

Nonperiodic Oscillation of Bright Solitons in Condensates with a Periodically Oscillating Harmonic Potential

Zhang-Ming He^a, Deng-Long Wang^a, Yan-Chao She^a, Jian-Wen Ding^{a,b}, and Xiao-Hong Yan^b

^a Department of Physics, Xiangtan University, Xiangtan 411105, China

^b College of Electronic Science Engineering, Nanjing University of Post & Telecommunication, Nanjing, 211106, China

Reprint requests to D. W.; E-mail: dlwang@xtu.edu.cn

Z. Naturforsch. **67a**, 723 – 728 (2012) / DOI: 10.5560/ZNA.2012-0085

Received April 2, 2012 / revised August 18, 2012 / published online November 14, 2012

Considering a periodically oscillating harmonic potential, we explored the dynamic properties of bright solitons in a Bose–Einstein condensate by using Darboux transformation. It is found that the soliton movement exhibits a nonperiodic oscillation under a slow oscillating potential, while it is hardly affected under a fast oscillating potential. Furthermore, the head-on and/or ‘chase’ collisions of two solitons have been obtained, which could be controlled by the oscillation frequency of the potential.

Key words: Bose–Einstein Condensates; Oscillating Solitons; Periodically Oscillating Potential Trap; Darboux Transformation.

PACS numbers: 05.45.Yv; 03.75.Lm; 03.75.Kk

1. Introduction

Bose–Einstein condensates (BECs) in weakly interacting alkali atomic gases have been proved to be an ideal laboratory system for investigating fundamental nonlinear phenomena, such as bright solitons [1–3], dark solitons [4–8], and vortices [9]. Especially, the bright solitons in BECs open possibilities for future applications in coherent atomic optics, atom interferometry, and atom transport [2]. Recently, soliton oscillations have been obtained experimentally [10–13], which boost an immense theoretical interest in the nonlinear matter waves.

Theoretically, it was shown that external potentials have an important effect on oscillating properties of solitons in one-dimensional BECs [14–20]. For example, when a bright soliton is loaded into an attractive harmonic potential, it executes harmonic oscillations with the oscillating frequency depending on the trapping frequency [15, 19]. For an optical potential, when the energy of the bright soliton is lower than the height of the potential, it can exhibit an os-

cillating behaviour around the bottom of the potential notch with an oscillating frequency depending on both the lattice spacing and the height of the potential [19]. Similar results have also been obtained when the bright solitons are loaded into a tanh-shaped potential [19]. Usually, the periodic oscillation of bright solitons can be expected under a spatially nonuniform potential trap [15, 19]. In fact, a periodically oscillating external potential is easily achieved in BEC experiments [21, 22]. In this case, how about the dynamic behaviours of bright solitons? To our knowledge, this is an open subject.

In this paper, we explore the oscillating properties of bright solitons in BECs with a periodically oscillating harmonic potential. A nonperiodic oscillating behaviour of bright soliton is obtained under a slow oscillating potential, different from that under a time-independent harmonic potential. Also, the head-on and/or ‘chase’ collisions of two solitons have been observed, which could be controlled by the oscillating frequency of a harmonic potential. The results could be useful for future applications of BECs in accurate atomic clocks and other devices.

2. Model and Soliton Solutions

A periodically oscillating harmonic potential can be given by $V(r) = m\omega_{\perp}^2(Y^2 + Z^2) + m\omega_{\parallel}^2[X - k\sin(\omega_p T)]^2/2$ [23–25]. Here m is the atomic mass; ω_{\perp} and ω_{\parallel} are the radial and transverse trapping frequencies, respectively; k and ω_p are the oscillating amplitude and oscillating frequency of the external potential, respectively. If $\omega_{\perp} \gg |\omega_{\parallel}|$, it is reasonable to reduce the Gross–Pitaevskii (GP) equation into a one-dimensional nonlinear Schrödinger equation with an oscillating harmonic potential,

$$i\psi_t + \psi_{xx} + 2g|\psi|^2\psi - \frac{\omega_{\parallel}^2}{\omega_{\perp}^2}[x - k\sin(\omega t)]^2\psi = 0, \quad (1)$$

where $g = -2Na_s/a_{\perp}$, and the time t and the coordinate x are measured, respectively, in units of $2/\omega_{\perp}$ and a_{\perp} , with $a_{\perp} = \sqrt{\hbar/m\omega_{\perp}}$. Here, the oscillating frequency of the external potential $\omega = 2\omega_p/\omega_{\perp}$; N is the atom number and a_s the s -wave scattering length (SL) [26–28]. The tuning of SL can be achieved by Feshbach resonance [29, 30]. We consider the time-dependent SL and choose $g(t) = c\exp(\gamma t)$ (where c is a constant and $\gamma^2 = -4\omega_{\parallel}^2/\omega_{\perp}^2$ with γ a real constant number) [31, 32].

