$ such that $d(x, \Pi(X, S)) = d(x, p)$.*

Proof. The lemma is deduced from the property that $\Pi(X, S)$ is the S-Urysohn universal ultrametric space (see the property (P1) and [28, Propositions 20.2 and 21.1]). □

Recall that $\omega_0 = \mathbb{Z}_{\geq 0}$ as a set.

Example 2.1. We give an example of the R-petaloid space. Let R be a range set. We also denote by $\mathrm{G}(R, \omega_0)$ the set of all functions $f : R \to \omega_0$ such that $f(0) = 0$ and the set $\{0\} \cup \{x \in R | f(x) \neq 0\}$ is tenuous. For $f, g \in \mathrm{G}(R, \omega_0)$, we define an R-valued ultrametric $\triangle$ on $\mathrm{G}(R, \omega_0)$ by $\triangle(f, g) = \max\{r \in R | f(r) \neq g(r)\}$ if $f \neq g$; otherwise, $\triangle(f, g) = 0$. Then, the space $(\mathrm{G}(R, \omega_0), \triangle)$ is R-petaloid. Remark that the S-piece $\Pi(\mathrm{G}(R, \omega_0), S)$ of the R-petal of $(\mathrm{G}(R, \omega_0), \triangle)$ is defined by the subset of $\mathrm{G}(R, \omega_0)$ consisting of all functions $f : R \to \omega_0$ such that $\{x \in R | f(x) \neq 0\} \subseteq S$. Hence, all R-petaloid ultrametric spaces are isometric to $(\mathrm{G}(R, \omega_0), \triangle)$ (see [16, Theorem 1.3]). For more information of this construction, we refer the readers to [9,11,23].

Remark 2.2. Even if R is finite or countable, the R-Urysohn universal ultrametric space satisfies all of the properties (P1)–(P4) in the definition of the R-petaloid space (Definition 2.1). Thus, the concept of the petaloid spaces can be naturally considered as a generalization of separable Urysohn universal ultrametric spaces.

3 Characterizations of injectivity

In this section, we provide characterizations of the injectivity for finite ultrametric spaces.

Lemma 3.1. *Let R be a range set. Then, the following statements are true*:
(1) *An ultrametric space (X, d) is (R, ω_0)-haloed if and only if (X, d) is (R, n)-haloed for all $n \in \mathbb{Z}_{\geq 1}$.*
(2) *An ultrametric space (X, d) is (R, ω_0)-avoidant if and only if (X, d) is (R, n)-avoidant for all $n \in \mathbb{Z}_{\geq 1}$.*
(3) *An ultrametric space (X, d) is $\mathcal{N}(R, \omega_0)$-injective if and only if (X, d) is $\mathcal{N}(R, n)$-injective for all $n \in \mathbb{Z}_{\geq 1}$.*

Proof. The statements (2) and (3) follow from Definitions 1.4 and 1.2, respectively (see also Remark 1.1).

Let us prove (1). First, we assume that (X, d) is (R, ω_0)-haloed. Then, Definition 1.3 shows that (X, d) is (R, n)-haloed for every $n \in \mathbb{Z}_{\geq 1}$.

Our purpose is to prove that (X, d) is $\mathcal{N}(R, \omega_0)$-injective. Take $a \in X$ and $r \in R \setminus \{0\}$. We define a relation $\sim_r$ on $\mathrm{B}(a, r)$ by declaring that $x \sim_r y$ means $d(x, y) < r$. From the strong triangle inequality, it follows that $\sim_r$ is an equivalence relation on $\mathrm{B}(a, r)$. Let Q denote the quotient set of $\mathrm{B}(a, r)$ by $\sim_r$. Since (X, d) is (R, n)-haloed for all $n \in \mathbb{Z}_{\geq 1}$, we note that $n \leq \mathrm{Card}(Q)$ for all $n \in \mathbb{Z}_{\geq 1}$. Therefore, Q is infinite. Take a complete system W of representatives of Q. Then, W becomes an infinite r-equidistant subset of $\mathrm{B}(a, r)$. This proves that (X, d) is $\mathcal{N}(R, \omega_0)$-injective. □

The next three lemmas play key roles to prove our first main result.

Lemma 3.2. *Let R be a range set and $n \in \mathbb{Z}_{\geq 1}$. If an ultrametric space (X, d) is (R, n)-haloed, then it is (R, n)-avoidant.*

Proof. To show that (X, d) is (R, n)-avoidant, take $a \in X, r \in R$ and take a subset A of $\mathrm{B}(a, r)$ with $\mathrm{Card}(A) < n$. Since (X, d) is (R, n)-haloed, we can find an r-equidistant subset H of $\mathrm{B}(a, r)$ with $n \leq \mathrm{Card}(H)$. Since d is an ultrametric, for every $x \in A$, we obtain $\mathrm{Card}(\{y \in H | d(x, y) < r\}) \leq 1$. Thus, due to $\mathrm{Card}(A) < \mathrm{Card}(H)$, we can take $p \in H$ such that $r \leq d(p, x)$ for all $x \in A$. In this situation, statement, (2) in Lemma 2.1 implies that $d(x, p) = r$ for all $x \in A$. □

Lemma 3.3. *Let R be a range set and $n \in \mathbb{Z}_{\geq 1}$. If an ultrametric space (X, d) is (R, n)-avoidant, then it is $\mathcal{N}(R, n + 1)$-injective.*

Proof. Take an arbitrary R-valued ultrametric space $(Y \sqcup \{\theta\}, e)$ in $\mathcal{N}(R, n + 1)$, where θ is a point with $\theta \notin Y$ (i.e., we think a one-point extension $Y \sqcup \{\theta\}$ of Y). We also take an isometric embedding $\phi : Y \to X$. Note that $\mathrm{Card}(Y \sqcup \{\theta\}) < n + 1$. To show that (X, d) is $\mathcal{N}(R, n + 1)$-injective, we only need to verify that there exists an isometric embedding $\Phi : Y \sqcup \{\theta\} \to X$ with $\Phi|_Y = \phi$. Put $r = \min_{y \in Y} e(y, \theta)$ and take $q \in Y$ such that
(i) $r = e(q, \theta)$.

In this situation, we have $r \in R$. Put $A = \phi(Y) \cap \mathrm{B}(\phi(q), r)$. Note that $\mathrm{Card}(A) < n$. Since (X, d) is (R, n)-avoidant and since $\mathrm{Card}(A) < n$, there exists $t \in \mathrm{B}(\phi(q), r)$ such that
(ii) $d(a, t) = r$, for all $a \in A$.

We now prove the following statement:
(A) We have $d(\phi(y), t) = e(y, \theta)$, for all $y \in Y$.

We divide proof of statement (A) into two cases.

Case 1. $[e(y, q) \leq r]$: Then, $e(y, \theta) \leq e(y, q) \vee e(q, \theta) \leq r$. By the minimality of r, we conclude that $e(y, \theta) = r$. From (ii), it follows that $d(\phi(y), t) = r$. Thus, $d(\phi(y), t) = e(y, \theta)$.

Case 2. $[r < e(y, q)]$: in this case, equality (i) yields $e(q, \theta) < e(y, q)$. Using (1) in Lemma 2.1, we have
(iii) $e(y, \theta) = e(y, q)$.

Since ϕ is isometric, we obtain
(iv) $e(y, q) = d(\phi(y), \phi(q))$.

The inequality $r < e(y, q)$ and (iv) imply $r < d(\phi(y), \phi(q))$, which means that $\phi(y) \notin \mathrm{B}(\phi(q), r)$. From (2) in Lemma 2.1, we deduce that $\mathrm{B}(\phi(q), r) = \mathrm{B}(t, r)$. Hence, $\phi(y) \notin \mathrm{B}(t, r)$. Thus, $r < d(\phi(y), t)$. Using $r = d(\phi(q), t)$ (see the definition of t and eqaulity (ii)), we have the inequality $d(\phi(q), t) < d(\phi(y), t)$, and hence, statement (1) in Lemma 2.1 implies that
(v) $d(\phi(y), \phi(q)) = d(\phi(y), t)$.

Combining equalities (iii)–(iv), we conclude that $d(\phi(y), t) = e(y, \theta)$. Then, the proof of the statement (A) is complete.

Next, we define a map $\Phi : Y \sqcup \{\theta\} \to X$ by $\Phi(x) = \phi(x)$ for all $x \in Y$ and $\Phi(\theta) = t$. Statement (A) shows that Φ is an isometric embedding with $\Phi|_Y = \phi$. Therefore, the space (X, d) is $\mathcal{N}(R, n + 1)$-injective. □

Lemma 3.4. *Let R be a range set and $n \in \mathbb{Z}_{\geq 1}$. If an ultrametric space (X, d) is $\mathcal{N}(R, n+1)$-injective, then it is (R, n)-haloed.*

Proof. Take $a \in X$, and $r \in R \setminus \{0\}$. By recursion, we will construct $\{A_i\}_{i=1}^{n}$ such that
(1) we have $A_i \subseteq \mathrm{B}(a, r)$ for all $i \leq n$;
(2) for every $i \in \{1, \dots, n\}$, we have $\mathrm{Card}(A_i) = i$;
(3) for every $i \in \{1, \dots, n-1\}$, we have $A_i \subseteq A_{i+1}$;
(4) each A_i is r-equidistant.

