Existence of Solutions for Stochastic Differential Equations under G-Brownian Motion with Discontinuous Coefficients

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The existence theory for the vector valued stochastic differential equations under G-Brownian motion (G-SDEs) of the type $X_t = X_0 + \int_0^t f(v,X_v)\,\mathrm{d}v + \int_0^t g(v,X_v)\,\mathrm{d}\langle B\rangle_v + \int_0^t h(v,X_v)\,\mathrm{d}B_v, t \in [0,T]$, with first two discontinuous coefficients is established. It is shown that the G-SDEs have more than one solution if the coefficient g or the coefficients f and g simultaneously, are discontinuous functions. The upper and lower solutions method is used and examples are given to explain the theory and its importance.

Key words: Stochastic Differential Equations; G-Brownian Motion; Discontinuous Coefficients; Existence; Upper and Lower Solutions.
Mathematics Subject Classification 2000: 60H10, 60H20

1. Introduction

Measuring risk in finance is an important problem, and in the last twelve years, many kind of risk measures have been presented by various authors such as the coherent risk measures, the convex risk measures, and the law invariant risk measures [1 – 3]. In the recent time, the notion of a sublinear expectation is introduced by S. Peng for measuring risk in finance under volatile uncertainty [4]. He developed the notions of G-Brownian motion and the theory of stochastic calculus under sublinear expectation [4 – 6]. Since under volatile uncertainty the corresponding uncertain probabilities are singular from each other, they produce a serious problem for the related path analysis. The traditional classical techniques provide a limited understanding of these types of problems. For example path-dependent derivatives under a classical probability space. For such type of problems, G-Brownian motion provides a powerful tool and can be easily treated.

Under his stochastic calculus, Peng established the existence and uniqueness of solutions for the stochastic differential equations under G-Brownian motion (G-SDEs) with Lipschitz continuous coefficients [4, 5]. Motivated from the importance of discontinuous functions, Faizullah and Piao established the existence of solutions for the G-SDEs with a discontinuous drift coefficient [7]. Now here, we develop the existence theory when the coefficient g or the coefficients f and g simultaneously are discontinuous functions such as in the following scalar G-SDEs:

$$\mathrm{d}X_t = \mathrm{d}t + H(X_t)\,\mathrm{d}\langle B\rangle_t + \mathrm{d}B_t,$$
$$\mathrm{d}X_t = H(X_t)\,\mathrm{d}t + \{X_t\}\,\mathrm{d}\langle B\rangle_t + \mathrm{d}B_t,$$

where the unit step function or Heaviside function $H : \mathbb{R} \to \mathbb{R}$ is defined by [8, 9]

$$H(x) = \begin{cases} 0, & \text{if } x < 0; \\ 1, & \text{if } x \geq 0. \end{cases}$$

This is an important function in science and is considered to be a fundamental function in engineering; for example, the switching process of voltage in an electrical circuit is mathematically described by the unit step function and arises in many discontinuous ordinary differential equations [10]. The sawtooth function or fractional part function $\{x\} : \mathbb{R} \to [0,1)$ has discontinuities at the integers and is defined by

$$\{x\} = x - \lfloor x \rfloor, \quad x \in \mathbb{R},$$

where $\lfloor x \rfloor$ is the floor function [11]. The importance of this function is clear from the sawtooth waves which are used in music and computer graphics. Moreover, the impressed voltage on a circuit could be represented by the sawtooth function [12].

In this article, the following stochastic differential equation under G-Brownian motion is studied:

$$X_t = X_0 + \int_0^t f(v,X_v)\,\mathrm{d}v + \int_0^t g(v,X_v)\,\mathrm{d}\langle B\rangle_v + \int_0^t h(v,X_v)\,\mathrm{d}B_v\,,\ \ t\in[0,T]\,, \tag{1}$$

where $X_0 \in \mathbb{R}^n$ is a given constant initial condition and $(\langle B\rangle_t)_{t\geq 0}$ is the quadratic variation process of the G-Brownian motion $(B_t)_{t\geq 0}$. All the coefficients $f(t,x)$, $g(t,x)$, and $h(t,x)$ are in the space $M_G^2(0,T;\mathbb{R}^n)$ [6]. A process X_t belonging to the mentioned space and satisfying the G-SDE (1) is said to be its solution. It is assumed that $f(t,x)$ and $g(t,x)$ are discontinuous functions whereas $h(t,x)$ is Lipschitz continuous for all $x\in\mathbb{R}^n$.

This paper is organized in the following manner. In Section 2, some basic notions and definitions are given. In Section 3, the upper and lower solutions method for the stochastic differential equations under G-Brownian motion is introduced. In Section 4, the comparison theorem is established, while the existence of solutions for the G-SDEs with simultaneous discontinuous coefficients f and g is developed in Section 5. The proof is given in the appendix.

