Huizhang Yang*

Symmetry Reductions and Exact Solutions to the Kudryashov–Sinelshchikov Equation

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Abstract: In this article, based on the compatibility method, some nonclassical symmetries of Kudryashov–Sinelshchikov equation are derived. By solving the corresponding characteristic equations associated with symmetry equations, some new exact explicit solutions of Kudryashov–Sinelshchikov equation are obtained. For the exact explicit traveling wave solutions, the exact parametric representations are investigated by the integral bifurcation method.

Keywords: Compatibility Method; Exact Solutions; Integral Bifurcation Method; Kudryashov–Sinelshchikov Equation; Symmetry Reductions.

MSC: 34C37; 35G05; 37G10; 37L20.

1 Introduction

In 2010, Kudryashov and Sinelshchikov presented the following equation [1]:

$$u_t+\alpha uu_x+u_{xxx}-(uu_{xx})_x-\beta u_x u_{xx}=0, \qquad (1)$$

where u is a density and which are the terms which model heat transfer and viscosity, α, β are real parameters. Equation (1) is called Kudryashov–Sinelshchikov equation and is for describing the pressure waves in a mixture liquid and gas bubbles taking into consideration the viscosity of liquid and the heat transfer, it is a generalisation of the KdV and the BKdV equation and similar but not identical to the Camassa–Holm equation. Equation (1) was studied by many researchers in various methods. Ryabov found four families of solitary of (1) when $\beta=-3$ or $\beta=-4$ by using a modification of the truncated expansion method [2]; Randrüüt obtained a kind of new periodic wave solution which is called meandering solution in a more straightforward manner [3]; Li et al. investigated bifurcations of travelling wave solutions of the Kudryashov–Sinelshchikov equation and proved the existence of the peakon, solitary waves, and smooth and nonsmooth periodic waves [4]. He et al. [5] investigated the periodic loop solutions and their limit forms. Different kinds of other exact solutions are obtained in [6–8]. In [9], Chen et al. studied travelling wave solutions of (1) in the special case $\beta=2$. Here, we will investigate the travelling wave solutions of (1) for the other special cases.

Our aim in this article is to perform the Kudryashov–Sinelshchikov equation with the help of the compatibility method ([10, 11]) and the integral bifurcation method [9]. The remainder of this article is organized as follows: in Section 2, two types of nonclassical symmetries of the Kudryashov–Sinelshchikov equation (1) are presented. In Section 3, different types of symmetry reductions of the Kudryashov–Sinelshchikov equation (1) are obtained. In Section 4, some new exact explicit solutions that include travelling wave solutions and nontravelling wave solutions of the Kudryashov–Sinelshchikov equation (1) are derived from the reduced equations. In Section 5, some new exact travelling wave solutions of the Kudryashov–Sinelshchikov equation (1) are derived by integral bifurcation method. The last section is a short summary and discussion.

2 Symmetries of the Kudryashov–Sinelshchikov Equation

The main purpose of the compatibility method is to seek the nonclassical symmetry of (1) in the form

$$u_t=a(x,t)u_x+b(x,t)u+\gamma(x,t), \qquad (2)$$

where $a(x,t)$, $b(x,t)$, and $\gamma(x,t)$ are functions to be determined later by the compatibility of (1) and (2). Firstly, we can obtain the highest-order derivative term u_{xxx} of (1) by substituting (2) into (1) as follows:

$$u_{xxx}=-\frac{au_x+bu+\gamma+\alpha uu_x-(1+\beta)u_x u_{xx}}{1-u}. \qquad (3)$$

***Corresponding author: Huizhang Yang,** College of Mathematics of Honghe University, Mengzi Yunnan 661199, P.R. China, E-mail: yanghuizhangyn@163.com; yhz928@126.com

From the equality of u_t in (1), we can get at the derivatives of (1) and (2) with respect to t. Then in terms of the equality of u_{tt} of (1) and (2), one can get

$$-(\alpha uu_x+(1-u)u_{xxx}-(1+\beta)u_xu_{xx})_t=(au_x+bu+\gamma)_t. \qquad (4)$$