Take $s \in \mathrm{B}(a, r)$ and put $A_1 = \{s\}$. Fix $l < n$ and assume that we have already constructed $\{A_i\}_{i=1}^{l}$. We shall construct A_{l+1}. Take a point θ such that $\theta \notin X$ and define an ultrametric e on $A_l \sqcup \{\theta\}$ by $e|_{A_l^2} = d$ and $d(x, \theta) = r$ for all $x \in A_l$. We also define an isometric embedding $\phi : A_l \to X$ by $\phi(a) = a$. Using the $\mathcal{N}(R, n+1)$-injectivity together with $\mathrm{Card}(A_l \sqcup \{\theta\}) < n+1$, we obtain $p \in \mathrm{B}(a, r)$ such that $d(\phi(x), p) = e(x, \theta) = r$ for all $x \in A_l$. Put $A_{l+1} = A_l \sqcup \{p\}$. Then, the resulting set A_n is r-equidistant and satisfies $\mathrm{Card}(A_n) = n$. Hence, (X, d) is (R, n)-haloed. □

Now, we are able to prove Theorem 1.1.

Proof of Theorem 1.1. Combining Lemmas 3.2–3.4, we obtain the implications $(A1) \Rightarrow (A2)$, $(A2) \Rightarrow (A3)$, and $(A3) \Rightarrow (A1)$. Thus, the three statements (A1)–(A3) are equivalent to each other.

The equivalences between (B1)–(B3) are deduced from Lemma 3.1 and the former part of the theorem. This completes the proof of Theorem 1.1. □

Even if a range set R is at most countable, the R-Urysohn universal ultrametric space has a petal structure.

Lemma 3.5. *If R is an at most countable range set and (X, d) is the R-Urysohn universal ultrametric space, then there exists a family $\{\Pi(X, S)\}_{S \in \mathbf{TEN}(R)}$ of subsets of X satisfying the conditions $(P1)$–$(P4)$ in Definition 2.1.*

Proof. This lemma follows from the constructions of the R-Urysohn universal ultrametric space and constructions of petal structures (see Example 2.1 in the case where R is finite or countable). We also refer the readers to [15] and [16, Remark 1.1]. □

Based on Lemma 3.5, even when R is at most countable, we can take a petal structure of the R-Urysohn universal ultrametric space.

Using Theorem 1.1, we clarify the metric structure of the Cartesian product of the Urysohn universal ultrametric spaces.

Lemma 3.6. *If S and T are range sets, and if ultrametric space (X, d) and (Y, e) are (S, ω_0)-haloed and (T, ω_0)-haloed, respectively, then the space $(X \times Y, d \times_\infty e)$ is $(S \cup T, \omega_0)$-haloed.*

Proof. To prove that $(X \times Y, d \times_\infty e)$ is $(S \cup T, \omega_0)$-haloed, take $(a, b) \in X \times X$ and $r \in (S \cup T) \setminus \{0\}$. We first consider the case of $r \in S$. Since (X, d) is (S, ω_0)-haloed, we can find a countably infinite r-equidistant subset E of $\mathrm{B}(a, r)$. By the definition of $d \times_\infty e$, the set $E \times \{b\}$ is a countably infinite r-equidistant subset of $\mathrm{B}((a, b), r; d \times_\infty e)$. In the case of $r \in T$, by replacing the roles of S and b with those of T and a, we can find a countable r-equidistant subset of $\mathrm{B}((a, b), r; d \times_\infty e)$. □

Subsequently, we show Theorem 1.2.

Proof of Theorem 1.2. We divide the proof into two cases depending on the cardinalities of R_0 and R_1.

Case 1. [R_0 and R_1 are finite or countable]: in this case, (X, d) is a separable complete $\mathcal{N}(R_0, \omega_0)$-injective R_0-valued ultrametric space (resepectively, (Y, e) is a separable complete $\mathcal{N}(R_1, \omega_0)$-injective R_1-valued ultrametric space). In particular, $(X \times Y, d \times_\infty e)$ is separable and complete. Lemma 3.6 shows that $(X \times Y, d \times_\infty e)$ is $(R_0 \cup R_1, \omega_0)$-haloed, and hence, it is $\mathcal{N}(R_0 \cup R_1, \omega_0)$-injective by Theorem 1.1. Thus, the product $(X \times Y, d \times_\infty e)$ is the $(R_0 \cup R_1)$-Urysohn universal ultrametric space.

Case 2. [Either of R_0 or R_1 is uncountable]: we may assume that R_0 is uncountable. In this setting, (X, d) is the R_0-petaloid ultrametric space (Definition 1.1). Whether R_1 is at most countable or not, the space (Y, e) has a petal structure due to Lemma 3.5. Now, we define a petal structure on $(X \times Y, d \times_\infty e)$ by

$$\Pi(X \times Y, S) = \Pi(X, R_0 \cap S) \times \Pi(Y, R_1 \cap S),$$

for $S \in \mathbf{TEN}(R_0 \cup R_1)$. This structure satisfies property (P3). Indeed, for $S, T \in \mathbf{TEN}(R_0 \cup R_1)$, using property (P3) for (X, d) and (Y, e), by the purely set-theoretic equality $(E \cap P) \times (F \cap Q) = (E \times F) \cap (P \times Q)$ for arbitrary sets E, F, P, and Q, we have

$$\begin{aligned}\Pi(X \times Y, S \cap T) &= \Pi(X, R_0 \cap S \cap T) \times \Pi(Y, R_1 \cap S \cap T)\\ &= \Pi(X, (R_0 \cap S) \cap (R_0 \cap T)) \times \Pi(Y, (R_1 \cap S) \cap (R_1 \cap T))\\ &= (\Pi(X, R_0 \cap S) \cap \Pi(X, R_0 \cap T)) \times (\Pi(Y, R_1 \cap S) \cap \Pi(Y, R_1 \cap T))\\ &= (\Pi(X, R_0 \cap S) \times \Pi(Y, R_1 \cap S)) \cap (\Pi(X, R_0 \cap T) \times \Pi(Y, R_1 \cap T))\\ &= \Pi(X \times Y, S) \cap \Pi(X \times Y, T).\end{aligned}$$

Property (P1) follows from the separable case proven earlier. We now verify (P2). For every pair $S, T \in \mathbf{TEN}(R_0 \cup R_1)$, we have

$$\Pi(X, R_0 \cap S) \cup \Pi(X, R_0 \cap T) \subseteq \Pi(X, R_0 \cap (S \cup T)), \tag{3.1}$$

$$\Pi(Y, R_1 \cap S) \cup \Pi(Y, R_1 \cap T) \subseteq \Pi(Y, R_1 \cap (S \cup T)), \tag{3.2}$$

(see [16, Lemma 2.1]). Then, for arbitrary $S, T \in \mathbf{TEN}(R_0 \cup R_1)$, we also have

$$\Pi(X, R_0 \cap S) \times \Pi(Y, R_1 \cap T) \subseteq \Pi(X, R_0 \cap (S \cup T)) \times \Pi(Y, R_1 \cap (S \cup T)). \tag{3.3}$$

According to property (P2) for (X, d) and (Y, e) and $\mathbf{TEN}(R_0) \cup \mathbf{TEN}(R_1) \subseteq \mathbf{TEN}(R_0 \cup R_1)$, we observe that

$$X = \bigcup_{S \in \mathbf{TEN}(R_0 \cup R_1)} \Pi(X, R_0 \cap S), \tag{3.4}$$

$$Y = \bigcup_{T \in \mathbf{TEN}(R_0 \cup R_1)} \Pi(Y, R_1 \cap T). \tag{3.5}$$

Thus, we obtain

$$\begin{aligned}X \times Y &\subseteq \bigcup_{S, T \in \mathbf{TEN}(R_0 \cup R_1)} \Pi(X, R_0 \cap S) \times \Pi(Y, R_1 \cap T)\\ &\subseteq \bigcup_{S \cup T \in \mathbf{TEN}(R_0 \cup R_1)} \Pi(X, R_0 \cap (S \cup T)) \times \Pi(Y, R_1 \cap (S \cup T))\\ &= \bigcup_{U \in \mathbf{TEN}(R_0 \cup R_1)} \Pi(X \times Y, U) \subseteq X \times Y.\end{aligned}$$

This means that (P2) is valid.

