

# Contents

Preface	vii
<b>1 Elliptic curves and equations</b>	1
1.1 A general overview .....	1
1.2 Elliptic curves and the Mordell–Weil Theorem .....	5
<b>2 Heights</b>	9
2.1 Notations and facts .....	9
2.2 Absolute values in a number field .....	11
2.3 Heights: Absolute and logarithmic .....	13
2.4 A formula for the absolute logarithmic height .....	18
2.5 Heights of points on an elliptic curve .....	20
2.6 The canonical height .....	23
<b>3 Weierstrass equations over <math>\mathbb{C}</math> and <math>\mathbb{R}</math></b>	29
3.1 The Weierstrass $\wp$ function .....	29
3.2 The Weierstrass equation .....	31
3.3 $\psi : E(\mathbb{C}) \mapsto \mathbb{C}/\Lambda$ .....	33
3.4 Weierstrass equations with real coefficients .....	36
3.4.1 $\Delta > 0$ .....	38
3.4.2 $\Delta < 0$ .....	40
3.4.3 Explicit expressions for the periods .....	41
3.4.4 Computing $\omega_1$ and $\omega_2$ in practice .....	44
3.5 $\psi : E(\mathbb{R}) \mapsto \mathbb{C}/\Lambda$ and $\iota : E(\mathbb{R}) \longrightarrow \mathbb{R}/\mathbb{Z}\omega_1$ .....	47
<b>4 The elliptic logarithm method</b>	54
<b>5 Linear form for the Weierstrass equation</b>	57
<b>6 Linear form for the quartic equation</b>	60
<b>7 Linear form for simultaneous Pell equations</b>	69

<b>8 Linear form for the general elliptic equation</b>	78
8.1 A short Weierstrass model . . . . .	78
8.2 Puiseux series . . . . .	80
8.3 Large solutions . . . . .	84
8.4 The elliptic integrals . . . . .	86
8.5 Computing in practice $B_1$ of Proposition 8.3.2 . . . . .	89
8.6 Computing in practice $B_2$ and $c_9$ of Proposition 8.4.2 . . . . .	91
8.7 The linear form $L(P)$ and its upper bound . . . . .	94
<b>9 Bound for the coefficients of the linear form</b>	98
9.1 Lower bound for linear forms in elliptic logarithms . . . . .	98
9.2 Computational remarks . . . . .	105
9.3 Weierstrass equation example . . . . .	107
9.4 Quartic equation example . . . . .	110
9.5 Simultaneous Pell equations example . . . . .	114
9.6 General elliptic equation: A quintic example . . . . .	118
<b>10 Reducing the bound obtained in Chapter 9</b>	121
10.1 Reduction using the LLL-algorithm . . . . .	122
10.2 Examples . . . . .	125
10.2.1 Weierstrass equation . . . . .	125
10.2.2 Quartic equation . . . . .	127
10.2.3 System of simultaneous Pell equations . . . . .	131
10.2.4 General elliptic equation: A quintic example . . . . .	134
<b>11 <math>S</math>-integer solutions of Weierstrass equations</b>	137
11.1 The formal group of $C$ and $p$ -adic elliptic logarithms . . . . .	137
11.2 Points with coordinates in $\mathbb{Z}_S$ . . . . .	144
11.3 The $p$ -adic reduction . . . . .	154
11.4 Example . . . . .	158
List of symbols	165
Bibliography	173
Index	177