To obtain the exact solutions of (1), we make use of the Darboux transformation [33–35]. The seed solution of (1) can be chosen as $\psi_0 = \sqrt{g}Q\exp[i c\exp(2g\gamma)/(2\gamma)]$, where $Q = \exp(-i\gamma x^2/2 - iDt - iE)$, with $D = [k\gamma^3\sin(\omega t) + k\gamma^2\omega\cos(\omega t)]/(\gamma^2 + \omega^2)$ and $E = \{\sin(2\omega t)[k^2\gamma^4\omega^2 - k^2\gamma^6 - 4k^2(\gamma^2 + \omega^2)/2 + k^2\gamma^5\omega\sin^2(\omega t)]/[8\omega(\gamma^2 + \omega^2)^2] + k^2(5\gamma^2 + 4\omega^2)t/(8\gamma^2 + 8\omega^2)\}$. Subsequently, the Lax pair of (1) is presented as

$$\Phi_x = U\Phi, \quad \Phi_t = V\Phi, \quad (2)$$

where $\Phi = (\phi_1, \phi_2)^T$; the superscript ‘T’ denotes the matrix transpose. Here, $U = \begin{pmatrix} \lambda & p \\ -\bar{p} & -\lambda \end{pmatrix}$, and $V = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}$, with $p = \sqrt{g}\psi\bar{Q}$, $A = 2i\lambda^2 + i\lambda(\gamma x - D)g|\psi|^2$, $B = 2i\sqrt{g}\psi\bar{Q}\lambda + i\sqrt{g}\psi_x\bar{Q} + \sqrt{g}\psi\bar{Q}(\gamma x - D)/2$, and $C = -2i\sqrt{g}\bar{\psi}Q\lambda - i\sqrt{g}\bar{\psi}_xQ - \sqrt{g}\bar{\psi}Q(\gamma x - D)/2$; the overbar denotes the complex conjugate. From the compatibility condition $(\partial^2\Phi)/(\partial x\partial t) = (\partial^2\Phi)/(\partial t\partial x)$, one has $U_t - V_x + UV - VU = 0$. By

performing the Darboux transformation [33–35]

$$\psi_1 = \psi_0 + 2(\lambda_1 + \bar{\lambda}_1) \frac{Q\phi_1\bar{\phi}_2}{\sqrt{g}|\phi_1|^2 + |\phi_2|^2}, \quad (3)$$

we can obtain a single soliton solution of (1)

$$\psi_1 = \left[-1 + 2 \frac{(\lambda_0^2 - 1)\cos(\varphi) + i\lambda_0\sqrt{\lambda_0^2 - 1}\sin(\varphi)}{\lambda_0^2\cosh(\theta) + \lambda_0\sqrt{\lambda_0^2 - 1}\sinh(\theta) - \cos(\varphi)} \right] \psi_0. \quad (4)$$

Here $\theta = 2c\exp(\gamma t)\sqrt{\lambda_0^2 - 1}[x - k\gamma^2\sin(\omega t)/(\gamma^2 + \omega^2) + c\exp(\gamma t)/\gamma]$ and $\varphi = 2c^2\exp(2\gamma t)\sqrt{\lambda_0^2 - 1}/\gamma$ with λ_0 constant.

Then, by repeating the Darboux transformation N times, we can obtain the N th-order solution

$$\psi_n = \psi_0 + 2 \sum_{n=1}^N (\lambda_n + \bar{\lambda}_n) \frac{\phi_1[n, \lambda_n]\bar{\phi}_2[n, \lambda_n]Q}{\sqrt{g}\Phi[n, \lambda_n]^T \bar{\Phi}[n, \lambda_n]} \quad (5)$$

with $\Phi[n, \lambda] = (\lambda I - S[n-1]) \cdots (\lambda I - S[1])\Phi[1, \lambda]$. Here $S_{l_1 l_2}[n'] = (\lambda_{n'} + \bar{\lambda}_{n'})[\phi_{l_1}[n', \lambda_{n'}]\bar{\phi}_{l_2}[n', \lambda_{n'}]]/[\phi_1[n', \lambda_{n'}]^2 + \phi_2[n', \lambda_{n'}]^2] - \bar{\lambda}_{n'}\delta_{l_1 l_2}$, $l_1, l_2 = 1, 2, n' = 1, 2, \dots, n-1$, and $n = 2, 3, \dots, N$. For $N = 2$, one can obtain the two-soliton solution of (1)