Next, we confirm property (P4). Take $S, T \in \mathbf{TEN}(R_0 \cup R_1)$ and take $(x, y) \in \Pi(X \times Y, T)(=\Pi(X, R_0 \cap T)) \times (\Pi(Y, R_1 \cap T))$. We shall show that $(d \times_\infty e)((x, y), \Pi(X \times Y, S)) \in \{0\} \cup (T \backslash S)$. First, we obtain

$$(d \times_\infty e)((x, y), \Pi(X \times Y, S)) = \inf_{(u,v) \in \Pi(X \times Y, S)} (d \times_\infty e)((x, y), (u, v)) = \inf_{(u,v) \in \Pi(X \times Y, S)} \max\{d(x, u), e(y, v)\},$$

i.e.,

$$(d \times_\infty e)((x, y), \Pi(X \times Y, S)) = \inf_{(u,v) \in \Pi(X \times Y, S)} \max\{d(x, u), e(y, v)\}. \tag{3.6}$$

Since $\Pi(X \times Y, S) = \Pi(X, R_0 \cap S) \times \Pi(Y, R_1 \cap S)$, by (3.6), we see that

$$(d \times_\infty e)((x, y), \Pi(X \times Y, S)) \geq \max\{d(x, \Pi(X, R_0 \cap S)), e(y, \Pi(Y, R_1 \cap S))\}. \tag{3.7}$$

Using Lemma 2.3 twice, we can find $p \in \Pi(X, R_0 \cap S)$ and $q \in \Pi(Y, R_1 \cap S)$ such that

$$d(x, \Pi(X, R_0 \cap S)) = d(x, p), \tag{3.8}$$

$$e(y, \Pi(Y, R_1 \cap S)) = e(x, q). \tag{3.9}$$

Thus, equation (3.6) implies

$$(d \times_\infty e)((x, y), \Pi(X \times Y, S)) \leq \max\{d(x, p), e(y, q)\} = \max\{d(x, \Pi(X, R_0 \cap S)), e(y, \Pi(Y, R_1 \cap S))\}. \tag{3.10}$$

Inequalities (3.7) and (3.10) yield

$$(d \times_\infty e)((x, y), \Pi(X \times Y, S)) = \max\{d(x, \Pi(X, R_0 \cap S)), e(y, \Pi(Y, R_1 \cap S))\}. \tag{3.11}$$

From property (P4) for (X, d) and (Y, e), it follows that

$$d(x, \Pi(Y, R_1 \cap S)) \in \{0\} \cup (R_1 \cap (T \setminus S)) \subseteq \{0\} \cup (T \setminus S)$$

and

$$e(y, \Pi(Y, R_1 \cap S)) \in \{0\} \cup (R_0 \cap (T \setminus S)) \subseteq \{0\} \cup (T \setminus S).$$

Hence, (3.11) shows that $(d \times_\infty e)((x, y), \Pi(X \times Y, S)) \in \{0\} \cup (T \setminus S)$. Namely, property (P4) is fulfilled.

Therefore, the product $(X \times Y, d \times_\infty e)$ is the $(R_0 \cup R_1)$-petaloid space. The proof of Theorem 1.2 is complete. □

We say that a metric space (X, d) is *(isometrically) homogeneous* if for every pair $x, y \in X$, there exists an isometric bijection $f : X \to X$ such that $f(x) = y$.

As an application, we obtain the next proposition.

Proposition 3.7. *Let R be a range set, $n \in \mathbb{Z}_{\geq 1}$, and (X, d) be an R-valued ultrametric space. Assume that the following conditions*:

(1) *there exists $a \in X$ such that for every $r \in R \setminus \{0\}$, there exists an r-equidistant subset A of* $\mathrm{B}(a, r)$ *with* $\mathrm{Card}(A) \geq n$;

(2) *(X, d) is isometrically homogeneous.*

Then, (X, d) is $\mathcal{N}(R, n + 1)$-injective.

Proof. We first show that (X, d) is (R, n)-haloed. Take an arbitrary point $p \in X$. Let a be a point stated in (1). By assumption (2), we can take an isometric bijection $f : X \to X$ such that $f(a) = p$. Using (1), we see that for every $r \in R \setminus \{0\}$, there exists an r-equidistant subset A of $\mathrm{B}(a, r)$ with $\mathrm{Card}(A) \geq n$. Since f is isometric and bijective, we have $f(\mathrm{B}(a, r)) = \mathrm{B}(p, r)$, and hence, $f(A)$ is an r-equidistant subset of $\mathrm{B}(p, r)$ with $\mathrm{Card}(f(A)) \geq n$. Since p and r are arbitrary, (X, d) is (R, n)-haloed. Theorem 1.1 proves that the space (X, d) is $\mathcal{N}(R, n + 1)$-injective. □

Remark 3.1. In [9], it is shown that an ultrametric space (X, d) is isometrically homogeneous if and only if it is *ultrahomogeneous*, i.e., for every finite subset A of X and for every isometric embedding $I : A \to X$, there exists a bijective isometry $H : X \to X$ such that $H|_A = I$.

Lemma 3.8. *Let X be a set and $w : X \times X \to [0, \infty)$ be a symmetric function, i.e., $w(x, y) = w(y, x)$ whenever $x, y \in X$. If $w(x, x) = 0$ for all $x \in X$ and if there exists $L \in (0, \infty)$ such that $w(x, y) \in [L, 2L]$ for all distinct $x, y \in X$, then w satisfies the triangle inequality.*

Proof. To prove the triangle inequality for w, take three points $x, y, z \in X$. If two of x, y, and z are identical, then at least one element of $w(x, y)$, $w(x, z)$, and $w(z, y)$ is 0. In this case, the inequality $w(x, y) \leq w(x, z) + w(z, y)$ holds. Thus, we may assume that x, y, and z are mutually distinct. By the assumption on the range of w, we have

$$w(x, y) \leq 2L = L + L \leq w(x, z) + w(z, y).$$

This completes the proof. □

The proof of Theorem 1.3 is based on linear algebra.

Proof of Theorem 1.3. Take $s_0, s_1 \in \mathbb{U}$ with $\rho(s_0, s_1) = 2^{-1/p}$ if $p < \infty$; otherwise, $\rho(s_0, s_1) = 1$. Put $A = \{s_0, s_1\} \times \{s_0, s_1\}$. In other words, $A = \{(s_0, s_0), (s_0, s_1), (s_1, s_0), (s_1, s_1)\}$. Of course, A is a subset of $\mathbb{U} \times \mathbb{U}$. We now consider a one-point extension of $(A, \rho \times_p \rho)$. Let θ be a point such that $\theta \notin A$. For each $\mathbf{r} = (r_{00}, r_{01}, r_{10}, r_{11}) \in [2^{-1}, 1]^4$, we define a metric $e_\mathbf{r}$ on $A \sqcup \{\theta\}$ by $e_\mathbf{r}|_{A^2} = \rho \times_p \rho$ and $e_\mathbf{r}((s_i, s_j), \theta) = r_{ij}$. Since each r_{ij} belongs to $[2^{-1}, 1]$ and $\rho \times_p \rho$ on A takes only values in $\{0, ,2^{-1/p}, 1\}$ and since $\{2^{-1/p}, 1\} \subseteq [2^{-1}, 1]$, the function $e_\mathbf{r} : (A \sqcup \{\theta\})^2 \to \mathbb{R}$ actually satisfies the triangle inequality due to Lemma 3.8 for $L = 2^{-1}$.

To establish that $(\mathbb{U} \times \mathbb{U}, \rho \times_p \rho)$ is not $\mathcal{F}$-injective, we shall show that we can take a point $\mathbf{r} \in [2^{-1}, 1]^4$ such that there is no point $(\alpha, \beta) \in \mathbb{U} \times \mathbb{U}$ satisfying that

(I) $e_\mathbf{r}((s_i, s_j), \theta) = (\rho \times_p \rho)((s_i, s_j), (\alpha, \beta))$, for all $(s_i, s_j) \in A$.

Suppose, contrary to our claim, that for every $\mathbf{r} \in [2^{-1}, 1]^4$, there exists $(\alpha, \beta) \in \mathbb{U} \times \mathbb{U}$ satisfying the condition (I). Put $x = \rho(s_0, \alpha)$, $y = \rho(s_1, \alpha)$, $z = \rho(s_0, \beta)$, and $w = \rho(s_1, \beta)$.

In the case of $p = \infty$, take $\mathbf{r} = (r_{00}, r_{01}, r_{10}, r_{11}) \in [2^{-1}, 1]^4$ so that $r_{01} < r_{11}$, and $r_{10} < r_{11}$. By the definition of $\rho \times_\infty \rho$, the values x, y, z, and w should satisfy $x \vee z = r_{00}$, $x \vee w = r_{01}$, $y \vee z = r_{10}$, and $y \vee w = r_{11}$. Then, we have $y \le r_{10}$ and $w \le r_{01}$, which imply that $y \vee w \le r_{10} \vee r_{01} < r_{11}$. This is a contradiction to $y \vee w = r_{11}$.

In the case of $p < \infty$, by the definition of $\rho \times_p \rho$ and condition (I), the values x, y, z, and w should satisfy $x^p + z^p = r_{00}^p$, $x^p + w^p = r_{01}^p$, $y^p + z^p = r_{10}^p$, and $y^p + w^p = r_{11}^p$. Namely, we obtain

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^p \\ y^p \\ z^p \\ w^p \end{pmatrix} = \begin{pmatrix} r_{00}^p \\ r_{01}^p \\ r_{10}^p \\ r_{11}^p \end{pmatrix}. \tag{3.12}$$

Since $\mathbf{r} \in [2^{-1}, 1]^4$ is arbitrary, we note that the set $[2^{-p}, 1]^4$ is contained in the image of the matrix in (3.12). However, the rank of the matrix is three (since the dimension of the kernel is one). This result contradicts the fact that the topological dimension of $[2^{-p}, 1]^4$ is four (or we can choose four specific linearly independent vectors from this set). This completes the proof of Theorem 1.3. □

4 Hyperspaces as universal spaces

In this section, we investigate the metric structures of hyperspaces of Urysohn universal ultrametric spaces.

