Advanced Calculus: Differential Calculus and Stokes' Theorem
Answers to Selected Exercises

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Chapter 1

Section 1.1

(1a) $E = \{x \in \mathbb{R} \mid x = n^2,\ n \in \mathbb{Z}\}$.

(1d) $B = \{p/q \mid p, q \in \mathbb{Z},\ q \leq 30\}$.

(3a) No, $5 \in E$, but $5 \notin F$.

(3b) Yes. because $\cos n\pi = \pm 1$.

Section 1.2

(4a) Lin. indep. $\mathrm{span}(v_1, v_2) = \mathbb{R}^2$.

(4b) Lin. dep., $v_1 = v_3 - v_2$: $\mathrm{span}(v_1, v_2, v_3) = \{(2\alpha_1, \alpha_2, 4\alpha_1 + 3\alpha_2) \mid \alpha_1, \alpha_2 \in \mathbb{R}\}$.

(5a) Yes.

(5c) No.

(7) $(3, 2, -1)$.

Section 1.3

(3) $r = \sin(2\theta)$.

(6) $\cos\phi = \pm\sin\phi$.

Section 1.4

(1a) $\{(x, y \in \mathbb{R}^2 \mid x \pm y\}$.

(1c) (ρ, ϕ, θ) have no restrictions.

(2a) $\dfrac{\partial f}{\partial x} = -y\tan(xy)$, $\dfrac{\partial f}{\partial y} = -x\tan(xy)$.

(2d) $\dfrac{\partial f}{\partial x} = 2x\cos\left(\dfrac{1}{\sqrt{x^2+y^2+z^2}}\right) + x(x^2+y^2+z^2)^{-1/2}\sin\left(\dfrac{1}{\sqrt{x^2+y^2+z^2}}\right)$. The other partial derivatives have a similar form.

(4a) Neither.

(4b) Closed.

(4c) Open.

Section 1.5

(4a) $\dfrac{x^2}{\left(\sqrt{\frac{3}{2}}\right)^2} + \dfrac{y^2}{\left(\sqrt{\frac{3}{5}}\right)^2} = 1.$

(5b) No intersection between the conics.

Section 1.6

(1a) Hyperboloid of two sheets $-\dfrac{x^2}{\left(\sqrt{\frac{3}{2}}\right)^2} + \dfrac{y^2}{\left(\sqrt{3}\right)^2} - \dfrac{z^2}{\left(\sqrt{3}\right)^2} = 1.$

(1b) elliptic paraboloid $x = y^2 + \dfrac{z^2}{\left(\sqrt{\frac{1}{2}}\right)^2}$.

(1d) hyperboloid of one sheet $-\dfrac{x^2}{(2\sqrt{2})^2} - \dfrac{y^2}{\left(\sqrt{2}\right)^2} + \dfrac{z^2}{\left(\frac{2}{3}\right)^2} = 1.$

(2a) horizontal trace , $y^2 - 2x^2 = 3 + K_3^2$ vertical traces $y^2 - z^2 = 2K_1^2 + 3$ and $2x^2 + z^2 = K_2^2 - 3$.

(2b) horizontal trace $x = y^2 + 2K_3^2$, vertical traces $y^2 + 2z^2 = K_1$ and $x = 2z^2 + K_2^2$.

(2d) horizontal trace $\dfrac{x^2}{2^2} - y^2 = 2 - 3K_1^2$.

(5) Parametric representation of intersection curve: $x(t) = t$, $y(t)^{\pm} = \pm\sqrt{(3(1/2 + t^2)}$ and $z(t) = \pm\sqrt{5t^2 + 3/2}$, domain $t \in \mathbb{R}$. The curve is made up of four pieces: $x(t), y^+(t), z^+(t)$, $x(t), y^+(t), z^-(t)$, $x(t), y^-(t), z^+(t)$ and $x(t), y^-(t), z^-(t)$. This is a curve which projects to a hyperbola in the xy-plane.

(8) $x(t) = t^2$, $y(t) = t$, $z(t) = +\sqrt{4 - t^2}$ and $x(t) = t^2$, $y(t) = t$, $z(t) = -\sqrt{4 - t^2}$. Domain $-2 \leq t \leq 2$.

(9) $x(t) = t^3$, $y(t) = t^2$ and $z(t) = \pm\sqrt{t^6 + t^4}$.

Chapter 2

Section 2.1

(1a) differentiable at all $t \in [0, 1]$, $\mathbf{r}'(t) = (2t - 2, 2\pi\cos(2\pi t)$. The function is smooth.

