

INTRODUCTION

These constitute lecture notes of graduate courses given by the authors at Indiana University (1985-86) and the University of Chicago (1986-87), respectively.

In recent years there has been considerable progress in some of the questions related to the Navier-Stokes equations and their relation to finite-dimensional phenomena. For instance, the upper bound for the dimension of the universal attractor for 2D Navier-Stokes equations has been lowered from an estimate of the type $G^2 \exp G^4$ to an estimate of the type $G^{2/3} \log G^{1/3}$, where G is a nondimensional number, typically of the order of 100-1000. This most recent estimate can be understood in terms of the Kraichnan length and seems to be optimal for general body forces.

We try in these lecture notes to give an almost self-contained treatment of the topics we discuss. These notes are by no means an exhaustive treatise on the subject of Navier-Stokes equations. It has been our choice to present results using the most elementary techniques available. Thus, for instance, the regularity theory for the Stokes system (Chapter 3) is an adaptation of the classical L^2 regularity theory for a single elliptic equation of [A1]; our adaptation is inspired from [G]. Another example of our desire to illustrate the general results, while avoiding excessive technicalities, is the way we describe the asymptotic behavior of the eigenvalues of the Stokes operator (Chapter 4). For general bounded domains we provide a lower

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bound, using essentially elementary means. The lower bounds are all we really need later on. For completeness we give the elementary proof of the exact asymptotic behavior in the periodic case. The same asymptotic behavior for general domains, while true, would have required considerably more effort to describe. Questions regarding the notions of weak and strong solutions and their relations to classical solutions are studied in some detail. We prove that strong solutions are as smooth as the data permit; thus, loss of regularity can only occur if the solution ceases to be strong. We then show how, if there is an initial datum leading to loss of regularity in infinite time, there exists another one which leads to loss of regularity in finite time. We give the argument of Scheffer and Leray estimating the Hausdorff dimension of the singular times of a weak solution to be not more than $1/2$. A simple argument is used to prove that, in the absence of boundaries, the vanishing viscosity limit of the Navier-Stokes equation is the Euler equation for incompressible fluids. The same technique can be used to show that as long as the solution to the incompressible Euler equation is smooth, solutions to small viscosity Navier-Stokes equations with the same initial data remain smooth. We provide a proof of time analyticity and consequent backward uniqueness for the initial value problem for the Navier-Stokes equations.

The importance of contact element transport is emphasized in the last chapter. We study first (Chapter 13) the decay of volume elements and give optimal lower bounds for the dimension at which this process starts. These bounds use inequalities of Lieb-Thirring. The construction of the universal attractor for 2D Navier-Stokes is given in Chapter 14. The fractal and Hausdorff dimensions of the universal attractor are

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estimated making the connection with the Kaplan-Yorke formula involving global Lyapunov exponents. Upper bounds for the fractal dimension of bounded invariant sets for 3D Navier-Stokes are given also. The final chapter deals with the concept of inertial manifolds for an artificial viscosity perturbation of the Navier-Stokes equation. The spectral blocking property and consequent cone invariance are illustrated in detail. These are ideas of independent interest and were successfully used to construct inertial manifolds for several physically significant equations. As of this writing the question of the existence of inertial manifolds for the Navier-Stokes equations remains open.

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