Recall from Section 1 the spaces $\mathcal{K}(X)$, $\mathcal{K}_{m,l}(X)$, $\mathcal{K}_{m,\infty}(X)$ and the Hausdorff distance $\mathcal{HD}_d$.

We begin with the following well-known lemma on the Hausdorff distance. We omit the proof.

Lemma 4.1. *For every metric space (X, d), and for every pair $E, F \in \mathcal{K}(X)$, the Hausdorff distance $\mathcal{HD}_d(E, F)$ is equal to the infimum of all $r \in (0, \infty)$ such that $E \subseteq \bigcup_{b \in F} \mathrm{B}(b, r)$ and $F \subseteq \bigcup_{a \in E} \mathrm{B}(a, r)$.*

For the sake of self-containedness, we give a proof of the next basic lemma.

Lemma 4.2. *Take $m \in \mathbb{Z}_{\ge 2} \sqcup \{\infty\}$, and $l \in (0, \infty]$. Let (X, d) be a metric space. Then, the next two statements are true*:

(1) *If (X, d) is separable, then so is $(\mathcal{K}_{m,l}(X), \mathcal{HD}_d)$.*

(2) *If (X, d) is complete, then so is $(\mathcal{K}_{m,l}(X), \mathcal{HD}_d)$.*

Proof. To prove (1), take a countable dense subset Q of X. Then, the set of all finite non-empty subsets of Q is dense in $(\mathcal{K}(X), \mathcal{HD}_d)$. Thus, it is separable. Since $\mathcal{K}_{m,l}(X)$ is a metric subspace of $\mathcal{K}(X)$, the space $(\mathcal{K}_{m,l}(X), \mathcal{HD}_d)$ is separable.

Next, we show (2). The completeness of $(\mathcal{K}(X), \mathcal{HD}_d)$ can be proven in a similar way to [5, Proposition 7.3.7]. Thus, it suffices to verify that $\mathcal{K}_{m,l}(X)$ is closed in $(\mathcal{K}(X), \mathcal{HD}_d)$ for all $m \in \mathbb{Z}_{\geq 2}$ and $l \in (0, \infty)$. Take a sequence $\{K_i\}_{i \in \mathbb{Z}_{\geq 0}}$ in $\mathcal{K}_{m,l}(X)$ and assume that it converges to a set $L \in \mathcal{K}(X)$. Since the map $D : \mathcal{K}(X) \to \mathbb{R}$ defined by $D(E) = \operatorname{diam} E$ is continuous (in fact, it is 2-Lipschitz), we obtain $\operatorname{diam} L \leq l$. We next show $\operatorname{Card}(L) \leq m$. For the sake of contradiction, suppose that $m < \operatorname{Card}(L)$. Then, there exists a subset E of L with $\operatorname{Card}(E) = m + 1$. Put $r = \min\{d(x, y) | x, y \in E, x \neq y\}$, and take a sufficiently large $n \in \mathbb{Z}_{\geq 0}$ so that $\mathcal{HD}_d(L, K_n) < r/3$. In this case, Lemma 4.1 implies that for every $x \in E$, there exists $y_x \in K_n$ such that $d(x, y_x) < r/3$. By the definition of r, we have $y_x \neq y_{x'}$ for all distinct $x, x' \in E$. Thus, the set $F = \{y_x \in K_n | x \in E\}$ has cardinality $m + 1$. This is impossible due to $\operatorname{Card}(K_n) \leq m$. Therefore, $\operatorname{Card}(L) \leq m$. Subsequently, we obtain $L \in \mathcal{K}_{m,l}(X)$. This completes the proof. □

Let (X, d) be an ultrametric space, and $r \in (0, \infty)$. For $E \in \mathcal{K}(X)$ we denote by $\langle E \rangle_{c(r)}$ the set $\{B(a, r) | a \in E\}$. For convenience, we also define $\langle E \rangle_{c(0)} = \{\{a\} | a \in E\}$, which is the case of $r = 0$. By (3) in Lemma 2.1 and the compactness of E, the set $\langle E \rangle_{c(r)}$ is finite for all $r \in (0, \infty)$. The next proposition is an analogue of [22, Theorem 5.1] and [15, Corollary 2.30] for the Hausdorff distance.

Proposition 4.3. *Let (X, d) be an ultrametric space. Then, for every pair of subsets E and F of X, the value $\mathcal{HD}_d(E, F)$ is equal to the minimum $r \in R$ such that $\langle E \rangle_{c(r)} = \langle F \rangle_{c(r)}$.*

Proof. Let W denote the set of all $r \in R$ such that $\langle E \rangle_{c(r)} = \langle F \rangle_{c(r)}$. Put $u = \inf W$. First, we shall prove that u is equal to the Hausdorff distance between E and F. For every $r \in W$, the equality $\langle E \rangle_{c(r)} = \langle F \rangle_{c(r)}$ is true. This result means that $\{B(a, r) | a \in E\} = \{B(b, r) | b \in F\}$. This equality yields $E \subseteq \bigcup_{b \in F} B(b, r)$ and $F \subseteq \bigcup_{a \in E} B(a, r)$. Thus, Lemma 4.1 implies that $\mathcal{HD}_d(A, B) \leq u$. To obtain the opposite inequality, take an arbitrary number l with $\mathcal{HD}_d(A, B) < l$. Then, $E \subseteq \bigcup_{b \in F} B(b, l)$ and $F \subseteq \bigcup_{a \in E} B(a, l)$. To show $\langle E \rangle_{c(l)} \subseteq \langle F \rangle_{c(l)}$, take $B(s, l) \in \langle E \rangle_{c(l)}$. From $E \subseteq \bigcup_{b \in F} B(b, l)$, it follows that there exists $b \in F$ such that $s \in B(b, l)$. Thus, statement (3) in Lemma 2.1 yields $B(s, l) = B(b, l)$. Namely, $B(s, l) \in \langle F \rangle_{c(l)}$, and hence, $\langle E \rangle_{c(l)} \subseteq \langle F \rangle_{c(l)}$. Replacing the role of $\langle E \rangle_{c(l)}$ with that of $\langle F \rangle_{c(l)}$, we also obtain $\langle F \rangle_{c(l)} \subseteq \langle E \rangle_{c(l)}$. As a result, we have $\langle E \rangle_{c(l)} = \langle F \rangle_{c(l)}$. Therefore, $u \leq \mathcal{HD}_d(E, F)$, and we then obtain $u = \mathcal{HD}_d(E, F)$.

Next, we will show that $u = \min W$. If $u = 0$, then $u \in R$ and $E = F$. Thus, $\langle E \rangle_{c(0)} = \langle F \rangle_{c(0)}$. Namely, $u = \min W$. In what follows, we may assume that $u > 0$.

Now we show that $u \in R$. Put $S = d(E^2) \cup d(F^2)$. Then, S is tenuous (see [15, Corollary 2.28]) and $S \subseteq R$. To obtain a contradiction, suppose that $u \notin S$. Then, since S is tenuous, we can take $s, t \in S \setminus \{0\}$ such that $u \in (s, t)$ and $(s, t) \cap S = \emptyset$. In this setting, we obtain the following statement:

(B) For every $x \in E \cup F$ and for every $r \in (s, t)$, we have $B(x, r) = B(x, s)$.

Since $u = \inf W$, there exists $v \in W$ such that $v \in [u, t)$. Due to $v \in W$, we have $\langle E \rangle_{c(v)} = \langle F \rangle_{c(v)}$. This equality and statement (B) show that $\langle E \rangle_{c(s)} = \langle F \rangle_{c(s)}$, and hence, $s \in W$, which is a contradiction to $u = \inf W$ and $s < u$. Therefore, $u \in S$, and hence, $u \in R$.

To verify that u attains the minimum, we will show $u \in W$. Take a sequence $\{r_i\}_{i \in \mathbb{Z}_{\geq 0}}$ in W such that $u \leq r_i$ for all $i \in \mathbb{Z}_{\geq 0}$, and $\lim_{i \to \infty} r_i = u$. If $r_i = u$ for some $i \in \mathbb{Z}_{\geq 0}$, we have $u \in W$. We may assume that $r_i \neq u$ for any $i \in \mathbb{Z}_{\geq 0}$. Since S is tenuous and $u \in S \setminus \{0\}$, we can take $q \in S$ such that $(u, q) \cap S = \emptyset$. For a sufficiently large $i \in \mathbb{Z}_{\geq 0}$, the value r_i belongs to (u, q). Using the same argument as the proof of $u \in R$, similar to statement (B), we note that every $x \in E \cup F$ satisfies $B(x, r_i) = B(x, u)$. Consequently, we obtain $\langle E \rangle_{c(u)} = \langle F \rangle_{c(u)}$, which means that $u \in W$. As a result, we conclude that u is minimal in W. This completes the proof. □

As a consequence of Proposition 4.3, we obtain the following well-known statement.