(1b) differentiable at all $t \neq 0$. $\mathbf{r}'(t) = (6t, -\sin(t), (2/t^3)e^{-1/t^2})$. The vector function is smooth

(1d) differentiable at all $t \in [1, 3]$. $\mathbf{r}'(t) = (12t^2, 2t) \neq (0, 0)$ for all $t \in [1, 3]$. The vector function is smooth.

(2a) $\int \mathbf{r}(t)\,dt = (t^3/3 - t^2, -(1/2\pi)\cos(2\pi t), t^2/2)$,

(2c) $\int \mathbf{r}(t)\,dt = (3e^t(t - 1), (3/5)t^{5/3}, t^3/3)$.

(3a) velocity given above, $||\mathbf{r}'(t)|| = \sqrt{(2(t-1))^2 + 4\pi^2\cos^2(2\pi t) + 1}$ and acceleration $\mathbf{r}''(t) = (2, -4\pi^2\sin(2\pi t), 0)$,

(3c) velocity given above, $||\mathbf{r}'(t)|| = \sqrt{(3e^t(t+1))^2 + (4/9)t^{-2/3} + 4t^2}$ and acceleration $\mathbf{r}''(t) = (6e^t + 3te^t, (-2/9)t^{-4/3}, 2)$.

(5) If $y = f(x)$ and f is differentiable, then $\mathbf{r}(t) = (t, f(t))$ and $\mathbf{r}'(t) = (1, f'(t)) \neq (0, 0)$ for all t.

(8) $\mathbf{r}'(t) = (v_{01}, v_{02}, -9.8t + v_{03})$ and $\mathbf{r}(t) = (v_{01}t, v_{02}t, -4.9t^2 + v_{03}t + h_0)$. Final velocity occurs at $t = t_0$ that solves $-4.9t^2 + v_{03}t + h_0 = 0$; $\mathbf{r}'(t_0)$ is the final velocity vector.

Section 2.2

(1a) $\ell(s) = (t^2 - 2t, \sin(2\pi t), t) + s(2t-2, 2\pi\cos(2\pi t), 1)$ and at $t_0 = \pi$, $\ell(s) = (\pi^2 - 2\pi, \sin(2\pi^2), \pi) + s(2(\pi-1), 2\pi\cos(2\pi^2), 1)$

(1c) $\ell(s) = (3te^t, t^(2/3), t^2) + s(3e^t + 3te^t, (2/3)t^{-1/3}, 2t)$ and at $t_0 = 1$, $\ell(s) = (3e, 1, 1) + s(6e, 2/3, 2)$.

Section 2.3

(1b) $\tilde{\mathbf{r}}(s) = (\cos(s), \sin(s))$ with $s \in [0, 32\pi]$, speed

$$||\tilde{\mathbf{r}}'(s)|| = \sqrt{\sin^2(s) + \cos^2(s)} = 1$$

arc-length parametrization

(1c) $\tilde{\mathbf{r}}(s) = (s, s^2/\sqrt{2}, s^3/3)$ with $s \in [1, 10]$, speed $||\tilde{\mathbf{r}}'(s)|| = \sqrt{1 + 2s^2 + s^4} \neq 1$.

(2b) $s(t) = \int_0^t ||\mathbf{r}'(\tau)||\, d\tau = \int_0^t \sqrt{e^{2\tau} + 4e^{4\tau} + 16e^{8\tau}}\, d\tau$.

(2d) $s(t) = \int_0^t ||\mathbf{r}'(\tau)||\, d\tau = \int_0^t \sqrt{5e^{-\tau}}\, d\tau = -2\sqrt{5}e^{-t/2}$, $t = \phi(s) = -2\ln(-s/2\sqrt{5})$.

Chapter 3

Section 3.1

(1b) $T_pC = \{\alpha(2, 8, 64) \mid \alpha \in \mathbb{R}\}$

(1c) $T_pC = \{\alpha(-8\sin(8t), 8\cos(8t)) \mid \alpha \in \mathbb{R}\}$.

(3b) $\tau_1(p) = (1, 0, -\pi/2)$, $\tau_2(p) = (0, 1, -1)$,

$$T_pS = \{(\alpha, \beta, -\alpha\pi/2 - \beta) \mid \alpha, \beta \in \mathbb{R}\}$$

(3c) $\tau_1(p) = (1, 0, 0)$, $\tau_2(p) = (0, 1, 0)$, $T_pS = \{(\alpha, \beta, 0) \mid \alpha, \beta \in \mathbb{R}\}$.