2. Preliminaries

The book [6] and papers [4, 5, 13–16] are good references for the material of this section. Let Ω be a (non-empty) basic space and $\mathcal{H}$ be a linear space of real valued functions defined on Ω such that any arbitrary constant $c\in\mathcal{H}$ and if $X\in\mathcal{H}$ then $|X|\in\mathcal{H}$. We consider that $\mathcal{H}$ is the space of random variables.

Definition 1. (Sublinear Expectation). A functional $\mathbb{E}:\mathcal{H}\to\mathbb{R}$ is called sublinear expectation, if $\forall X,Y\in\mathcal{H}$, $c\in\mathbb{R}$, and $\lambda\geq 0$ it satisfies the following properties:

(i) (Monotonicity): If $X\geq Y$ then $\mathbb{E}[X]\geq\mathbb{E}[Y]$.
(ii) (Constant Preserving): $\mathbb{E}[c]=c$.
(iii) (Subadditivity): $\mathbb{E}[X+Y]\leq\mathbb{E}[X]+\mathbb{E}[Y]$ or $\mathbb{E}[X]-\mathbb{E}[Y]\leq\mathbb{E}[X-Y]$.
(iv) (Positive Homogeneity): $\mathbb{E}[\lambda X]=\lambda\mathbb{E}[X]$.

The triple $(\Omega,\mathcal{H},\mathbb{E})$ is called a sublinear expectation space.

Consider the space of random variables $\mathcal{H}$ such that if $X_1,X_2,\dots,X_n\in\mathcal{H}$ then $\varphi(X_1,X_2,\dots,X_n)\in\mathcal{H}$ for each $\varphi\in\mathbb{C}_{\text{l.Lip}}(\mathbb{R}^n)$, where $\mathbb{C}_{\text{l.Lip}}(\mathbb{R}^n)$ is the space of linear functions φ defined as

$$\mathbb{C}_{\text{l.Lip}}(\mathbb{R}^n) = \{\varphi:\mathbb{R}^n\to\mathbb{R}\,|\,\exists C\in\mathbb{R}^+,\ m\in\mathbb{N}\text{ s.t. } |\varphi(x)-\varphi(y)|\leq C(1+|x|^m+|y|^m)|x-y|\},$$

for $x,y\in\mathbb{R}^n$.

Definition 2. (Independence). An n-dimensional random vector $Y=(Y_1,Y_2,\dots,Y_n)$ is said to be independent from an m-dimensional random vector $X=(X_1,X_2,\dots,X_m)$ if

$$\mathbb{E}[\varphi(X,Y)] = \mathbb{E}[\mathbb{E}[\varphi(x,Y)]_{x=X}]\,,\quad \forall\,\varphi\in\mathbb{C}_{\text{l.Lip}}(\mathbb{R}^m\times\mathbb{R}^n)\,.$$

Definition 3. (Identical Distribution). Two n-dimensional random vectors X and $\hat{X}$ defined, respectively, on the sublinear expectation spaces $(\Omega,\mathcal{H},\mathbb{E})$ and $(\hat\Omega,\hat{\mathcal{H}},\hat{\mathbb{E}})$ are said to be identically distributed, denoted by $X\sim\hat{X}$ or $X=^d\hat{X}$, if

$$\mathbb{E}[\varphi(X)] = \hat{\mathbb{E}}[\varphi(\hat{X})]\quad \forall\,\varphi\in\mathbb{C}_{\text{l.Lip}}(\mathbb{R}^n)\,.$$

Definition 4. (G-Normal Distribution). Let $(\Omega,\mathcal{H},\mathbb{E})$ be a sublinear expectation space and $X\in\mathcal{H}$ with

$$\bar\sigma^2=\mathbb{E}[X^2],\quad \underline{\sigma}^2=-\mathbb{E}[-X^2]\,.$$

Then X is said to be G-distributed or $\mathcal{N}(0;[\bar\sigma^2,\underline{\sigma}^2])$-distributed, if $\forall\,a,b\geq 0$, we have

$$aX+bY\sim\sqrt{a^2+b^2}X$$

for each $Y\in\mathcal{H}$ which is independent to X and $Y\sim X$.