Corollary 4.4. *Let R be a range set and (X, d) be an ultrametric. The following two statements are true*:
(1) *The Hausdorff metric $\mathcal{HD}_d$ is an ultrametric on $\mathcal{K}(X)$.*
(2) *If (X, d) is R-valued, then so is $(\mathcal{K}(X), \mathcal{HD}_d)$.*

Proposition 4.5. *Let R be a range set, and (X, d) be an (R, ω_0)-haloed ultrametric space. Take $m \in \mathbb{Z}_{\geq 2} \sqcup \{\infty\}$, and $l \in (0, \infty]$. Then, the space $(\mathcal{K}_{m,l}(X), \mathcal{HD}_d)$ is (R, ω_0)-haloed.*

Proof. Take $A \in \mathcal{K}_{m,l}(X)$ and $r \in R$. In the case of $m < \infty$, put $E = A$. In the case of $m = \infty$, take a finite set E with $\mathcal{HD}_d(E, A) \leq r$. Fix $x \in E$, and take a countable r-equidistant subset H of $\mathrm{B}(x, r;\ d)$. We now construct an r-equidistant subset of $\mathrm{B}(A, r; \mathcal{HD}_d)$.

For each $q \in H$, put $F_q = (E \backslash \mathrm{B}(x, r;\ d)) \cup \{q\}$. Note that if $l < r$, then $F_q = \{q\}$. Since E is a finite set, each F_q is closed in X. We also note that $\mathrm{Card}(F_q) \leq m$ and $\mathrm{diam}(F_q) \leq l$ for all $q \in H$. Namely, for every $q \in H$, we have $F_q \in \mathcal{K}_{m,l}(X)$. By the fact that H is r-equidistant, and by the definition of F_q, we obtain $\mathcal{HD}_d(F_q, F_{q'}) = r$ for all distinct $q, q' \in H$. Proposition 4.3 shows that $\mathcal{HD}_d(F_q, E) \leq r$, and hence, $\mathcal{HD}_d(F_q, A) \leq r$ for all $q \in H$. Thus, $\{F_q | q \in H\}$ is a countable r-equidistant subset of $\mathrm{B}(A, r;\ \mathcal{HD}_d)$. This proves the proposition. □

Remark 4.1. As a sophisticated version of Proposition 4.5, we can prove the following statement: Let $n \in \mathbb{Z}_{\geq 0}$, $m \in \mathbb{Z}_{\geq 2} \sqcup \{\infty\}$, and R be a range set. If (X, d) is $\mathcal{N}(R, n)$-injective, then the space $(\mathcal{K}_{m,\infty}(X), \mathcal{HD}_d)$ is $\mathcal{N}(R, n + 1)$-injective. Based on this statement and Theorem 1.1, the operations of the hyperspace and symmetric products can be considered as a Katětov functor in the category of ultrametric spaces (for Katětov functors in a category of ordinary metric spaces, see [19]).

We now clarify the metric structure of the hyperspace of the R-Urysohn universal ultrametric space in the case where R is finite or countable.

Theorem 4.6. *If a range set R is finite or countable and (X, d) is the R-Urysohn universal ultrametric space, then for every $m \in \mathbb{Z}_{\geq 2} \sqcup \{\infty\}$ and for every $l \in (0, \infty]$, the space $(\mathcal{K}_{m,l}(X), \mathcal{HD}_d)$ is the R-Urysohn universal ultrametric space. In particular, it is isometric to (X, d).*

Proof. According to Corollary 4.4, Proposition 4.5, and Theorem 1.1, the hyperspace $(\mathcal{K}_{m,l}(X), \mathcal{HD}_d)$ is an R-valued $\mathcal{N}(R, \omega_0)$-injective ultrametric space. On account of Lemma 4.2, we note that $(\mathcal{K}_{m,l}(X), \mathcal{HD}_d)$ is separable and complete. Therefore $(\mathcal{K}_{m,l}(X), \mathcal{HD}_d)$ is the R-Urysohn universal ultrametric space and isometric to (X, d). □

For the uncountable case, we need the following lemma.

Lemma 4.7. *Let R be an uncountable range set and (X, d) be the R-petaloid ultrametric space. Then, for every $S \in \mathbf{TEN}(R)$, and for every finite subset A of X, there exists a subset M of $\Pi(X, S)$ such that*
(1) $\mathcal{HD}_d(A, M) \leq \max_{a \in A} d(a, \Pi(X, S))$;
(2) $\mathrm{Card}(M) \leq \mathrm{Card}(A)$;
(3) $\mathrm{diam}(M) \leq \mathrm{diam}(A)$.

Proof. The lemma follows from Lemma 2.3 and [2, Theorem 4.6] stating that every proximal subset of an (generalized) ultrametric space is a 1-Lipschitz retract of the whole space. For the sake of self-containedness, we briefly explain a construction of M based on the proof of [2, Theorem 4.6]. For each $a \in A$, we put $l_a = d(a, \Pi(X, S))$ and we denote by $E[a]$ the set of all $y \in \Pi(X, S)$ such that $d(a, y) = l_a$. For each set $E[a]$, take a point $p_{E[a]} \in E[a]$. Note that if $E[a] = E[b]$, then $p_{E[a]} = p_{E[b]}$. Lemma 2.3 implies that $E_a \neq \emptyset$ for any $a \in A$. Define $M = \{p_{E[a]} | a \in A\}$. An important part of the construction of M is that we choose $p_{E[a]}$ so that it depends on the set $E[a]$ rather than the point a. Since $d(a, p_{E[a]}) = d(a, \Pi(X, S))$ for all $a \in A$, property (1) is true. Property (2) follows from the definition of M. Next, we verify (3). It is sufficient to prove that

$d(p_{E[a]}, p_{E[b]}) \le d(a,b)$ for all $a, b \in A$. If $l_a \vee l_b \le d(a,b)$, then the strong triangle inequality implies $d(p_{E[a]}, p_{E[b]}) \le d(a,b)$. If $d(a,b) < l_a \vee l_a$, then we can prove $l_a = l_b$ and $E[a] = E[b]$. Thus, $p_{E[a]} = p_{E[b]}$. In this situation, of course, we have $d(p_{E[a]}, p_{E[b]}) \le d(a,b)$. □

For a metric space (X, d) and for $r \in (0, \infty)$, a subset of A is called an *r-net* if for all distinct $x, y \in A$ we have $r < d(x, y)$. An r-net is *maximal* if it is maximal with respect to inclusion $\subseteq$. Let us prove Theorem 1.4.

Proof of Theorem 1.4. The case where R is finite or countable is proven in Theorem 4.6. We now treat the case where R is uncountable. Remark that in this case, (X, d) is the R-petaloid space (Definition 1.1). Corollary 4.4 implies that $(\mathcal{K}_{m,l}(X), \mathcal{HD}_d)$ is an R-valued ultrametric space. We define a petal on $(\mathcal{K}_{m,l}(X), \mathcal{HD}_d)$ as follows. For every $S \in \mathbf{TEN}(R)$, we define

$$\Pi(\mathcal{K}_{m,l}(X), S) = \{E \in \mathcal{K}_{m,l}(X) | E \subseteq \Pi(X, S)\}.$$

By this definition, property (P3) is satisfied.

Since for every $S \in \mathbf{TEN}(R)$ the space $(\Pi(X, S), d)$ is the S-Urysohn universal ultrametric space (property (P1) for (X, d)), according to Theorem 4.6, the space $(\Pi(\mathcal{K}_{m,l}(X), S), \mathcal{HD}_d)$ is the S-Urysohn universal ultrametric space. Namely, the property (P1) is true for $(\mathcal{K}_{m,l}(X), \mathcal{HD}_d)$. Due to [16, Proposition 2.8], property (P2) is satisfied.

We now show that property (P4) is valid for $(\mathcal{K}_{m,l}(X), \mathcal{HD}_d)$. Take $S, T \in \mathbf{TEN}(R)$, and take $E \in \Pi(\mathcal{K}_{m,l}(X), T)$. Let L stand for the distance between the point E and the set $\Pi(\mathcal{K}_{m,l}(X), S)$. We shall prove that L belongs to $\{0\} \cup (T \setminus S)$. We may assume that $E \notin \Pi(\mathcal{K}_{m,l}(X), S)$. Then, $E \subseteq \Pi(X, T)$, $E \not\subseteq \Pi(X, S)$, and $L > 0$. We put $H = \{d(y, \Pi(X, S)) | y \in E\}$. Using property (P4) for (X, d), we see that $H \subseteq \{0\} \cup (T \setminus S)$. Put $h = \max H$ and take $z \in E$ such that $d(z, \Pi(X, S)) = h$. The existence of h is guaranteed by the fact that H is a subset of the tenuous set T. Note that $h \in T \setminus S$ and $h > 0$.

From now on, we shall confirm that $L = h$. To prove $h \le L$, for the sake of contradiction, suppose that there exists $F \in \Pi(\mathcal{K}_{m,l}(X), S)$ with $\mathcal{HD}_d(E, F) < h$. Put $u = \mathcal{HD}_d(E, F)$. Then, Lemma 4.1 and $u < h$ imply that there exists $f \in F$ such that $d(z, f) < h$. This is a contradiction to $h = \inf_{x \in \Pi(X,S)} d(z, x)$. Thus, for every $F \in \Pi(\mathcal{K}_{m,l}(X), S)$, we have $h \le \mathcal{HD}_d(E, F)$. Hence, $h \le L$.