Substituting (2) and (3) into the expansion of (4) yields a polynomial of u and its derivatives. Setting all the coefficients of this polynomial to zero yields a set of differential equations with regard to unknown functions a, b, γ as following

$$\begin{aligned}
&b_x(\beta+4)+3a_{xx}=0,\\
&\gamma_{xxx}+\gamma_t+\gamma^2-3a_x\gamma=0,\\
&(b+\gamma)(\beta+1)=0,\\
&-\gamma_x(\beta+1)+3a_{xx}+3b_x=0,\\
&(\beta+1)(a_{xx}+2b_x)=0,\\
&4b_{xx}+a_{xxx}+2\alpha a_x+\beta b_{xx}=0,\\
&-6a_{xx}-b_x(\beta+7)+\gamma_x(\beta+1)=0,\\
&\gamma_{xxx}-\alpha\gamma_x+3a_xb+\alpha b_x-2b_{xxx}-b_t+b^2=0,\\
&-2\gamma_{xxx}+b_{xxx}-\gamma_t+\alpha\gamma_x+3a_x(\gamma-b)+b_t+2b\gamma=0,\\
&a_{xxx}+3b_{xx}-\gamma_{xx}(\beta+1)+a_t-3aa_x+\gamma(a+\alpha)=0,\\
&-2a_{xxx}-b_{xx}(\beta+7)+\gamma_{xx}(\beta+1)-a_t+a_x(3a-2\alpha)\\
&\quad+b(a+\alpha)=0.
\end{aligned}$$

Solving the above-mentioned system of differential equations, we can get the following results

Case 1 $\alpha=0,\quad \beta=\beta,\quad a=\dfrac{-C_1x+3C_3-x}{3C_1t+3C_2},\quad b=\dfrac{1}{C_1t+C_2},$
$\gamma=-\dfrac{1}{C_1t+C_2},$
and the corresponding nonclassical symmetry of (1) expressed by

$$\sigma_1=-(C_1t+C_2)u_t+\frac{-C_1x+3C_3-x}{3}u_x+u-1, \qquad (5)$$

where β, C_1, C_2 and C_3 are arbitrary constants.

Case 2 $\alpha=\alpha,\quad \beta=\beta,\quad a=-\alpha+\dfrac{C_2}{t+C_1},\quad b=-\dfrac{1}{t+C_1},$
$\gamma=\dfrac{1}{t+C_1}.$

and the corresponding nonclassical symmetry of (1) expressed by

$$\sigma_2=-(t+C_1)u_t+(-\alpha t-\alpha C_1+C_2)u_x-u+1, \qquad (6)$$

where α, β, C_1, and C_2 are arbitrary constants.

3 Symmetry Reductions for the Kudryashov–Sinelshchikov Equation

In order to obtain the invariant transformation, we write the characteristic equation in the form

$$\frac{dx}{a}=\frac{dt}{-1}=\frac{du}{-bu-\gamma} \qquad (7)$$

3.1 Symmetry Reductions for Case 1

The determining equations for similarity variables of $\sigma_1=0$ are

$$\frac{dx}{-C_1x+3C_3-x}=\frac{dt}{-3(C_1t+C_2)}=\frac{du}{-3(u-1)}, \qquad (8)$$

here C_1, C_2, and C_3 are arbitrary constants.

Case 3.1.1 let $C_1=C_2=C_3=0$, solving the above system (8), we can obtain u as the following form

$$u=1+x^3f(t), \qquad (9)$$

$f=f(t)$ are similarity variables. Substituting (9) into (1), we get the reduced equation of the Kudryashov–Sinelshchikov equation as

$$f'(t)-6(4+3\beta)f^2(t)=0. \qquad (10)$$

Case 3.1.2 let $C_1=C_3=0$, $C_2\neq 0$, solving the system (8), we can obtain u as the following form

$$u=1+e^{\frac{t}{C_2}}f(\xi), \qquad (11)$$

where $\xi=xe^{-\frac{t}{3C_2}}$ and $f=f(\xi)$ are similarity variables. Substituting (11) into (1), we reduce the Kudryashov–Sinelshchikov equation to the following ODE

$$f(\xi)-\frac{\xi}{3}f'(\xi)-C_2f(\xi)f'''(\xi)-(1+\beta)C_2f'(\xi)f''(\xi)=0. \qquad (12)$$