(4) (a) none, (b) none, (c) none, (d) $q \in T_pS$ problem 3(c).

(6) $(r_0, \theta_0) \in (\sqrt{2}/2, \pi/4)$, $f_1(p) = (1, 0, (\partial f/\partial r)(r_0, \theta_0)) = (1, \pi/4, -1)$ and $f_2(p) = (0, 1, (\partial f/\partial\theta)(r_0, \theta_0)) = (0, 1, 0)$. The tangent space is the same as in problem 3(d). In 3(d) we have, $\tau_1(p) = (1, 0, -\sqrt{2}/2)$ and $\tau_2(p) = (0, 1, -\sqrt{2}/2)$. We show that $(1, \pi/4, 1)$ and $(0, 1, 0)$ are linear combinations of $\tau_1(p)$ and $\tau_2(p)$. The vector $(1, \pi/4, -1)$ in Cartesian coordinates is given by

$$(\sqrt{2}/2, \sqrt{2}/2, -1) = \sqrt{2}/2(1, 0, -\sqrt{2}/2) + \sqrt{2}/2(0, 1, -\sqrt{2}/2).$$

The polar coordinates path R given by $(r_0, \theta_0 + t) = (\sqrt{2}/2, \pi/4 + t)$ becomes in Cartesian coordinates the path C given by

$$\mathbf{r}(t) = (\sqrt{2}/2\cos(\pi/4+t), \sqrt{2}/2\sin(\pi/4+t))$$

and so the vector $(0,1)$ at $T_{(r_0,\theta_0)=(\sqrt{2}/2,\pi/4)}R$ is mapped to $T_{(x,y)=(\sqrt{2}/2,\sqrt{2}/2)}C$ by $\mathbf{r}'(0) = (-1/2, 1/2)$. Thus, $(0,1,0)$ is mapped to

$$(-1/2, 1/2, 0) = -1/2(1, 0, -\sqrt{2}/2) + 1/2(0, 1, -\sqrt{2}/2).$$

(7a) $\mathbf{r}'(1) = (2,1)$ so $v = 5$.

Section 3.2

(1a) $df = (1/y)\,dx - (x/y^2)\,dy$

(1c) $df = -3yz\sin(xyz)\,dx - 3xz\sin(xyz)\,dy - 3xy\sin(xyz)\,dz$.

(3a) $dt(v_j) = t_{j+1} - t_j$, (b) $T_{p_j}C = \{\alpha\mathbf{r}'(t_j) \mid \alpha \in \mathbb{R}\}$ with $\alpha = t_{j+1} - t_j$ this means W_j is in $T_{p_j}C$, (c) $dx(W_j) = x'(t_j)\,dt(v_j)$, $dy(W_j) = y'(t_j)\,dt(v_j)$.

(5b) $dr(1,-1) = 1$, $d\theta(1,-1) = -1$ and

$$dr(v_x, v_y) = \frac{1}{2}\left(\frac{1-\sqrt{3}}{2}\right) - \frac{\sqrt{3}}{2}\left(\frac{-(1+\sqrt{3})}{2}\right) = 1$$

$$d\theta(v_x, v_y) = \frac{\sqrt{3}}{2}\left(\frac{1-\sqrt{3}}{2}\right) + \frac{1}{2}\left(\frac{-(1+\sqrt{3})}{2}\right) = -1.$$

(7)

$$d\rho = \frac{x}{\sqrt{x^2+y^2+z^2}}\,dx + \frac{y}{\sqrt{x^2+y^2+z^2}}\,dy + \frac{z}{\sqrt{x^2+y^2+z^2}}\,dz.$$

$d\theta$ same as for polar coordinates.

$$\begin{aligned} d\phi &= \frac{-xz(x^2+y^2+z^2)^{-3/2}}{\sqrt{1-(z^2/(x^2+y^2+z^2))}}\,dx + \frac{-yz(x^2+y^2+z^2)^{-3/2}}{\sqrt{1-(z^2/(x^2+y^2+z^2))}}\,dy \\ &\quad + \frac{(x^2+y^2+z^2)^{-1/2} - z^2(x^2+y^2+z^2)^{-3/2}}{\sqrt{1-(z^2/(x^2+y^2+z^2))}}\,dz. \end{aligned}$$

Section 3.3

(1) (a) -11, (c) 10 , (e) $-5\pi/4\sqrt{2}$.

(2) $F(x,y) = x^2y$. Adding any constant is also a solution.