G-Expectation and G-Brownian Motion. Let $\Omega=C_0(\mathbb{R}^+)$, that is, the space of all $\mathbb{R}$-valued continuous paths $(w_t)_{t\in\mathbb{R}^+}$ with $w_0=0$ equipped with the distance

$$\rho(w^1,w^2)=\sum_{k=1}^{\infty}\frac{1}{2^k}\left(\max_{t\in[0,k]}|w_t^1-w_t^2|\wedge 1\right),$$

and consider the canonical process $B_t(w)=w_t$ for $t\in[0,\infty)$, $w\in\Omega$, then for each fixed $T\in[0,\infty)$, we have

$$\mathrm{L_{ip}}(\Omega_T)=\{\varphi(B_{t_1},B_{t_2},\dots,B_{t_n}):\ t_1,\dots,t_n\in[0,T]\,,\varphi\in\mathbb{C}_{\text{l.Lip}}(\mathbb{R}^n)\,,n\in\mathbb{N}\}\,,$$

where $\mathrm{L_{ip}}(\Omega_t)\subseteq\mathrm{L_{ip}}(\Omega_T)$ for $t\leq T$ and $\mathrm{L_{ip}}(\Omega)=\cup_{m=1}^{\infty}\mathrm{L_{ip}}(\Omega_m)$.

Consider a sequence $\{\xi_i\}_{i=1}^{\infty}$ of n-dimensional random vectors on a sublinear expectation space $(\hat{\Omega}, \hat{\mathcal{H}}_p, \hat{\mathbb{E}})$ such that ξ_{i+1} is independent of $(\xi_1, \xi_2, \ldots, \xi_i)$ for each $i = 1, 2, \ldots, n-1$, and ξ_i is G-normally distributed. Then a sublinear expectation $\mathbb{E}[.]$ defined on $\mathrm{L_{ip}}(\Omega)$ is introduced as follows.

For $0 = t_0 < t_1 < \ldots < t_n < \infty$, $\varphi \in \mathbb{C}_{\mathrm{l.Lip}}(\mathbb{R}^n)$ and each

$$X = \varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}) \in \mathrm{L_{ip}}(\Omega),$$
$$\mathbb{E}[\varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}})]$$
$$= \hat{\mathbb{E}}[\varphi(\sqrt{t_1 - t_0}\xi_1, \ldots, \sqrt{t_n - t_{n-1}}\xi_n)].$$

Definition 5. (G-Brownian Motion). The sublinear expectation $\mathbb{E} : \mathrm{L_{ip}}(\Omega) \to \mathbb{R}$ defined above is called a G-expectation, and the corresponding canonical process $(B_t)_{t\geq 0}$ is called a G-Brownian motion.

The completion of $\mathrm{L_{ip}}(\Omega)$ under the norm $\|X\|_p = (\mathbb{E}[|X|^p])^{1/p}$ for $p \geq 1$ is denoted by $L^p_{\mathrm{G}}(\Omega)$ and $L^p_{\mathrm{G}}(\Omega_t) \subseteq L^p_{\mathrm{G}}(\Omega_T) \subseteq L^p_{\mathrm{G}}(\Omega)$ for $0 \leq t \leq T < \infty$. The filtration generated by the canonical process $(B_t)_{t\geq 0}$ is denoted by $\mathcal{F}_t = \sigma\{B_s, 0 \leq s \leq t\}$, $\mathcal{F} = \{\mathcal{F}_t\}_{t\geq 0}$.

Itô's Integral of G-Brownian Motion. For any $T \in \mathbb{R}^+$, a finite ordered subset $\pi_T = \{t_0, t_1, \ldots, t_N\}$ such that $0 = t_0 < t_1 < \ldots. < t_N = T$ is a partition of $[0,T]$ and

$$\mu(\pi_T) = \max\{|t_{i+1} - t_i| : i = 0, 1, \ldots, N-1\}.$$

A sequence of partitions of $[0,T]$ is denoted by $\pi_T^N = \{t_0^N, t_1^N, \ldots, t_N^N\}$ such that $\lim_{N\to\infty} \mu(\pi_T^N) = 0$.

Consider the following simple process: Let $p \geq 1$ be fixed. For a given partition $\pi_T = \{t_0, t_1, \ldots, t_N\}$ of $[0,T]$,

$$\eta_t(w) = \sum_{i=0}^{N-1} \xi_i(w) I_{[t_i, t_{i+1}]}(t), \tag{2}$$

where $\xi_i \in L^p_{\mathrm{G}}(\Omega_{t_i})$, $i = 0, 1, \ldots, N-1$. The collection containing the above type of processes is denoted by $M^{p,0}_{\mathrm{G}}(0,T)$. The completion of $M^{p,0}_{\mathrm{G}}(0,T)$ under the norm $\|\eta\| = \{\int_0^T \mathbb{E}[|\eta_v|^p]\,\mathrm{d}v\}^{1/p}$ is denoted by $M^p_{\mathrm{G}}(0,T)$ and for $1 \leq p \leq q$, $M^p_{\mathrm{G}}(0,T) \supset M^q_{\mathrm{G}}(0,T)$.