Case 3.1.3 let $C_1=-1$, $C_2=0$, $C_3\neq 0$, solving the system (8), we can obtain u as the following form

$$u=1+\frac{f(\xi)}{t}, \qquad (13)$$

where $\xi=te^{-\frac{x}{C_3}}$ and $f=f(\xi)$ are similarity variables. Substituting (13) into (1), we obtain the reduced equation of the Kudryashov–Sinelshchikov equation as

$$C_3^3\xi f'(\xi)-C_3^3 f(\xi)+\xi f(\xi)(f'''(\xi)\xi^2+3f''(\xi)\xi+f'(\xi))$$
$$+(1+\beta)f'(\xi)\xi^2(f''(\xi)\xi+f'(\xi))=0. \quad (14)$$

Case 3.1.4 let $C_1\neq 0$ and $C_1\neq -1$, $C_2=C_3=0$, solving the system (8), we can obtain u as the following form

$$u=1+t^{\frac{1}{C_1}}f(\xi), \quad (15)$$

where $\xi=xt^{-\frac{1}{3}\left(1+\frac{1}{C_1}\right)}$ and $f=f(\xi)$ are similarity variables. Substituting (15) into (1), one can see that the reduced equation of the Kudryashov–Sinelshchikov equation is in the form

$$f(\xi)-\frac{1+C_1}{3}\xi f'(\xi)-C_1 f(\xi)f'''(\xi)-(1+\beta)C_1 f'(\xi)f''(\xi)=0. \quad (16)$$

3.2 Symmetry Reductions for Case 2

The determining equations for similarity variables of $\sigma_2=0$ are

$$\frac{\mathrm{d}x}{-\alpha t-\alpha C_1+C_2}=\frac{\mathrm{d}t}{-(t+C_1)}=\frac{\mathrm{d}u}{u-1}, \quad (17)$$

here C_1, C_2 are arbitrary constants.

Case 3.2.1 let $C_1=C_2=0$, we obtain $u=1+\frac{f(\xi)}{t}$, $\xi=x-\alpha t$ by solving the system of (17). Substituting it into (1), we derive the following reduced ODE

$$-f(\xi)+\alpha f(\xi)f'(\xi)-(1+\beta)f'(\xi)f''(\xi)-f(\xi)f'''(\xi)=0. \quad (18)$$

Case 3.2.2 let $C_1=0$ and $C_2\neq 0$, by solving the system of (17), we obtain the following expression of u as

$$u=1+\frac{f(\xi)}{t},\ \xi=x-\alpha t-C_2\ln t. \quad (19)$$

Substituting (19) into (1), the reduced nonlinear partial differential equation is derived as

$$-C_2 f'(\xi)-f(\xi)+\alpha f(\xi)f'(\xi)-(1+\beta)f'(\xi)f''(\xi)$$
$$-f(\xi)f'''(\xi)=0. \quad (20)$$

Case 3.2.3 let $C_1\neq 0$ and $C_2=0$, we obtain the following expression of u by solving the system of (17)

$$u=1+\frac{f(\xi)}{t-c_1},\ \xi=x-\alpha t. \quad (21)$$

Substituting (21) into (1), the reduced nonlinear partial differential equation is derived as same as (18).

4 Exact Explicit Solutions to the Kudryashov–Sinelshchikov Equation

For Case 3.1.1, according to simple calculation, obviously we obtain the rational function solution of the reduced equation (10) as

$$f(t)=-\frac{1}{(24+18\beta)t+C}, \quad (22)$$

where C is an arbitrary constant. Therefore, when $\alpha=0$, we have the exact solution of (1) as follows

$$u(x,\ t)=1-\frac{x^3}{(24+18\beta)t+C}. \quad (23)$$

For Case 3.1.2, we will give the exact analytic solutions to the reduced equation (12) using the power series method ([12, 13]). Now, we seek a solution of (12) in a power series of the following form