Next, we show $L = h$. Take a maximal finite h-net Q of E such that $Q \subseteq E$. Note that $Q \in \Pi(\mathcal{K}_{m,l}(X), S)$. Lemma 4.7 enables us to take a subset V of $\Pi(X, S)$ such that $\mathcal{HD}_d(Q, V) \le h$, $\mathrm{Card}(V) \le \mathrm{Card}(Q)$, and $\mathrm{diam}(V) \le \mathrm{diam}(Q)$. Since $Q \in \Pi(\mathcal{K}_{m,l}(X), S)$, we note that $V \in \mathcal{K}_{m,l}(X)$, and hence, $V \in \Pi(\mathcal{K}_{m,l}(X), S)$. By the fact that Q is a maximal h-net of E, we obtain $\mathcal{HD}_d(E, Q) \le h$. Since $\mathcal{HD}_d$ is an ultrametric (statement (1) in Corollary 4.4), combining $\mathcal{HD}_d(Q, V) \le h$ and $\mathcal{HD}_d(E, Q) \le h$, the strong triangle inequality shows that $\mathcal{HD}_d(E, V) \le h$. Hence, $L \le h$. By $V \in \Pi(\mathcal{K}_{m,l}(X), S)$ and $h \le L$, we conclude that $L = h$. Therefore, $L \in T \setminus S$. This completes the proof of Theorem 1.4. □

Remark 4.2. Theorem 1.4 can be considered as a non-Archimedean analogue of the statement that the Gromov-Hausdorff space is isometric to the quotient metric space of hyperspace of the Urysohn universal metric space (see [12, Exercise (b) in the page 83], [1, Theorem 3.4], and [32]).

Remark 4.3. Let R be a range set and (X, d) be the R-Urysohn universal ultrametric space. There does not seem to exist a natural isometry between (X, d) and $(\mathcal{K}(X), \mathcal{HD}_d)$. In the case of $R = \{0, 1\}$, the space (X, d) is the countable discrete space and all bijections between X and $\mathcal{K}(X)$ become isometries between them.

Remark 4.4. Theorem 1.4 implies that there is an (non-trivial) ultrametric space (X, d) such that $(\mathcal{K}(X), \mathcal{HD}_d)$ is isometric to (X, d). In contrast, in [31], it is shown that for any bounded metric space (X, d) with more than one point, the space $(\mathrm{CL}(X), \mathcal{HD}_d)$ is not isometric to (X, d), where $\mathrm{CL}(X)$ stands for the set of all non-empty closed bounded subsets of (X, d).

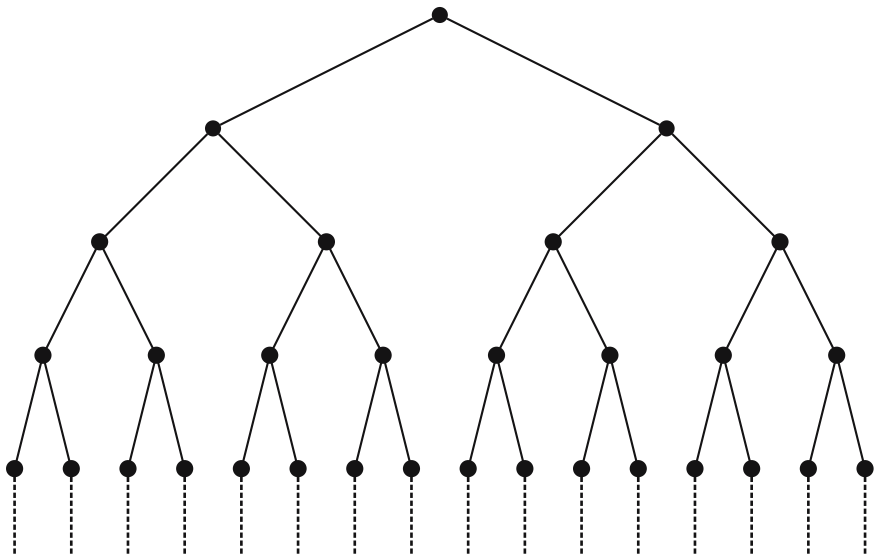

Figure 3: Dendrogram/tree/ultrametric space: in our discussion, for each $r \in R$, all branching points in this diagram have actually infinite edges with length r. For correspondences between ultrametrics and trees, see, for example, [9,10,13,21,32].

5 Urysohn universal ultrametric spaces as subsets

This section is devoted to the proof of Theorem 1.5. Throughout this section, the symbols R, (X, d) and K are the same objects as in the statement in Theorem 1.5. Namely, R is an uncountable range set, (X, d) is a complete $\mathcal{N}(R, \omega_0)$-injective ultrametric space, and K is a compact subset of X satisfying $d(K^2) \subseteq R$.

Before proving the theorem, we explain the plan of the proof. To obtain a subspace F stated in Theorem 1.5, we shall construct a petal structure consisting of subsets of X. For this purpose, we use the idea of a family tree, or a genealogical tree. This strategy is inspired by the fact that every ultrametric space can be seen as the end space of a tree. We fix a point $\varpi \in X$, and we regard ϖ as an original ancestor, or a protobiont. For every $S \in \mathbf{TEN}(R)$, we also define S-heirs of ϖ by sequences beginning from ϖ, which also can be regarded as descendants of ϖ passing through S. Considering the completion $\mathbb{E}(S)$ of the set $\mathbb{A}(S)$ of all S-heirs of ϖ, we can observe that the family $\{\mathbb{E}(S)\}_{S \in \mathbf{TEN}(R)}$ becomes a petal structure contained in X. Then, the space $F = \bigcup_{S \in \mathbf{TEN}(R)} \mathbb{E}(S)$ is as desired (Figure 3).

We begin with a concept that is a consequence of Theorem 1.1.

Definition 5.1. We say that a family $\{\Omega[a, r]\}_{a \in X, r \in R \setminus \{0\}}$ of subsets of X is an *R-seed of X* if the following conditions are satisfied for every $a \in X$, and for every $r \in R \setminus \{0\}$:
- we have $a \in \Omega[a, r]$;
- the set $\Omega[a, r]$ is r-equidistant;
- we have $\mathrm{Card}(\Omega[a, r]) = \omega_0$, i.e., the set $\Omega[a, r]$ has countably infinite cardinality.

Example 5.1. Fix a tenuous range set $R = \{0\} \cup \{\exp(-k) | k \in \mathbb{Z}_{\geq 0}\}$. Put $X = (\mathbb{Z}_{\geq 0})^{\omega_0}$ and define an R-valued ultrametric space d on X by

$$d(\{x_i\}_{i \in \mathbb{Z}_{\geq 0}}, \{y_i\}_{i \in \mathbb{Z}_{\geq 0}}) = \exp(-\min\{i \in \mathbb{Z}_{\geq 0} | x_i \neq y\}),$$

where we regard $\exp(-\infty) = 0$. A typical example of an R-seed is a family of the boundaries $\Omega[a, r] = \{x \in X | d(x, a) = r\}$ of balls of the space (X, d). An R-seed is corresponding to a family of countably many branches at level r in the tree with the root a (Figure 3).

Note that since (X, d) is (R, ω_0)-haloed (Theorem 1.1), there exists an R-seed, and note that each $\Omega[a, r]$ is not necessarily maximal with respect to inclusion $\subseteq$.

Definition 5.2. Fix an R-seed $\{\Omega[a,r]\}_{a\in X, r\in R\setminus\{0\}}$ and a point $\varpi \in X$. For $S \in \mathbf{TEN}(R)$, we say that a point $x \in X$ is an *S-heir of ϖ* if there exists sequences $\{v_i\}_{i=0}^{m}$ in X and $\{r_i\}_{i=0}^{m-1}$ in S such that:
(S1) $v_0 = \varpi$;
(S2) $v_m = x$;
(S3) $v_{i+1} \in \Omega[v_i, r_i]$, for all $i \in \{0, \dots, m-1\}$;
(S4) $v_i \neq v_{i+1}$, for all $i \in \{0, \dots, m-1\}$;
(S5) if $2 \leq m$, then $r_{i+1} < r_i$, for all $i \in \{0, \dots, m-2\}$.

In this case, the pair of $\{v_i\}_{i=0}^{m}$ and $\{r\}_{i=0}^{m-1}$ is called an *S-inheritance of x from ϖ with length m*. If $m = 0$, then we consider $\{r_i\}_{i=0}^{m-1}$ as the empty sequence. This definition is nothing but chasing the ancestors of x in the dendrogram (Figure 3) and identifying the sequence of those ancestors with its descendant x.

Definition 5.3. For an R-seed $\{\Omega[a,r]\}_{a\in X, r\in R\setminus\{0\}}$, and a point $\varpi \in X$, we denote by $\mathbb{A}(S)$ the set of all S-heirs of ϖ. We also denote by $\mathbb{E}(S)$ the closure of $\mathbb{A}(S)$ in X.

Note that since (X, d) is complete, the set $\mathbb{E}(S)$ is isometric to the completion of $\mathbb{A}(S)$ for all $S \in \mathbf{TEN}(R)$.