Definition 6. (Itô's Integral). For each $\eta_t \in M^{2,0}_{\mathrm{G}}(0,T)$, the Itô's integral of G-Brownian motion is defined as

$$I(\eta) = \int_0^T \eta_v \,\mathrm{d}B_v = \sum_{i=0}^{N-1} \xi_i (B_{t_{i+1}} - B_{t_i}),$$

where η_t is given by (2).

Definition 7. (Quadratic Variation Process). An increasing continuous process $(\langle B\rangle_t)_{t\geq 0}$ with $\langle B\rangle_0 = 0$, defined by

$$\langle B\rangle_t = B_t^2 - 2\int_0^t B_v \,\mathrm{d}B_v,$$

is called the quadratic variation process of G-Brownian motion.

Let $\mathcal{B}(\Omega)$ be the Borel σ-algebra of Ω. It was proved in [6, 13] that there exists a weakly compact family $\mathcal{P}$ of probability measures P defined on $(\Omega, \mathcal{B}(\Omega))$ such that

$$\mathbb{E}[X] = \sup_{P\in\mathcal{P}} E_P[X], \quad \forall X \in \mathrm{L_{ip}}(\Omega).$$

This makes the following definitions reasonable.

Definition 8. (Capacity). The capacity $\hat{c}(.)$ associated to the family $\mathcal{P}$ is defined by

$$\hat{c}(A) = \sup_{P\in\mathcal{P}} P(A), \quad A \in \mathcal{B}(\Omega).$$

Definition 9. (Polar Set and Quasi-Sure Property). A set A is said to be polar if its capacity is zero, that is, $\hat{c}(A) = 0$ and a property holds quasi-surely (q. s. in short) if it holds outside a polar set.

Through out the paper for $X = (x_1, x_2, \ldots, x_n)$, $Y = (y_1, y_2, \ldots, y_n)$, $X \leq Y$ means $x_i \leq y_i$, $i = 1, 2, \ldots, n$.

3. Upper and Lower Solutions Method

Recall that the concept of upper and lower solutions for the classical stochastic differential equations was established in [8, 9, 17, 18].

Definition 10. (Upper Solution). A process $U_t \in M^2_{\mathrm{G}}(0,T)$ is said to be an upper solution of the G-SDE (1) on the interval $[0,T]$ if for any fixed s the inequality (interpreted component wise)

$$U_t \geq U_s + \int_s^t f(v, U_v)\,\mathrm{d}v + \int_s^t g(v, U_v)\,\mathrm{d}\langle B\rangle_v + \int_s^t h(v, U_v)\,\mathrm{d}B_v, \quad 0 \leq s \leq t \leq T, \tag{3}$$

holds q. s.

Definition 11. (Lower Solution). A process $L_t \in M^2_{\mathrm{G}}(0,T)$ is said to be a lower solution of the

G-SDE (1) on the interval $[0,T]$ if for any fixed s the inequality (interpreted component wise)

$$L_t \le L_s + \int_s^t f(v,L_v)\,\mathrm{d}v + \int_s^t g(v,L_v)\,\mathrm{d}\langle B\rangle_v + \int_s^t h(v,L_v)\,\mathrm{d}B_v\,,\ \ 0 \le s \le t \le T\,, \tag{4}$$

holds q. s.

Example 1. Consider the following scalar stochastic differential equation: (G-SDE)

$$\mathrm{d}X_t = H(X_t)\,\mathrm{d}t + \{X_t\}\,\mathrm{d}\langle B\rangle_t + \mathrm{d}B_t\,,\ \ t \in [0,T]\,, \tag{5}$$

where the respective Heaviside and sawtooth functions $H(x)$ and $\{x\}$ are defined in the introduction.

Then $U_t = U_0 + \int_0^t \mathrm{d}v + \int_0^t \mathrm{d}\langle B\rangle_v + \int_0^t \mathrm{d}B_v$ and $L_t = L_0 + \int_0^t \mathrm{d}B_v$ for $t \in [0,T]$ are the upper and lower solutions of the G-SDE (5), respectively, which are shown as the following:

$$\begin{aligned} U_t &= U_0 + \int_0^t \mathrm{d}v + \int_0^t \mathrm{d}\langle B\rangle_v + \int_0^t \mathrm{d}B_v \\ &= U_s + \int_s^t \mathrm{d}v + \int_s^t \mathrm{d}\langle B\rangle_v + \int_s^t \mathrm{d}B_v \\ &\ge U_s + \int_s^t \{U_v\}\,\mathrm{d}v + \int_s^t H(U_v)\,\mathrm{d}\langle B\rangle_v \\ &\quad + \int_s^t \mathrm{d}B_v\,,\ \ 0 \le s \le t \le T\,, \end{aligned}$$