$$f(\xi)=\sum_{n=0}^{\infty}A_n\xi^n. \quad (24)$$

Substituting (24) into (12), we have

$$\sum_{n=0}^{\infty}A_n\xi^n-\frac{1}{3}\sum_{n=0}^{\infty}(n+1)A_{n+1}\xi^{n+1}-C_2\sum_{n=0}^{\infty}\sum_{k=0}^{n}(k+1)(k+2)(k+3)$$
$$A_{k+3}A_{n-k}\xi^n-(1+\beta)C_2\sum_{n=0}^{\infty}\sum_{k=0}^{n}(k+1)(k+2)(n-k+1)A_{k+2}$$
$$A_{n-k+1}\xi^n=0, \quad (25)$$

From (25), comparing coefficients, we obtain (for $n=0$)

$$A_0-6C_2A_0A_3-2(1+\beta)C_2A_1A_2=0, \quad (26)$$

with arbitrary chosen $A_0\neq 0$, A_1 and A_2, we have

$$A_3=\frac{A_0-2(1+\beta)C_2A_1A_2}{6C_2A_0}, \quad (27)$$

and (for $n=1$)

$$\frac{1}{3}A_1-6C_2A_1A_3-12C_2A_0A_4-2(1+\beta)C_2A_2^2-3\beta C_2A_0A_3=0. \quad (28)$$

Generally, for $n\geq 1$, in view of (25), we can get all the coefficients $A_n (n\geq 0)$ of the power series (24) as follows

$$A_{n+3}=\frac{\left(1-\frac{n}{3}\right)A_n-C_2\sum_{k=0}^{n-1}(k+1)(k+2)(k+3)A_{k+3}A_{n-k}}{(n+1)(n+2)(n+3)C_2A_0}$$
$$-\frac{\sum_{k=0}^{n}(k+1)(k+2)(n-k+1)A_{k+2}A_{n-k+1}}{(n+1)(n+2)(n+3)C_2A_0}. \quad (29)$$

This implies that for (12), there exists a power series solutions (24). In addition, it is easy to prove that the convergence of the power series (24) with the coefficients given by (25–27), we can see [12, 13] and references cited therein. Thus, this power series solution is an exact analytic solution.

Hence, the power series solution of (12) can be written as

$$f(\xi)=A_0+A_1\xi+A_2\xi^2+A_3\xi^3+\sum_{n=1}^{\infty}A_{n+3}\xi^{n+3}$$
$$=A_0+A_1\xi+A_2\xi^2+\frac{A_0-2(1+\beta)C_2A_1A_2}{6C_2A_0}\xi^3+\cdots$$

Thus, we can obtain that the power series solution of (1) as

$$u(x,t)=1+e^{\frac{t}{C_2}}\left[A_0+A_1\xi+A_2\xi^2+A_3\xi^3+\sum_{n=1}^{\infty}A_{n+3}\xi^{n+3}\right]$$
$$=1+e^{\frac{t}{C_2}}\left[A_0+A_1\xi+A_2\xi^2+\frac{A_0-2(1+\beta)C_2A_1A_2}{6C_2A_0}\xi^3+\cdots\right], \quad (30)$$

here $\xi=xe^{-\frac{t}{3C_2}}$, and C_2 is an arbitrary constant.

Remark 1. For Case 3.1.3 and Case 3.1.4, we can obtain the exact analytic solution of the reduced equations (14) and (16) by using the power series method similarly, here we do not list them for simplicity.

For Case 3.2.1, suppose that the solution of the reduced equation (18) is in the form

$$f(\xi)=a_o+a_1\phi+a_2\phi^2 \quad (31)$$

and ϕ satisfies the equation

$$\phi'=A+B\phi+C\phi^2, \quad (32)$$

where $\phi=\phi(\xi)$ and a_0, a_1, a_2, A, B, C are constants to be determined later. Substituting (31) and (32) into (18), and equating the coefficients of like powers of ϕ^i, $(i=0,1,2,\ldots)$ to zero, which give rise to the system of algebraic equations to a_0, a_1, a_2, A, B, C. With the aid of Maple, we obtain the following solutions