(5) $dW = \dfrac{mMg}{\sqrt{2}r^2}\left(-\cos(t) - \sin(t) - 1\right)\,dt$.

Chapter 4

Section 4.1

(2) $\int_C \omega = \displaystyle\int_0^1 -3t\,dt.$

(3) Using the parametrizations $\mathbf{r}_1(t) = (t, t^2)$ with $t \in [0,1]$, $\mathbf{r}_2(t) = (1-t)(1,1) + t(2,0) = (1+t, 1-t)$ and $\mathbf{r}_3(t) = (2(1-t), 0)$ we have

$$\int_C \omega = \int_0^1 (e^{t^3} + 2t^2 e^{t^2})\,dt + \int_0^1 (e^{1-t^2} - (1+t)e^{1-t})\,dt + \int_0^1 -2\,dt.$$

(5) $dW = xy^2\,dx + (xy+yz)\,dy + (-3z^3 + y^2)\,dz$ and $\int_C dW = \int_{C_1} dW + \int_{C_2} dW$ with

$$\int_{C_1} dW = \int_0^\pi (-\cos(t)\sin^3(t) + \sin(t)\cos(t)(\cos(t)+t) + (-3t^3 + \sin^2(t)))\,dt$$

and using $\mathbf{r}_1(t) = (1-t)(-1,0,\pi) + t(0,-1,0)$ with $t \in [0,1]$ then

$$\int_{C_2} dW = \int_0^1 (t^3 - 2\pi t^2 + (\pi-1)t + 3\pi^4(1-t)^3)\,dt.$$

(6a) Substitute $\mathbf{r}(t)$ in the equation.

(6b) $dW = -9.8\,dz$ so $\int_C dW = \int_0^2 -9.8\,dt = -19.6.$

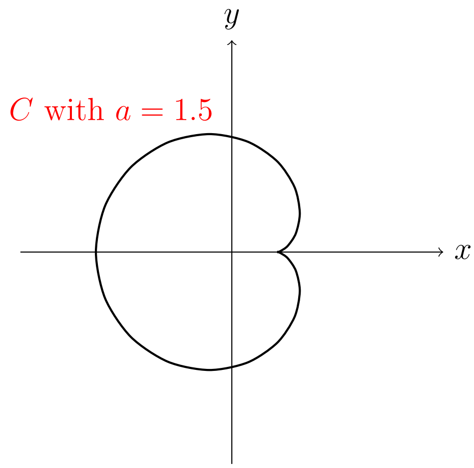
(6c) $dW = (-3z-1)\,dx$ so

$$\int_C dW = \int_0^2 (-3t-1)(2\pi\sqrt{2-t}\sin(2\pi t) - \frac{1}{2\sqrt{2-t}}\cos(2\pi t))\,dt.$$

Section 4.2.1

(1a) The paramterization is injective.

(3) $\displaystyle\int_C ds = \int_c^d \|r'(t)\| dt = \frac{1}{3}(2\sqrt{2} - 1).$



(5) $\int_0^{2\pi} 2a\sqrt{2(1-\cos(t))}dt = 16a$

(7) $\int_0^2 \sqrt{a^2 4\pi^2+1}dt = 2\sqrt{a^2 4\pi^2+1}$.

Section 4.2.2

(1) $\int_C x\,ds = \frac{1}{12}(5^{3/2}-1)$

(4) $\int_C \sqrt{x^2+y^2}\,ds = -\frac{\sqrt{2}}{2}(e^{-2\pi}-1)$

(5) $\overline{x} = \frac{1}{m}\int_C x\rho\,ds = \frac{1}{6\pi m}\sqrt{9+4\pi^2} = \overline{y} = \frac{1}{m}\int_C y\rho\,ds$,

$$\overline{z} = \int_C z\rho\,ds = \int_0^1 3t\cos(2\pi t)\sin(2\pi t)\sqrt{9+4\pi^2}\,dt = -\frac{3}{8m\pi}\sqrt{9+4\pi^2}$$

where $m = \sqrt{9+4\pi^2}/4\pi$.

Section 4.2.3

(1b) $\kappa = \frac{10}{(1+100t^2)^{\frac{3}{2}}}$.

(1d) $\kappa = \frac{\sqrt{2}}{\sqrt{2}} = 1$.

Section 4.3

(1b) $\int_1^3 t + \frac{\sqrt{2}}{2}\cos(\sqrt{2}t) + \ln(t)dt = \ln(27) + 2 + \frac{1}{2}\sin(3\sqrt{2}) - \frac{1}{2}\sin(\sqrt{2})$

(5) The potential function for $F(\mathbf{x})$ is $f(\mathbf{x}) = -\epsilon qQ/||x||$. The work done is $\displaystyle\int F\cdot d\mathbf{r} = f(0,10^{-8},0) - f(10^{012},0,0)$.