where $U_s = U_0 + \int_0^s \mathrm{d}v + \int_0^s \mathrm{d}\langle B\rangle_v + \int_0^s \mathrm{d}B_v$ for each fixed s such that $0 \le s \le t \le T$. On similar arguments as above, one can show that $L_t = L_0 + \int_0^t \mathrm{d}B_v$ is a lower solution of the scalar G-SDE (5). The existence of solutions of the G-SDE (5) will be discussed later in Section 5.

Suppose that U_t and L_t are the respective upper and lower solutions of the G-SDE

$$\mathrm{d}X_t = f(t,w)\,\mathrm{d}t + g(t,w)\,\mathrm{d}\langle B\rangle_t + h(t,X_t)\,\mathrm{d}B_t\,,\ \ t \in [0,T]\,. \tag{6}$$

Define two functions $p_{L,U}, q_{L,U} : [0,T] \times \mathbb{R}^n \times \Omega \to \mathbb{R}^n$ by

$$\begin{aligned} p_{L,U}(t,x,w) &= \max\{L_t(w), \min\{U_t(w),x\}\}\,, \\ q_{L,U}(t,x,w) &= p_{L,U}(t,x,w) - x\,, \end{aligned} \tag{7}$$

and consider the stochastic differential equation

$$\mathrm{d}X_t = \tilde{f}(t,X_t)\,\mathrm{d}t + \tilde{g}(t,X_t)\,\mathrm{d}\langle B\rangle_t + \tilde{h}(t,X_t)\,\mathrm{d}B_t\,,\ \ t \in [0,T] \tag{8}$$

with a given constant initial condition $X_0 \in \mathbb{R}^n$, where

$$\begin{aligned} \tilde{f}(t,x,w) &= f(t,w) + q_{L,U}(t,x,w)\,, \\ \tilde{g}(t,x,w) &= g(t,w) + q_{L,U}(t,x,w)\,, \\ \tilde{h}(t,x,w) &= h(p_{L,U}(t,x,w)) \end{aligned}$$

are Lipschitz continuous in x. It is known that the stochastic differential equation (8) has a unique solution $X_t \in M_G^2(0,T;\mathbb{R}^n)$ [4, 5, 14].

4. Comparison Theorem for the G-SDEs

Lemma 1. *Assume that the respective upper and lower solutions U_t and L_t of the G-SDE (6) satisfy $L_t \le U_t$ for $t \in [0,T]$. Then U_t and L_t are upper and lower solutions of the G-SDE (8), respectively.*

Proof. Since $L_t \le U_t$ yields $p_{L,U}(t,U_t) = U_t$ and $q_{L,U}(t,U_t) = 0$, therefore

$$\begin{aligned} &U_s + \int_s^t \tilde{f}(v,U_v)\,\mathrm{d}v + \int_s^t \tilde{g}(v,U_v)\,\mathrm{d}\langle B\rangle_v + \int_s^t \tilde{h}(v,U_v)\,\mathrm{d}B_v \\ &= U_s + \int_s^t \left[f(v,w) + q_{L,U}(v,U_v)\right]\mathrm{d}v + \int_s^t \Big[g(v,w) \\ &\quad + q_{L,U}(v,U_v)\Big]\,\mathrm{d}\langle B\rangle_v + \int_s^t h(v,p_{L,U}(v,U_v))\,\mathrm{d}B_v \\ &= U_s + \int_s^t f(v,w)\,\mathrm{d}v + \int_s^t g(v,w)\,\mathrm{d}\langle B\rangle_v \\ &\quad + \int_s^t h(v,U_v)\,\mathrm{d}B_v \le U_t\,. \end{aligned}$$

Hence U_t for $0 \le s \le t \le T$ is an upper solution of the G-SDE (8). In a similar way as above, one can show that L_t is a lower solution of the G-SDE (8). □

The following lemma can be found in [19]. For the proof see Section Appendix A.

Lemma 2. *Let $X_t, Y_t \in M_G^{1,0}([0,T];\mathbb{R}^n)$. If $X_t \le Y_t$ for $t \in [0,T]$ and any $w \in \Omega$, then*

$$\int_0^T X_t\,\mathrm{d}\langle B\rangle_t \le \int_0^T Y_t\,\mathrm{d}\langle B\rangle_t\,.$$

Theorem 1. *(Comparison Result). Suppose that*

(i) The functions $f(t,x)$ and $g(t,x)$ are measurable with $\int_0^t \mathbb{E}[|\phi(v,.)|^2]\,\mathrm{d}v < \infty$ for $\phi = f$ and g, respectively, where $h(t,x)$ is Lipschitz continuous in x.