$$a_0=a_0,\ a_1=\frac{1}{\alpha A},\ a_2=0,\ A=A,\ B=0,\ C=0. \quad (33)$$

From (32) and (33), it is easy to see that $\phi(\xi)=A\xi+C_0$, and C_0 is an arbitrary constant. The solution of the reduced equation (18) is obviously obtained as

$$f(\xi)=a_0+\frac{1}{\alpha A}(A\xi+C_0), \quad (34)$$

and the solution of (1) is expressed as

$$u(x,t)=1+\frac{a_0+\frac{1}{\alpha A}(A\xi+C_0)}{t}, \quad (35)$$

where $\xi=x-\alpha t$, and a_0, A, α, C_0 are arbitrary constants.

For Case 3.2.2, suppose that the solution of the reduced equation (20) is in the form of (31) and ϕ satisfies (32) and a_0, a_1, a_2, A, B, C are constants to be determined later. Substituting (31) and (32) into (20). Similar to Case 3.2.1, we obtain the following solutions

$$a_0=a_0,\ a_1=\frac{Ba_0}{A},\ a_2=0,\ A=A,\ B=B,\ C=0,\ \alpha=\alpha,$$
$$\beta=\frac{\alpha-2B^2}{B^2}. \quad (36)$$

From (32) and (36), it is easy to see that $\phi'(\xi)=A+B\phi$, solving this first-order ordinary differential equation, we have $\phi(\xi)=C_0e^{B\xi}-\frac{B}{A}$ and C_0 is an arbitrary constant. The solution of the reduced equation (20) is obviously obtained as

$$f(\xi)=a_0+\frac{Ba_0}{A}\left(C_0e^{B\xi}-\frac{B}{A}\right), \quad (37)$$

and the solution of (1) is expressed as

$$u(x,t)=1+\frac{a_0+\frac{Ba_0}{A}\left(C_0e^{B\xi}-\frac{B}{A}\right)}{t}, \quad (38)$$

where $\xi=x-\alpha t-C_2\ln t$, and a_0, A, B, α, and C_0 are arbitrary constants.

5 Exact Travelling Wave Solutions of (1) by Integral Bifurcation Method

In this section, we consider travelling wave solutions of (1) with the special case $\beta=1$ by using the integral bifurcation method.

Substituting $u(x, t) = 1 - f(\xi)$ with $\xi = x - ct$ into (1) and integrating it once, we have

$$ff'' = \frac{1}{2}\alpha f^2 + (c-\alpha)f + \frac{1}{2}g - \frac{1}{2}\beta(f')^2, \tag{39}$$

where "′" is the derivative with respect to ξ and g an integral constant.

Letting $y = \frac{df}{d\xi}$, we get the following planar system

$$\frac{df}{d\xi} = y, \quad \frac{dy}{d\xi} = \frac{\alpha f^2 + 2(c-\alpha)f + g - \beta y^2}{2f}. \tag{40}$$

Using the transformation $d\xi = 2f d\tau$, it carries (41) into the Hamiltonian system

$$\frac{df}{d\tau} = 2fy, \quad \frac{dy}{d\tau} = \alpha f^2 + 2(c-\alpha)f + g - \beta y^2. \tag{41}$$

For $\beta \neq -2, -1, 0$, system (40) and (41) have the same first integral

$$H(f, y) = f^\beta \left(y^2 - \frac{\alpha\beta(\beta+1)f^2 + 2\beta(\beta+2)(c-\alpha)f + (\beta+1)(\beta+2)g}{\beta(\beta+1)(\beta+2)} \right) = h. \tag{42}$$

Taking $\beta = 1$ and $h = 0$, (42) can be rewritten as follow:

$$y^2 = \frac{\frac{1}{3}\alpha f^3 + (c-\alpha)f^2 + gf}{f}. \tag{43}$$

From (43) and the first equation of (41), we have

$$\frac{df}{d\tau} = 2\sqrt{gf^2 + (c-\alpha)f^3 + \frac{1}{3}\alpha f^4}. \tag{44}$$