Chapter 5

Section 5.1

(1b) Let $A = \begin{matrix} a_1 & a_2 \\ a_3 & a_4 \end{matrix}$, then $\det(A) = a_1a_4 - a_2a_3$.

$$\operatorname{graph}(g) = \{(a_1, a_2, a_3, a_4, a_1a_4 - a_2a_3) \mid (a_1, a_2, a_3, a_4) \in \mathbb{R}^4\}.$$

(2a) The level sets $\cos x \cos y = c$ are empty for $|c| > 1$.

Section 5.2

(1b) $\lim\limits_{(x,y,z)\to(0,0,0)} \frac{e^z}{1+x^2+y^2} = 1$

(1d)

$$\lim_{(x,y)\to(-1,1)} \begin{pmatrix} 3 & 7 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 5 \\ 11 \end{pmatrix}$$

(5f) $\lim\limits_{(\theta,\phi)\to(\pi,\frac{\pi}{2})} (2\sin\phi\cos\theta, 2\sin\phi\sin\theta, 2\cos\phi = (-2, 0, 0)$

(2a) Set $y = mx$. The value of the limit depends on the line of approach towards the origin and so the limit does not exist.

(2b) The limit depends on the line of approach towards the origin and so the limit does not exist.

(2d) The value of the limit depends on two lines of approach towards the origin. The limit does not exist.

(3b) Since $\sqrt{(x-1)^2 + (y+1)^2} < \delta$, then $|x - 1\| < \delta$, $\|y + 1\| < \delta$ and let $\delta = \sqrt{\epsilon}$. Thus, $\|xy + x - y - 1\| = \|(x-1)(y+1)\| < \delta^2$. Therefore $\|xy + x - y - 1\| < \delta^2 = (\sqrt{\epsilon})^2 = \epsilon$.

(3c) We know $\|(x, y)\| = \sqrt{x^2 + y^2} < \delta$ and set $\delta = \sqrt{\epsilon}$ then

$$\|\frac{(x, y)||(x, y)||}{1 + ||(x, y)||}\| < ||(x, y)||^2 < \delta^2 = \epsilon.$$

(5b)

$$\lim_{(x,y)\to(0,0)} \begin{pmatrix} 3 & x \\ 2 - y & x \end{pmatrix}^{-1} = \lim_{(x,y)\to(0,0)} \begin{pmatrix} \frac{1}{1+y} & \frac{-1}{1+y} \\ \frac{y-2}{x(1+y)} & \frac{3}{x(1+y)} \end{pmatrix}.$$

Because two entries in the matrix are undefined at $(x, y) = (0, 0)$, the function cannot be continuous at $(0, 0)$.

(6a) Let $f = (f_1, f_2)$ with $f_1 = \frac{x^2}{3x+4xy} = \frac{x}{3+4y}$, $f_2 = e^{\frac{-1}{xy}}$. Because $f_2(x,y)$ is not continuous at $(0,0)$, f is not continuous at $(0,0)$.

Section 5.3

(1b) Let $\epsilon > 0$ and suppose $(x^2+y^2)^{\frac{1}{2}} < \delta$. Set $\delta = \sqrt[5]{\epsilon}$, therefore $|(x^2+y^2)^{\frac{5}{2}}| < \delta^5 = (\sqrt[5]{\epsilon})^5 = \epsilon$. $L = (0,0)$ gives the best linear approximation.

(1d) The function is not continuous because $y - \frac{x}{|x|}$ does not approach 0.

(2c)

$$Jf(x,y) = \begin{pmatrix} y^2 & 2xy \\ ye^{xy} & xe^{xy} \\ \frac{1}{2\sqrt{x-y}} & \frac{-1}{2\sqrt{x-y}} \end{pmatrix}$$

(2d)

$$Jf(x_1,x_2,x_3,x_4) = \begin{pmatrix} 2x_1 & x_3 & x_2 & 0 \\ x_2 & x_1 + x_4 & 0 & x_2 \\ x_3 & x_4 & x_1 & x_2 \\ 0 & x_3 & x_2 & 2x_4 \end{pmatrix}$$

Section 5.3.1

(1)

$$Jf(x,y,z) = \begin{pmatrix} 3\cos(yz) & -3xz\sin(yz) & -3xy\sin(yz) \end{pmatrix}$$

f and all its partial derivatives are continuous for all $(x,y,z) \in \mathbb{R}^3$. f is differentiable for all $(x,y,z) \in \mathbb{R}^3$

(2) $\frac{\partial f_1}{\partial x}$ and $\frac{\partial f_1}{\partial y}$ are both continuous and defined at $(0,0)$. However, $\frac{\partial f_2}{\partial x}$ and $\frac{\partial f_2}{\partial y}$ are neither continuous nor defined at $(0,0)$. The information is not sufficient to decide whether $Df(0,0)$ exists.