Remark 5.1. The point ϖ is the unique S-heir from ϖ that has an S-inheritance with length 0 for all $S \in \mathbf{TEN}(R)$. Thus, we have $\mathbb{A}(\{0\}) = \{\varpi\}$ and $\mathbb{E}(\{0\}) = \{\varpi\}$.

In the next proposition, we observe that inheritances of two points determine a distance between these two points.

Proposition 5.1. *Let $S, T \in \mathbf{TEN}(R)$, and x and y be S-heir and T-heir of ϖ, respectively. Let $(\{v_i\}_{i=0}^{m}, \{r_i\}_{i=0}^{m-1})$ and $(\{w_i\}_{i=0}^{n}, \{l_i\}_{i=0}^{n-1})$ be S-inheritance and T-inheritance of x and y, respectively. In this situation, the next three statements are true*:
(1) *If $k \in \mathbb{Z}_{\geq 0}$ satisfies that $k \leq \min\{m, n\}$ and $v_i = w_i$, for all $i \leq k$, then $r_i = l_i$, for all $i < k$.*
(2) *If $k \in \mathbb{Z}_{\geq 0}$ satisfies that*
  (*a*) *$v_i = w_i$, for all $i \leq k$;*
  (*b*) *$v_{k+1} \neq w_{k+1}$,*
  then we have $\max\{r_k, l_k\} = d(x, y)$.
(3) *If $m < n$ and $v_i = w_i$, for all $i \leq m$, then we have $d(x, y) = l_m$.*

Proof. Since $r_i = d(v_i, v_{i+1})$ and $l_i = d(w_i, w_{i+1})$ (see condition (S3) in Definition 5.2), under the assumption of (1) we see that $r_i = l_i$ for all $i < k$. This completes the proof of (1).

We next prove (2). Put $z = v_k$. Then, $z = w_k$ by the assumption. By the definition of heirs, the strong triangle inequality, and by the conditions (S2) and (S5) in Definition 5.2, we have

$$\begin{aligned} d(v_{k+1}, x) &\leq d(v_k, v_{k+1}) \vee \cdots \vee d(v_{m-1}, v_m) \\ &= \max\{r_i | i \in \{k+1, \dots, m-1\}\} < r_k, \end{aligned}$$

and hence, $d(v_{k+1}, x) < r_k$. Similarly, we also have $d(w_{k+1}, y) < l_k$. Since $d(v_k, v_{k+1}) = r_k$ and $d(w_k, w_{k+1}) = l_k$, statement (1) in Lemma 2.1 yields $d(z, x) = d(v_k, x) = r_k$ and $d(z, y) = d(w_k, y) = l_k$. If $r_k \neq l_k$, then we obtain $d(x, y) = \max\{r_k, l_k\}$ using statement (1) in Lemma 2.1 again. If $r_k = l_k$, then $v_{k+1}, w_{k+1} \in \Omega[z, r_k]$ (see (S3) in Definition 5.2). Thus, due to $v_{k+1} \neq w_{k+1}$, we have $d(v_{k+1}, w_{k+1}) = r_k(=l_k)$. Using statement (1) in Lemma 2.1 once again, and by $d(v_{k+1}, x) < r_k$ and $d(w_{k+1}, y) < r_k$, we obtain $d(x, y) = r_k(=l_k)$. This proves statement (2).

Under the assumption of (3), we obtain $d(w_m, w_{m+1}) = l_m$ and $d(w_{m+1}, y) < l_m$. Condition (S2) states that $w_m = x$. Thus, statement (1) in Lemma 2.1 yields $d(x, y) = d(w_m, y) = l_m$. The proof is comlete. □

Remark 5.2. Let $S \in \mathbf{TEN}(R)$ and fix $x \in \mathbb{A}(S)$. Proposition 5.1 implies the uniqueness of an S-inheritance $(\{v_i\}_{i=0}^{m}, \{r_i\}_{i=0}^{m-1})$ of x from ϖ.

Lemma 5.2. *For every $S \in \mathbf{TEN}(R)$, the set $\mathbb{A}(S)$ is countable.*

Proof. For each $n \in \mathbb{Z}_{\geq 0}$, let A_n be the set of S-heirs from ϖ with length n. Note that $A_0 = \{\varpi\}$. Then, $\mathbb{A}(S) = \bigcup_{n\in\mathbb{Z}_{\geq 0}} A_n$. Remark that A_{n+1} can be regarded as a subset of the set $\bigcup_{a\in A_n, r\in S}\{a\} \times \Omega[a, r]$ by identifying a point $x \in A_{n+1}$ with its S-inheritance. Since S and each $\Omega[a, r]$ are countable, by induction, we see that each A_n is countable. Thus, $\mathbb{A}(S)$ is countable. □

We now prove that each $(\mathbb{E}(S), d)$ is the S-Urysohn universal ultrametric space.

Lemma 5.3. *For every $S \in \mathbf{TEN}(R)$, the space $(\mathbb{E}(S), d)$ is the S-Urysohn universal ultrametric space.*

Proof. Since $\mathbb{A}(S)$ is countable (Lemma 5.2), the space $\mathbb{E}(S)$ is separable. From (2) and (3) in Proposition 5.1, it follows that $(\mathbb{E}(S), d)$ is S-valued. We now verify that $(\mathbb{E}(S), d)$ is (S, ω_0)-haloed. It suffices to show that for every $a \in \mathbb{A}(S)$ and for every $r \in S\backslash\{0\}$, there exists an infinite r-equidistant subset of $\mathrm{B}(a, r)$. Let $(\{v_i\}_{i=0}^{m}, \{r_i\}_{i=0}^{m-1})$ be an S-inheritance of a. We divide the proof into three cases.

Case 1. $[r_0 < r]$: in this case, the set $\Omega[\varpi, r]$ is contained in $\mathbb{E}(S)$. Indeed, for every $q \in \Omega[\varpi, r]\backslash\{\varpi\}$, define $\{w_i\}_{i=0}^{1}$ and $\{l_i\}_{i=0}^{0}$ by $w_0 = \varpi$, $w_1 = q$, and $l_0 = r$. Then, $(\{w_i\}_{i=0}^{1}, \{l_i\}_{i=0}^{0})$ is an S-inheritance of q from ϖ. Hence, $\Omega[\varpi, r] \subseteq \mathbb{E}(S)$. By statement (2) in Proposition 5.1, we obtain $\Omega[\varpi, r] \subseteq \mathrm{B}(a, r)$. Thus, the set $\Omega[\varpi, r]$ is as desired.

Case 2. [There exists k such that $r_{k+1} < r < r_k$]: to prove $\Omega[v_k, r] \subseteq \mathbb{E}(S)$, for each $q \in \Omega[v_k, r]\backslash\{v_k\}$, we define $(\{w_i\}_{i=0}^{k+1}, \{l_i\}_{i=0}^{k})$ by

$$w_i = \begin{cases} v_i, & \text{if } i \leq k, \\ q, & \text{if } i = k+1, \end{cases}$$

and

$$l_i = \begin{cases} r_i, & \text{if } i \leq k, \\ r, & \text{if } i = k+1. \end{cases}$$

Then, $(\{w_i\}_{i=0}^{k+1}, \{l_i\}_{i=0}^{k})$ is an S-inheritance of q from ϖ, and hence, $\Omega[v_k, r] \subseteq \mathbb{E}(S)$. By statement (2) in Proposition 5.1, we obtain $\Omega[v_k, r] \subseteq \mathrm{B}(a, r)$. Thus, $\Omega[v_k, r]$ is an infinite r-equidistant subset as required.

Case 3. $[r < r_i$ for all $i \in \{0, \dots, m-1\}]$: in a similar way to the cases explained earlier, we observe that $\Omega[a, r] \subseteq \mathrm{B}(a, r)$.

Thus, we conclude that $(\mathbb{E}(S), d)$ is (S, ω_0)-haloed. Due to Theorem 1.1, the space $(\mathbb{E}(S), d)$ is $\mathcal{N}(S, \omega_0)$-injective, and hence, it is the S-Urysohn universal ultrametric space. □

Lemma 5.4. *Let $S, T \in \mathbf{TEN}(R)$. Let x be a T-heir of ϖ, and $(\{v_i\}_{i=0}^{m}, \{r_i\}_{i=0}^{m-1})$ be a T-inheritance of x. If $x \notin \mathbb{E}(S)$, then there exists $i \in \{0, \dots, m\}$ such that $r_i \in T\backslash S$.*

Proof. If all r_i were in S, then we would have $x \in \mathbb{E}(S)$. □

The following lemma is a preparation for the proof of (P4).