Denote that $\Delta = \frac{1}{3}\left(\frac{1}{3}\alpha - 4g\right)\alpha$, $\epsilon = \pm 1$, with the aid of Table 1 of [14], the following exact solutions of (44) are obtained.

$$f_1(\tau) = \frac{g(\alpha-c)\operatorname{sech}^2(\sqrt{g}\tau)}{(\alpha-c)^2 - \frac{1}{3}\alpha g(1+\epsilon\tanh(\sqrt{g}\tau))^2}, \ g>0, \tag{45}$$

$$f_2(\tau) = \frac{g(c-\alpha)\operatorname{csch}^2(\sqrt{g}\tau)}{(c-\alpha)^2 - \frac{1}{3}\alpha g(1+\epsilon\coth(\sqrt{g}\tau))^2}, \ g>0, \tag{46}$$

$$f_3(\tau) = \frac{2g\operatorname{sech}(2\sqrt{g}\tau)}{\epsilon\sqrt{\Delta} - (c-\alpha)\operatorname{sech}(2\sqrt{g}\tau)}, \ g>0, \ \Delta>0, \tag{47}$$

$$f_4(\tau) = \frac{2g\sec(2\sqrt{-g}\tau)}{\epsilon\sqrt{\Delta} - (c-\alpha)\sec(2\sqrt{-g}\tau)}, \ g<0, \ \Delta>0, \tag{48}$$

$$f_5(\tau) = \frac{2g\csc(2\sqrt{-g}\tau)}{\epsilon\sqrt{\Delta} - (c-\alpha)\csc(2\sqrt{-g}\tau)}, \ g<0, \ \Delta>0, \tag{49}$$

$$f_6(\tau) = \frac{g\operatorname{sech}^2(\sqrt{g}\tau)}{(\alpha-c) + 2\epsilon\sqrt{\frac{1}{3}\alpha g}\tanh(\sqrt{g}\tau)}, \ g>0, \ \alpha>0, \tag{50}$$

$$f_7(\tau) = \frac{g\sec^2(\sqrt{-g}\tau)}{(\alpha-c) + 2\epsilon\sqrt{-\frac{1}{3}\alpha g}\tan(\sqrt{-g}\tau)}, \ g<0, \ \alpha>0, \tag{51}$$

$$f_8(\tau) = \frac{g\operatorname{csch}^2(\sqrt{g}\tau)}{(c-\alpha) + 2\epsilon\sqrt{\frac{1}{3}\alpha g}\coth(\sqrt{g}\tau)}, \ g>0, \ \alpha>0, \tag{52}$$

$$f_9(\tau) = \frac{g\csc^2(\sqrt{-g}\tau)}{(\alpha-c) + 2\epsilon\sqrt{-\frac{1}{3}\alpha g}\cot(\sqrt{-g}\tau)}, \ g<0, \ \alpha>0, \tag{53}$$

$$f_{10}(\tau) = \frac{g}{12g-c}(1+\epsilon\tanh(\sqrt{g}\tau)), \ g>0, \tag{54}$$

$$f_{11}(\tau) = \frac{g}{12g-c}(1+\epsilon\coth(\sqrt{g}\tau)), \ g>0, \tag{55}$$

$$f_{12}(\tau) = \frac{16ge^{2\epsilon\sqrt{g}\tau}}{(e^{2\epsilon\sqrt{g}\tau} - 4(c-\alpha))^2 - \frac{64}{3}\alpha g}, \ g>0, \tag{56}$$

$$f_{13}(\tau) = \frac{16\epsilon ge^{2\epsilon\sqrt{g}\tau}}{1 - \frac{64}{3}\alpha ge^{4\epsilon\sqrt{g}\tau}}, \ g>0, \ c=\alpha. \tag{57}$$