(5b) (i) $U = \{(x,y,z,u) \in \mathbb{R}^4 \mid xy > 0\}$, (ii)

$$Jf(x,y,z,u) = \begin{pmatrix} \frac{y}{2\sqrt{xy}} & \frac{x}{2\sqrt{xy}} & 4z & 0 \\ ue^{yz} & uxze^{yz} & uxye^{yz} & xe^{yz} \\ -2x\sin(x^2-y^2) & 2y\sin(x^2-y^2) & 0 & 0 \end{pmatrix}.$$

(iii) The Jacobian is not defined at all the points in the domain.

Section 5.4

(1b) $T_pS = \{(\alpha, \beta, \alpha - 2\beta, 7\alpha + 6\beta) | \alpha, \beta \in \mathbb{R}\}$

(1c) Take $\alpha = 2$, and $\beta = 1$. Then, $w = (2, 1, 0, 20) \in T_pS$

(2a)

$$Df(x, y, z) = \begin{pmatrix} 2xy + yz & x^2 - 2y + xz & xy \\ \\ 3y + yz^2 & 3x + xz^2 & 2xy \end{pmatrix}$$

Df is the zero matrix if $(x, y, z) = (0, 0, 0)$.

(2b)

$$\begin{aligned} T_pS &= \{(\alpha, \beta, \gamma, \alpha y(2x + z) + \beta(x^2 - 2y + xz) + \gamma(xy), \\ &\quad \alpha(3y + yz^2) + \beta(3x + xz^2) + \gamma(2xy) | \alpha, \beta, \gamma \in \mathbb{R}\} \end{aligned}$$

Section 5.5

(1b)

$$D(g \circ f)(x, y) = \begin{pmatrix} 6x^2y^2 - y^4 & 2x^2y - 4xy^3 \\ 2xy^4 & 4x^2y^3 \\ y^2 - 8x[2x^2 - y^2] & 2xy + 4y[2x^2 - y^2] \end{pmatrix}$$

(1d)

$$D(g \circ f)(t) = \begin{pmatrix} -2\sin(t)[\sin(t) + t\cos(t)] \\ -2[\sin^2(t) + 4t\sin(t)\cos(t) + 2t^2\cos^2(t)] \end{pmatrix}$$

(3a)

$$\begin{aligned} DF(x, y, z) &= Df(h(x), g(x, y), z) \begin{pmatrix} \frac{\partial h}{\partial x} & 0 & 0 \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= (2x\cos x + yz, 3y^2z + xz, y). \end{aligned}$$

Section 5.6

(3a)

$$D^2f = e^{xz} \begin{pmatrix} xy(1 + xz) + yz & 1 + xz & 2xy + x^2yz \\ 1 + xz & 0 & x^2 \\ 2xy + x^2yz & x^2 & x^3y \end{pmatrix}.$$

Section 5.7

(1a) $T_2 f(x,y) = -1 - 4x + 3y - 6x^2 + 12xy - 3y^2$.

(3b) $\frac{1}{3!}e(x-1)^3 + \frac{3e}{2!}(x-1)^2(y-1) + \frac{3e}{2!}(x-1)(y-1)^2 + \frac{1}{3!}(y-1)^3$.

Chapter 6

Section 6.1

(1a) Critical points $(\pm 1, 0)$ are saddles.

(1c) Critical point $(-4, 2, -2)$ is a saddle.

(1f) Critical point at $(0,0,0,0)$ is a local minimum.

Section 6.2

(1a) $\Phi(r,\theta) = (r\cos\theta, r\sin\theta)$ with $D = \{(r,\theta) \mid 1 \le r \le 2,\ 0 \le \theta \le 3\pi/2\}$.

(1b) $\Phi(x,y) = (x,y)$ with $D = \{(x,y) \mid 0 \le y \le 1,\ 0 \le x \le y+1\}$.