Lemma 5.5. *Let $S, T \in \mathbf{TEN}(R)$, and $x \in \mathbb{E}(T)$. Then, the value $d(x, \mathbb{E}(S))$ belongs to $\{0\} \cup (T\backslash S)$.*

Proof. We may assume that $x \notin \mathbb{E}(S)$. Put $L = d(x, \mathbb{E}(S))$. Then, $L > 0$. Take a point $z \in \mathbb{A}(T)$ sufficiently close to x so that $d(z, \mathbb{E}(S)) = L$ and $z \in \mathbb{E}(T)\backslash\mathbb{E}(S)$. Let $(\{v_i\}_{i=0}^{m}, \{r_i\}_{i=0}^{m-1})$ be a T-inheritance of z from ϖ. Since $L > 0$ and $\varpi \in \mathbb{E}(S)$, we note that $z \neq \varpi$. Thus, $0 < m$. Let k be the minimal number such that $r_k \in T\backslash S$. The existence of k is guaranteed by Lemma 5.4. From (2) in Proposition 5.1, it follows that every $y \in \mathbb{A}(S)$ satisfies $d(z, y) \geq r_k$. If $k = 0$, we define $q = \varpi$; otherwise, $q = v_{k-1}$. Then, $q \in \mathbb{E}(S)$ and statement (3) in Proposition 5.1 implies that $d(z, q) = r_k$. As a result, we have $L = r_k$. Since $r_k \in T\backslash S$, we conclude that $L \in T\backslash S$. □

We now give the proof of Theorem 1.5.

Proof of Theorem 1.5. It suffices to show the latter part of Theorem 1.5. Choose a point ϖ such that $\varpi \in K$ and choose an R-seed $\{\Omega[a, r]\}_{a\in X, r\in R\setminus\{0\}}$ so that the following condition is true:
(M) If $a \in K$, then for every $r \in R\setminus\{0\}$, the set $\Omega[a, r] \cap K$ is a maximal r-equidistant set of K.

This is possible since K is compact and $d(K^2) \subseteq R$. Note that the following statements are true:
(a) For every $a \in K$ and every $r \in R$, the set $\Omega[a, r] \cap K$ is contained in $\mathrm{B}(a, r) \cap K$;
(b) For every $a \in K$, for every $r \in R$, and for every $p \in \mathrm{B}(a, r) \cap K$, there exists $w \in \Omega[a, r] \cap K$ such that $d(p, w) < r$.

Define $F = \bigcup_{S\in \mathbf{TEN}(R)} \mathbb{E}(S)$ and define a petal of F by $\Pi(F, S) = \mathbb{E}(S)$. By this definition, we can confirm that properties (P2) and (P3) are satisfied. Using Lemma 5.3, property (P1) is true. Lemma 5.5 proves property (P4). Therefore, (F, d) is the R-petaloid ultrametric space.

Next, we prove $K \subseteq F$. Take an arbitrary point $x \in K$. If $x \in \mathbb{A}(S)$ for some $S \in \mathbf{TEN}(R)$, then $x \in F$. Thus, we may assume that $x \notin \mathbb{A}(S)$ for any $S \in \mathbf{TEN}(R)$. Put $T = d(K^2)$. In this setting, by recursion, due to assumption (M) (or statements (a) and (b)), we can define sequences $\{v_i\}_{i=0}^{\infty}$ in X and $\{r_i\}_{i=0}^{\infty}$ in $T\setminus\{0\}$ such that
(1) $v_0 = \varpi$;
(2) $v_{i+1} \in \Omega[v_i, r_i] \cap K$, for all $i \in \mathbb{Z}_{\geq 0}$;
(3) $v_i \neq v_{i+1}$, for all $i \in \mathbb{Z}_{\geq 0}$;
(4) we have $r_{i+1} < r_i$, for all $i \in \mathbb{Z}_{\geq 0}$;
(5) $d(x, v_i) < r_i$, for all $i \in \mathbb{Z}_{\geq 0}$.

Since T is tenuous (see [15, Corollary 2.28]), we have $r_i \to 0$ as $i \to \infty$, and hence, $v_i \to x$ as $i \to \infty$. By the definitions, for each $m \in \mathbb{Z}_{\geq 0}$, the pair $(\{v_i\}_{i=0}^{m}, \{r_i\}_{i=0}^{m-1})$ becomes a T-inheritance of v_m from ϖ. Subsequently, each v_m belongs to $\mathbb{A}(T)$. Therefore, from the completeness of $\mathbb{E}(T)$ and $v_i \to x$ as $i \to \infty$, it follows that $x \in \mathbb{E}(T)$. Hence, $K \subseteq F$. We complete the proof of Theorem 1.5. □

Remark 5.3. For $S \in \mathbf{TEN}(R)$, an element $x \in \mathbb{A}(S)$ (or an S-inheritance of x from ϖ) can be translated into a member in $\mathrm{G}(R, \omega_0)$ (Example 2.1) with finite support as follows. For every $a \in X$ and every $r \in R\setminus\{0\}$, fix an enumeration $\Omega[a, r] = \{q(a, r, k) | k \in \mathbb{Z}_{\geq 0}\}$ and let $(\{v_i\}_{i=0}^{m}, \{r_i\}_{i=0}^{m-1})$ be an S-inheritance of x from ϖ. Then, the point x is corresponding to the function f_x in $\mathrm{G}(S, \omega_0)$ defined by

$$f_x(s) = \begin{cases} k, & \text{if } s = r_i \text{ for some } i \text{ and } v_{i+1} = q(v_i, r_i, k), \\ 0, & \text{otherwise.} \end{cases}$$

Note that the point ϖ is corresponding to the zero function in $\mathrm{G}(R, \omega_0)$.

Remark 5.4. Theorem 1.5 is still valid even if K is separable and $d(K^2)$ is tenuous.

6 Questions

For an index set T, Borsík and Doboš [4] introduced the notion of a metric preserving function $F : [0, \infty)^T \to [0, \infty)$, and the product metric of a family $\{(X_t, d_t)\}_{t\in T}$ associated with F (for the precise definition, see [4]). This is a generalization of the ℓ^p-product metric for $p \in [1, \infty]$. In the case of two variables ($\mathrm{Card}(T) = 2$), we denote by $d_1 \times_F d_2$ their product metric associated with F.

As a counter part of of Theorem 1.2, we prove Theorem 1.3 stating that $(\mathbb{U} \times \mathbb{U}, \rho \times_F \rho)$ is not $\mathcal{F}$-injective. By generalizing the product metric, we raise the next question.

Question 6.1. Is there a metric preserving function $F : [0, \infty)^2 \to [0, \infty)$ of two variables such that $(\mathbb{U} \times \mathbb{U}, \rho \times_F \rho)$ is $\mathcal{F}$-injective?

As an Archimedean analogue of Theorem 1.4, we ask the following question.

Question 6.2. Is $(\mathcal{K}(\mathbb{U}), \mathcal{HD}_\rho)$ isometric to $(\mathbb{U}, \rho)$?

The author suspects that Question 6.2 is negative.

For a suitable range subset $R \subseteq [0, \infty)$, we can construct a (unique) separable complete metric space $(\mathbb{U}_R, \rho_R)$ associated with R injective for all finite metric spaces whose distances belong to R (see, for instance, [26]). Note that ρ_R takes only values in R. In the case of $R = [0, \infty)$, the space $(\mathbb{U}_R, \rho_R)$ is nothing but $(\mathbb{U}, \rho)$. The *random graph* (or the *Rado graph*) G is a (unique) countable graph with the following property.

(E) For all finite vertices $u_1, \dots, u_m$, and $v_1, \dots, v_n$ of G, there exists a vertex p of G which is adjunct $u_1, \dots, u_m$, and not adjunct to $v_1, \dots, v_n$.

For the definition using probability and characterizations, see, for example, [6]. The random graph is studied in graph theory and model theory. Even though G is defined purely in terms of graph theory, the graph G can be identified with the metric space $(\mathbb{U}_{\{0,1,2\}}, \rho_{\{0,1,2\}})$ by declaring that $x, y \in \mathbb{U}_{\{0,1,2\}}$ are adjunct if and only if $\rho_{\{0,1,2\}}(x, y) = 1$ (see [20, Exercise 5]), and property (E) is corresponding to the injectivity for the class of all finite metric spaces whose distances take values in $\{0, 1, 2\}$. We put $(G, h) = (\mathbb{U}_{\{0,1,2\}}, \rho_{\{0,1,2\}})$. Based on Theorems 1.2 and 1.4, it is worth asking whether the isometry problems of the Cartesian product and the hyperspace of the random graph are true. Remark that if $p \in [1, \infty)$, then $h \times_p h$ does not take vales in $\{0, 1, 2\}$, and hence, $(G \times G, h \times_p h)$ is not the random graph. In the case of $p = \infty$, similar to Theorem 1.3, the space $(G \times G, h \times_p h)$ is not isometric to (G, h).

Question 6.3. Is there a function $F : [0, \infty)^2 \to [0, \infty)$ of two variables such that $(G \times G, h \times_F h)$ is isometric to (G, h)?

Question 6.4. Is $(\mathcal{K}(G), \mathcal{HD}_h)$ isometric to (G, h)?

Acknowledgments: The author wishes to express his deepest gratitude to all members of Photonics Control Technology Team (PCTT) in RIKEN, where the majority of the articles were written, for their invaluable supports. Special thanks are extended to the Principal Investigator of PCTT, Satoshi Wada, for the encouragement and support that transcended disciplinary boundaries.

Funding information: This work was partially supported by JSPS KAKENHI Grant Number JP24KJ0182. The author would like to thank the referees for helpful comments and suggestions.

Author contribution: The author confirms the sole responsibility for the conception of the study, presented results and manuscript preparation.

Conflict of interest: The author states no conflict of interest.

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