Using (45–57), transformations $d\xi = 2f d\tau$ and $u(x, t) = 1 - f(\xi)$ with $\xi = x - ct$, we can obtain the exact travelling wave solutions of (1) as follows:

$$\begin{cases} u_1(x,\,t)=1-\dfrac{g(\alpha-c)\mathrm{sech}^2(\sqrt{g}\tau)}{(\alpha-c)^2-\frac{1}{3}\alpha g(1+\epsilon\tanh(\sqrt{g}\tau))^2}, \\ x-ct=\dfrac{6\epsilon}{\sqrt{-3\alpha g}}\arctan\left(\dfrac{\alpha g(1+\epsilon\tanh(\sqrt{g}\tau))}{(\alpha-c)\sqrt{-3\alpha g}}\right), \\ g>0,\ \alpha<0, \end{cases} \tag{58}$$

$$\begin{cases} u_2(x,\,t)=1-\dfrac{g(c-\alpha)\mathrm{csch}^2(\sqrt{g}\tau)}{(c-\alpha)^2-\frac{1}{3}\alpha g(1+\epsilon\coth(\sqrt{g}\tau))^2}, \\ x-ct=\dfrac{6\epsilon}{\sqrt{-3\alpha g}}\arctan\left(\dfrac{\alpha g(1+\epsilon\coth(\sqrt{g}\tau))}{(c-\alpha)\sqrt{-3\alpha g}}\right), \\ g>0,\ \alpha<0, \end{cases} \tag{59}$$

$$\begin{cases} u_3(x,\,t)=1-\dfrac{2g\mathrm{sech}(2\sqrt{g}\tau)}{\epsilon\sqrt{\Delta}-(c-\alpha)\mathrm{sech}(2\sqrt{g}\tau)}, \\ x-ct=\dfrac{4\sqrt{g}}{\sqrt{\Delta-(c-\alpha)^2}}\arctan\left(\dfrac{(\epsilon\sqrt{\Delta}+(c-\alpha))\tanh(\sqrt{g}\tau)}{\sqrt{\Delta-(c-\alpha)^2}}\right), \\ g>0,\ \Delta>(c-\alpha)^2, \end{cases} \tag{60}$$

$$\begin{cases} u_4(x,\,t)=1-\dfrac{2g\sec(2\sqrt{-g}\tau)}{\epsilon\sqrt{\Delta}-(c-\alpha)\sec(2\sqrt{-g}\tau)}, \\ x-ct=-\dfrac{4\sqrt{-g}}{\sqrt{\Delta-(c-\alpha)^2}}\mathrm{arctanh}\left(\dfrac{(\epsilon\sqrt{\Delta}+(c-\alpha))\tan(\sqrt{-g}\tau)}{\sqrt{\Delta-(c-\alpha)^2}}\right), \\ g<0,\ \Delta>(c-\alpha)^2, \end{cases} \tag{61}$$

$$\begin{cases} u_5(x,\,t)=1-\dfrac{2g\csc(2\sqrt{-g}\tau)}{\epsilon\sqrt{\Delta}-(c-\alpha)\csc(2\sqrt{-g}\tau)}, \\ x-ct=-\dfrac{4\sqrt{-g}}{\sqrt{\Delta-(c-\alpha)^2}}\mathrm{arctanh}\left(\dfrac{(c-\alpha)\tan(\sqrt{-g}\tau)-\epsilon\sqrt{\Delta}}{\sqrt{\Delta-(c-\alpha)^2}}\right), \\ g<0,\ \Delta>(c-\alpha)^2, \end{cases} \tag{62}$$

$$\begin{cases} u_6(x,\,t)=1-\dfrac{g\mathrm{sech}^2(\sqrt{g}\tau)}{(\alpha-c)+2\epsilon\sqrt{\frac{1}{3}\alpha g}\tanh(\sqrt{g}\tau)}, \\ x-ct=\dfrac{3\epsilon}{\sqrt{3\alpha}}\ln\left|(\alpha-c)+2\epsilon\sqrt{\frac{1}{3}\alpha g}\tanh(\sqrt{g}\tau)\right|, \\ g>0,\ \alpha>0, \end{cases} \tag{63}$$

$$\begin{cases} u_7(x,\,t)=1-\dfrac{g\sec^2(\sqrt{-g}\tau)}{(\alpha-c)+2\epsilon\sqrt{-\frac{1}{3}\alpha g}\tan(\sqrt{-g}\tau)}, \\ x-ct=-\dfrac{3\epsilon}{\sqrt{3\alpha}}\ln\left|(\alpha-c)+2\epsilon\sqrt{-\frac{1}{3}\alpha g}\tan(\sqrt{-g}\tau)\right|, \\ g<0,\ \alpha>0, \end{cases} \tag{64}$$