(3a) $\Phi(r,\theta,z) = (r\cos\theta, r\sin\theta, z)$ with $R = \{(r,\theta,z) \mid 0 \le r \le \sqrt{z_{int}},\ 0 \le \theta \le 2\pi,\ r^2 \le z \le \sqrt{1-r^2}, z_{int} = (-1+\sqrt{5})/2\}$.

(3b) $\Phi(r,y,\theta) = \{r\cos\theta, y, r\sin\theta)$ with $R = \{(r,y,\theta) \mid 0 \le r \le 1,\ -\pi/2 \le \theta \le \pi/2,\ -1 \le y \le 1\}$.

(5a) $\Phi(r,\theta) = (r\cos\theta, r\sin\theta, 4-r^2)$ with $D = \{(r,\theta) \mid 0 \le r \le 2,\ 0 \le \theta \le 2\pi\}$.

(5b) $\Phi(x,y) = (x, y, x^2 - 2y^2)$ with $D = \{(x,y) \mid x \ge 0,\ y \in \mathbb{R}\}$.

Section 6.4

(1a) $\Delta f = 0$ so f is harmonic.

(1d) $\Delta f = e^{yz}(4 + (x^2+w^2)(z^2+y^2))$.

(3a) $\operatorname{div} F = 4x^2 + 4y^2$. $\operatorname{curl} F = (0,0,2)$.

(3e) $\operatorname{div} F = y + y^2$. $\operatorname{curl} F = (2z(y-x), 0, z^2 - x)$.

Section 6.5

(1a) ω is not exact, the condition yields $(-y/x^2) - (x/y^2) \neq 0$.

(1c) ω is exact.

Chapter 7

Section 7.1

(1a) Negatively oriented. Area is 8.

(2b) Area is $\sqrt{176}$.

(4a) The volume is 4.

Section 7.2

(1a) The integral is 192.

(1c) The integral is $\frac{1}{2}(e-1)$.

(3a) Type II.

(3d) Type III.

(4b) The integral is 64/9.

(6) The density is $\rho(x,y) = 50f(x,y)$ kg/m^2.

Section 7.3

(1) $\{y=0\}$: $\mathbf{r}_1(t) = (0,-t)$; $\{y=x-2\}$: $\mathbf{r}_2(t) = (t,t-2)$, $\{x=0\}$: $\mathbf{r}_3(t) = (-t,0)$, in each case $t \in (0,2)$.

(3) $\mathbf{r}_1(t) = (\cos t, -2\sin t)$ and $\mathbf{r}_2(t) = (\sqrt{2}/2\cos t, (1/2)\sin t)$ with $t \in (0,2\pi)$.

(5) $\oint_C \omega = 6\pi a^2$.

(7) $\oint_C F \cdot \hat{n}\, ds = 17/6$.

Section 7.4

(1) The integral is 5.

(3b) The integral is $\frac{1}{4}(e^2-5)$.

(4a) Type I.

(4d) Type II.

(7) Use the fact the intersection curve in (y,z) is $(z+\frac{1}{2})^2 + (y-1)^2 = (5/2)^2$.

Chapter 8

Section 8.1

(1) $\omega_1 \wedge \omega_2 =$

$$\begin{aligned}
&(-5x_1x_4 + 6x_3^2x_5^2)\,dx_1 \wedge dx_2 - (5x_1x_3^3 + 2x_2^2x_3x_5)\,dx_1 \wedge dx_3 \\
&+(5x_1^2x_4x_5 + 2x_3^3x_4x_5)\,dx_1 \wedge dx_4 + 5x_1^2x_2^5\,dx_1 \wedge dx_5 - (3x_3^4x_5 + x_2^2x_4)\,dx_2 \wedge dx_3 \\
&+(3x_1x_3x_4x_5^2 + x_3^2x_4^2)\,dx_2 \wedge dx_4 + 3x_1x_3x_5^3\,dx_2 \wedge dx_5 + (x_3^5x_4 - x_1x_2^2x_4x_5)\,dx_3 \wedge dx_4 \\
&-x_1x_2^2x_5^2\,dx_3 \wedge dx_5 + x_1x_3^2x_4x_5\,dx_4 \wedge dx_5.
\end{aligned}$$

(3) $dV_{(x,y,z)} = 7dV_{(u,v,w)}$.

Section 8.2

(1) $\omega_1 \wedge \omega_2 = (x_3 - x_2)\,dx_1 \wedge dx_2 \wedge dx_3$.

(5a) $\omega(u,v) = \displaystyle\sum_{i=1}^{n} v_iw_i - u_iz_i$.