(ii) The respective upper and lower solutions U_t and L_t of the G-SDE (6) with $\mathbb{E}[|U_t|^2] < \infty$, $\mathbb{E}[|L_t|^2] < \infty$ satisfy $L_t \le U_t$ for $t \in [0,T]$.

(iii) Also $X_0 \in \mathbb{R}^n$ is a given initial value with $\mathbb{E}[|X_0|^2] < \infty$ and $L_0 \le X_0 \le U_0$.

Then there exists a unique solution $X_t \in M_G^2(0,T;\mathbb{R}^n)$ of the G-SDE (6) such that $L_t \le X_t \le U_t$ for $t \in [0,T]$ q. s.

Proof. Define the functions $p_{L,U}, q_{L,U} : [0,T] \times \mathbb{R}^n \times \Omega \to \mathbb{R}^n$ by (7) and consider the G-SDE (8).

Now the G-SDE (8) has a unique solution and by Lemma 1 if U_t and L_t are upper and lower solutions of the G-SDE (6), respectively, then they are the respective upper and lower solutions for the G-SDE (8). Also it is clear that any solution X_t of the modified G-SDE (8) such that

$$L_t \le X_t \le U_t\,, \quad t \in [0,T]\,, \tag{9}$$

q. s. is also a solution of the G-SDE (6). Thus we only need to show that any solution X_t of the problem (8) does satisfy the inequality (9).

Assume that there exists an arbitrary interval $(t_1,t_2) \subset [0,T]$ such that $X_{t_1} = L_{t_1}$ and $X_t < L_t$ for $t \in (t_1,t_2)$, then we have

$$\begin{aligned} X_t - L_t &= \int_{t_1}^t \tilde{f}(v,X_v)\,\mathrm{d}v + \int_{t_1}^t \tilde{g}(v,X_v)\,\mathrm{d}\langle B\rangle_v \\ &+ \int_{t_1}^t \tilde{h}(v,X_v)\,\mathrm{d}B_v - \int_{t_1}^t \tilde{f}(v,L_v)\,\mathrm{d}v \\ &- \int_{t_1}^t \tilde{g}(v,L_v)\,\mathrm{d}\langle B\rangle_v - \int_{t_1}^t \tilde{h}(v,L_v)\,\mathrm{d}B_v \\ &= \int_{t_1}^t \Big[f(v,w) + q_{L,U}(v,X_v)\Big]\,\mathrm{d}v + \int_{t_1}^t \Big[g(v,w) \\ &+ q_{L,U}(v,X_v)\Big]\,\mathrm{d}\langle B\rangle_v + \int_{t_1}^t h(v,p_{L,U}(v,X_v))\,\mathrm{d}B_v \\ &- \int_{t_1}^t [f(v,w) + q_{L,U}(v,L_v)]\,\mathrm{d}v - \int_{t_1}^t \Big[g(v,w) \\ &+ q_{L,U}(v,L_v)\Big]\,\mathrm{d}\langle B\rangle_v - \int_{t_1}^t h\Big(v,p_{L,U}(v,L_v)\Big)\,\mathrm{d}B_v\,. \end{aligned}$$

Since $L_t \le U_t$ gives $p_{L,U}(t,L_t) = L_t$ and $X_t < L_t$ implies $X_t < U_t$ which gives $p_{L,U}(t,X_t) = L_t$. Also $q_{L,U}(t,L_t) = 0$ and $q_{L,U}(t,X_t) = L_t - X_t > 0$. Thus

$$X_t - L_t = \int_{t_1}^t q_{L,U}(v,X_v)\,\mathrm{d}v + \int_{t_1}^t q_{L,U}(v,X_v)\,\mathrm{d}\langle B\rangle_v > 0\,,$$

which yields a contradiction. Thus $X_t \ge L_t$ for $t \in [0,T]$. By using the similar arguments as above one can show that $X_t \le U_t$ for $t \in [0,T]$. □

5. G-SDEs with Simultaneous Discontinuous Coefficients f and g

Now we take the G-SDE

$$\mathrm{d}X_t = f(t,X_t)\,\mathrm{d}t + g(t,X_t)\,\mathrm{d}\langle B\rangle_t + h(t,X_t)\,\mathrm{d}B_t\,, \quad t \in [0,T]\,, \tag{10}$$

where $f(t,x)$ and $g(t,x)$ do not need to be continuous but suppose that they are increasing, that is, if $x \ge y$ then $\phi(t,x) \ge \phi(t,y)$ for $\phi(t,.) = f(t,.)$ and $g(t,.)$, respectively (where the inequalities are interpreted component wise), and $h(t,x)$ is Lipschitz continuous in x.