$$\begin{cases} u_8(x,\,t)=1-\dfrac{g\mathrm{csch}^2(\sqrt{g}\tau)}{(c-\alpha)+2\epsilon\sqrt{\frac{1}{3}\alpha g}\coth(\sqrt{g}\tau)}, \\ x-ct=\dfrac{3\epsilon}{\sqrt{3\alpha}}\ln\left|(\alpha-c)+2\epsilon\sqrt{\frac{1}{3}\alpha g}\coth(\sqrt{g}\tau)\right|, \\ g>0,\ \alpha>0, \end{cases} \tag{65}$$

$$\begin{cases} u_9(x,\,t)=1-\dfrac{g\csc^2(\sqrt{-g}\tau)}{(\alpha-c)+2\epsilon\sqrt{-\frac{1}{3}\alpha g}\cot(\sqrt{-g}\tau)}, \\ x-ct=\dfrac{3\epsilon}{\sqrt{3\alpha}}\ln\left|(\alpha-c)+2\epsilon\sqrt{-\frac{1}{3}\alpha g}\cot(\sqrt{-g}\tau)\right|, \\ g<0,\ \alpha>0, \end{cases} \tag{66}$$

$$\begin{cases} u_{10}(x,\,t)=1-\dfrac{g}{12g-c}(1+\epsilon\tanh(\sqrt{g}\tau)), \\ x-ct=\dfrac{2\sqrt{g}}{12g-c}(\sqrt{g}\tau+\epsilon\ln|\cosh(\sqrt{g}\tau)|), \\ g>0, \end{cases} \tag{67}$$

$$\begin{cases} u_{11}(x,\,t)=1-\dfrac{g}{12g-c}(1+\epsilon\coth(\sqrt{g}\tau)), \\ x-ct=\dfrac{2\sqrt{g}}{12g-c}(\sqrt{g}\tau+\epsilon\ln|\sinh(\sqrt{g}\tau)|), \\ g>0, \end{cases} \tag{68}$$

$$\begin{cases} u_{12}(x,\,t)=1-\dfrac{16ge^{2\epsilon\sqrt{g}\tau}}{(e^{2\epsilon\sqrt{g}\tau}-4(c-\alpha))^2-\frac{64}{3}\alpha g}, \\ x-ct=\dfrac{6\epsilon}{\sqrt{3\alpha}}\mathrm{arctanh}\left(\dfrac{3(e^{2\epsilon\sqrt{g}\tau}-4(c-\alpha))}{8\sqrt{3\alpha g}}\right), \\ g>0,\ \alpha>0, \end{cases} \tag{69}$$

$$\begin{cases} u_{13}(x,\,t)=1-\dfrac{16\epsilon ge^{2\epsilon\sqrt{g}\tau}}{1-\frac{64}{3}\alpha ge^{4\epsilon\sqrt{g}\tau}}, \\ x-ct=\dfrac{6\epsilon}{\sqrt{3\alpha}}\mathrm{arctanh}\left(\dfrac{8\alpha ge^{2\epsilon\sqrt{g}\tau}}{\sqrt{3\alpha g}}\right), \\ g>0,\ c=\alpha>0, \end{cases} \tag{70}$$

Remark 2. For the cases $\beta = -4, -3$, we can obtain the travelling wave solutions of (1) similarly, we omit them here.

6 Conclusions

We have discussed the symmetry reductions and exact explicit solutions of the Kudryashov–Sinelshchikov equation in this article. It is shown that (1) can be reduced to constant coefficients partial differential equations (10), (18), (20) and variable coefficients partial differential equations (12), (14), and (16) by the compatibility method. Furthermore, some new exact explicit solutions of (1) can be constructed by solving the reduced nonlinear partial differential equations, including travelling wave solutions and nontravelling wave solutions. At last, we also obtain some new exact explicit travelling wave solutions by using the integral bifurcation method.

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