Section 8.3

(2) $d\omega = -y\,dy \wedge dz + (xy-1)\,dz \wedge dx - xz\,dx \wedge dy$.

(6a) $\mathbb{H}\omega(x,y) = x^2y/4$.

(6c) $\mathbb{H}\omega(x,y) = \dfrac{e^{xy}-1}{2y} + \dfrac{x}{y^2}((y^2 - 2y + 2)e^y - 2)$.

Chapter 9

Section 9.1

(1a) $\Phi^*\omega(u,v) = u^3\,du \wedge dv$

(1c) $\Phi^*\omega(u,v) = (3-u)u\,du \wedge dv$

(1e) $\Phi^*\omega(u,v) = -8\sin v\,du \wedge dv$

(1g) $\Phi^*\omega(u,v,w) = 6(u^2+v^2+w^2)\,du \wedge dv \wedge dw$

Section 9.2

(1a) $\displaystyle\int_{-\sqrt{2}}^{\sqrt{2}} \left(\int_{-\sqrt{4-2x^2}}^{\sqrt{4-2x^2}} \sqrt{1+16x^2+4y^2}\, dy \right) dx$

(1c) $\displaystyle\int_{-1}^{1} \left(\int_{0}^{1} u\sqrt{1+4v^2+v^4}\, dv \right) du.$

(1f) $\displaystyle\int_{0}^{2} \left(\int_{0}^{2\pi} \sqrt{2}r\, d\theta \right) dr.$

Section 9.3

(1a) $\displaystyle\int_{0}^{1} \left(\int_{-1}^{1} 2(u+v)u^3v(-2u^2) + 3(uv)^2(u+v)(2u) + u^2(u+v)^2(u-v)\, dv \right) du$

(1c) $\displaystyle\int_{0}^{2} \left(\int_{0}^{2\pi} (1+\cos\theta) \begin{vmatrix} \cos\theta & 0 \\ 0 & 1 \end{vmatrix} + \sin\theta \begin{vmatrix} 0 & 1 \\ -\sin\theta & 0 \end{vmatrix} dx \right) dz$

(1d) $\displaystyle\int_{0}^{1} \left(\int_{0}^{1} xe^{yz} \begin{vmatrix} 2(x-1) & 0 \\ 0 & 1 \end{vmatrix} + e^{xz} \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + z^2 \begin{vmatrix} 2(x-1) & 0 \\ 1 & 0 \end{vmatrix} dx \right) dz$

Section 9.4

(1) $\vec{n} = (2\cos\theta, 2\sin\theta, 0)$.

(4) $\vec{n} = (\sin(a\theta), -\cos(a\theta), a\rho)$.

Chapter 10

Section 10.1

(1) $\partial S = \{(2+\cos v), 0, -\sin v) \mid 0 \le v \le 2\pi\} \cup \{-(2+\cos v, 0, -\sin v) \mid 0 \le v \le 2\pi\}$

(3) ∂S is the union of $\{(2-t, t, 0) \mid 0 \le t \le 2\} \cup \{(0, 2-t, t) \mid 0 \le t \le 2\} \cup \{(t, 0, 2-t) \mid 0 \le t \le 2\}$ and with $\{(1+4\cos t, \frac{1}{2} - \frac{1}{4}\sin t, \frac{1}{2} - \frac{1}{4}\cos t + \frac{1}{4}\sin t) \mid 0 \le t \le 2\pi\}$

(5) $(0,-1,0)$, $(1,1,0)$ and $(\cos\theta, 0, \sin\theta)$

Section 10.2

(1a) $\displaystyle\oint_C \omega = 2\pi$

(3) $d\omega_1 = dx \wedge dy = -d\omega_2$.

(4) $\partial S = \emptyset$ implies the integral is 0.

(5b) $\displaystyle\iint_S d\omega = -6a^2\pi.$

(5d) $\displaystyle\iiint_E d\omega = \pi/4.$

(6b) $\displaystyle\iint_S \omega_F = \int_0^1 \left(\int_0^{1-x} (1-3y)\,dy\right)\,dx.$

(6c) $\displaystyle\iint_S \omega_F = \int_0^{\pi/2} \left(\int_0^{\pi/2} 8\cos^2\theta \sin^3\phi\,d\phi\right)\,d\theta.$

Section 10.3

(1b) $\displaystyle\oint_C F \cdot d\mathbf{r} = -3/2$

(3) $\displaystyle\oiint_{\partial E} F \cdot \hat{n}\,dS = 6\pi.$