Theorem 2. *Suppose that*

(i) The functions $f(t,x)$ and $g(t,x)$ are increasing in x, where $h(t,x)$ is Lipschitz continuous in x.

(ii) U_t and L_t are the respective upper and lower solutions of the G-SDE (10) with $\int_0^t \mathbb{E}[|\phi(U_v)|^2]\,\mathrm{d}v < \infty$, $\int_0^t \mathbb{E}[|\phi(L_v)|^2]\,\mathrm{d}v < \infty$ for $\phi = f$, g, respectively, and $L_t \le U_t$ for $t \in [0,T]$.

Then there exists at least one solution $X_t \in M_G^2(0,T;\mathbb{R}^n)$ of the G-SDE (10) such that $L_t \le X_t \le U_t$ for $t \in [0,T]$ q. s.

Proof. Define the space of all d-dimensional stochastic processes by $\hat{\mathcal{H}}_2$, that is, $\hat{\mathcal{H}}_2 = \{X = \{X_t, t \in [0,T]\} : \mathbb{E}[|X_t|^2] < \infty\}$ with the norm $\|X_t\|_2 = \{\int_0^t \mathbb{E}[|X_v|^2]\,\mathrm{d}v\}^{1/2}$ for all $t \in [0,T]$, which is a Banach space [4–6].

Denote the order interval $[L,U]$ in $\hat{\mathcal{H}}_2$ by $\mathcal{K}$, that is, $\mathcal{K} = \{X : X \in \hat{\mathcal{H}}_2$ and $L_t \le X_t \le U_t\}$ for $t \in [0,T]$, which is closed and bounded by the above norm. By using the monotone convergence theorem [13], one can prove the convergence of a monotone sequence that belongs to $\mathcal{K}$ in $\hat{\mathcal{H}}_2$. Thus $\mathcal{K}$ is a regularly ordered metric space with the above norm. It is clear that for any process $V \in \mathcal{K}$, U_t, and L_t are the respective upper and lower solutions for the G-SDE

$$\mathrm{d}X_t = f(t,V_t)\,\mathrm{d}t + g(t,V_t)\,\mathrm{d}\langle B\rangle_t + h(t,X_t)\,\mathrm{d}B_t\,, \quad t \in [0,T]\,. \tag{11}$$

Hence by Theorem 1, for any $X_0 \in \mathbb{R}^n$ with $\mathbb{E}[|X_0|^2] < \infty$ and $L_0 \le X_0 \le U_0$, the G-SDE (11) has a unique solution $X_t \in M_G^2(0,T;\mathbb{R}^n)$ such that $L_t \le X_t \le U_t$ for $t \in [0,T]$ q. s.

Define an operator $F : \mathcal{K} \to \mathcal{K}$ by $F(V) = X$, where X is the unique solution of the G-SDE (11). Now suppose that $V_t^{(1)} \leq V_t^{(2)}$ for all $t \in [0,T]$ and define $X^{(1)} = F(V^{(1)})$, $X^{(2)} = F(V^{(2)})$ where $V^{(1)}, V^{(2)} \in \mathcal{K}$. Since it is given that f and g are increasing functions, therefore $X_t^{(1)}$ is a lower solution of the G-SDE

$$X_t = X_0 + \int_0^t f(v, V_v^{(2)})\,\mathrm{d}v + \int_0^t g(v, V_v^{(2)})\,\mathrm{d}\langle B\rangle_v + \int_0^t h(v, X_v)\,\mathrm{d}B_v, \quad t \in [0,T]. \tag{12}$$

But this problem has an upper solution U_t. Hence by Theorem 1, the G-SDE (12) has a solution $X_t^{(2)}$ such that $X_t^{(1)} \leq X_t^{(2)} \leq U_t$ for $t \in [0,T]$. Thus F is an increasing mapping and by Theorem 3, it has a fixed point $X^{(*)} = F(X^{(*)}) \in \mathcal{K}$ such that $Y_t \leq X_t^{(*)} \leq U_t$ q. s., where

$$X_t^{(*)} = X_0 + \int_0^t f(v, X_v^{(*)})\,\mathrm{d}v + \int_0^t g(v, X_v^{(*)})\,\mathrm{d}\langle B\rangle_v + \int_0^t h(v, X_v^{(*)})\,\mathrm{d}B_v, \quad t \in [0,T]. \qquad \square$$

Now continuing Example 1 by the above Theorem 2, there exists at least one solution $X^{(*)}$ of the G-SDE (5) such that $L_0 + B_t \leq X_t^{(*)} \leq U_0 + t + \langle B\rangle_t + B_t$ for $t \in [0,T]$, where $L_t = L_0 + B_t$ and $U_t = U_0 + t + \langle B\rangle_t + B_t$ are the respective lower and upper solutions of (5).

Remark 1. As the above results (i. e. Theorem 1 and Theorem 2) for the existence theory of G-SDEs are more general, we therefore give the following sketch for the existence of solutions for G-SDEs with a discontinuous coefficient g. The rest of work can be obtained in a similar fashion. Assume that U_t and L_t are the respective upper and lower solutions of the G-SDE

$$\mathrm{d}X_t = f(t, X_t)\,\mathrm{d}t + g(t, w)\,\mathrm{d}\langle B\rangle_t + h(t, X_t)\,\mathrm{d}B_t, \quad t \in [0,T].$$

Define the functions $p_{L,U}$ and $q_{L,U}$ by (7) and consider the stochastic differential equation

$$\mathrm{d}X_t = \tilde{f}(t, X_t)\,\mathrm{d}t + \tilde{g}(t, X_t)\,\mathrm{d}\langle B\rangle_t + \tilde{h}(t, X_t)\,\mathrm{d}B_t, \quad t \in [0,T],$$

with a given constant initial condition $X_0 \in \mathbb{R}^n$, where

$$\tilde{f}(t,x,w) = f(p_{L,U}(t,x,w)),$$
$$\tilde{g}(t,x,w) = g(t,w) + q_{L,U}(t,x,w),$$
$$\tilde{h}(t,x,w) = h(p_{L,U}(t,x,w))$$

are Lipschitz continuous in x.

6. Discussion

The existence theory for the G-SDEs with a discontinuous coefficient h is still open. We are unable to discuss this case. Because like the classical Itô's integral, the monotonicity condition does not hold in G-Itô's integral with respect to the G-Brownian motion, i. e. $X_t \geq Y_t$ does not imply $\int_0^T X_t\,\mathrm{d}B_t \geq \int_0^T Y_t\,\mathrm{d}B_t$. However its quadratic variation process is an increasing continuous process starting from zero and satisfy the above stated property, see Lemma 2 [19].

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Appendix A Proof

For the following definition and theorem see [20].

Definition 12. An ordered metric space M is called regularly (resp. fully regularly) ordered, if each monotone and order (resp. metrically) bounded ordinary sequence of M converges.

Theorem 3. *If $[a,b]$ is a non-empty order interval in a regularly ordered metric space, then each increasing mapping $F : [a,b] \to [a,b]$ has the least and the greatest fixed point.*

Proof of Lemma (2).

Proof. Since $\{\langle B\rangle_t : t \geq 0\}$ is an increasing continuous process with $\langle B\rangle_0 = 0$. Therefore for any fixed $w \in \Omega$ and $t_{i+1} \geq t_i$, $\langle B\rangle_{t_{i+1}} - \langle B\rangle_{t_i} \geq 0$, $i = 0, 1, \ldots, N-1$. Also for $X_t, Y_t \in M_G^{1,0}([0,T];\mathbb{R}^n)$, $X_t = \sum_{i=0}^{N-1} \xi_i I_{[t_i,t_{i+1})}$ and $Y_t = \sum_{i=0}^{N-1} \tilde{\xi}_i I_{[t_i,t_{i+1})}$, where $\xi_i, \tilde{\xi}_i \in L_G^1(\Omega_i)$, $i = 0, 1, \ldots, N-1$. Then $X_t \leq Y_t$ implies that

$$\sum_{i=0}^{N-1} \xi_i I_{[t_i,t_{i+1})} \leq \sum_{i=0}^{N-1} \tilde{\xi}_i I_{[t_i,t_{i+1})},$$

which yields

$$\sum_{i=0}^{N-1} \xi_i \left[\langle B\rangle_{t_{i+1}} - \langle B\rangle_{t_i}\right] \leq \sum_{i=0}^{N-1} \tilde{\xi}_i \left[\langle B\rangle_{t_{i+1}} - \langle B\rangle_{t_i}\right].$$

Hence

$$\int_0^T X_t\,\mathrm{d}\langle B\rangle_t \leq \int_0^T Y_t\,\mathrm{d}\langle B\rangle_t. \qquad \square$$

Remark 2. The above lemma shows that G-Ito's integral with respect to the quadratic variation process satisfies the monotonicity property. Also if $X_t \leq 0$ then $\int_0^T X_t \, \mathrm{d}\langle B \rangle_t \leq 0$.

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