$$\psi_2 = \psi_0(1 + 2G/F), \quad (6)$$

where $F = (\lambda_{01} + \lambda_{02})^2(h_1 + k_1)(h_2 + k_2) - 4\lambda_{01}\lambda_{02}(h_1h_2 + k_1k_2 + j_1j_2) + 2\sqrt{\lambda_{01}^2 - 1}\sqrt{\lambda_{02}^2 - 1}\sin(\varphi_1)\sin(\varphi_2)$ and $G = 2\lambda_{02}(\lambda_{01}^2 - \lambda_{02}^2)(h_1 + k_1)[j_2 + i\sqrt{\lambda_{02}^2 - 1}\sin(\varphi_2)] + 2\lambda_{01}(\lambda_{01}^2 - \lambda_{02}^2)(h_2 + k_2)[j_1 + i\sqrt{\lambda_{01}^2 - 1}\sin(\varphi_1)]$, with $k_i = (2\lambda_i^2 - 1)\cosh(\theta_i) + 2\lambda_{0i}\sqrt{\lambda_i^2 - 1}\sinh(\theta_i) - \cos(\varphi_i)$, $h_i = \cosh(\theta_i) - \cos(\varphi_i)$, and $j_i = -\lambda_{0i}\cosh(\theta_i) - \sqrt{\lambda_{0i}^2 - 1}\sinh(\vartheta_i) + \lambda_{0i}\cos(\varphi_i)$ ($i = 1, 2$). Here $\theta_i = 2c\sqrt{(\lambda_{0i}^2 - 1)}\exp(\gamma t)[x - k\gamma^2\sin(\omega t)/(\gamma^2 + \omega^2) + c\exp(\gamma t)/\gamma]$ and $\varphi = 2c^2\exp(2\gamma t)\sqrt{\lambda_{0i}^2 - 1}/\gamma$ with λ_{0i} constant ($i = 1, 2$). From (4) and (6), we can explore in detail the dynamic behaviour of bright solitons.

3. Results and Discussion

As a typical example, we consider a BEC consisting of ^7Li . Based on the current experimental conditions, the radial and transverse trapping frequencies are chosen as $\omega_{\perp} = \pi \times 100$ Hz and $\omega_1 = 5\pi$ Hz, respectively [2]. So, the time and space units correspond to 6.4 ms and $5.4 \mu\text{m}$ in reality, respectively.

3.1. Oscillating Properties of a Single Bright Soliton

For exploring the oscillating properties of a single soliton in a ^7Li BEC with a periodically oscillating harmonic potential, we here propose that $\omega = 10$ and $\omega = 0.2$ represent the fast and slow oscillation of the harmonic potential, respectively. Figure 1 shows the space-time distribution of the density of a BEC under an oscillating harmonic potential.

For a fast oscillating potential, one can see from Figure 1a that a bright soliton appears at the initial time. With the time going on, the amplitude of the bright soliton increases but its width decreases. Meanwhile, the bright soliton propagates along the positive direction of the x -axis. This phenomenon is similar to that for $k = 0$ in [14]. This suggests that the propagation properties of bright solitons are hardly dependent on the fast oscillating potential.

For a slow oscillating potential, the dynamical properties of the single soliton are shown in Figure 1b. One can see that the bright soliton moves along the positive direction of the x -axis when the time increases from 0 to 11, which is similar to that of Figure 1a. With time increasing from 11 to 23, interestingly, it is observed that the bright soliton moves along the negative direction of the x -axis, while it can not come back to the initial position. When the time further increases, the bright soliton again moves along the positive direction of the x -axis. Obviously, the bright soliton exhibits a nonperiodic oscillation, different from the periodic oscillation under a time-independent harmonic potential.

Especially, as depicted in Figure 1, the nonperiodic oscillating behaviour of a bright soliton strongly depends on the oscillating frequency of the harmonic potential. Thus, by tuning the oscillation frequency and amplitude of this potential, the moving behaviours of the bright soliton can be controlled. In Figure 2, we show the corresponding center positions of the bright

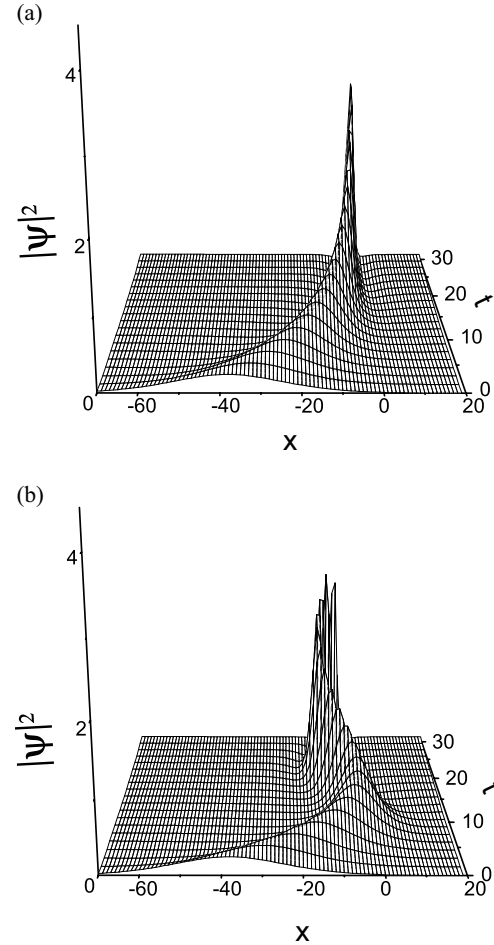


Fig. 1. Space-time distributions of the density of a BEC with a harmonic potential showing (a) a faster oscillation ($\omega = 10$), (b) a slower oscillation ($\omega = 0.2$). The other parameters used are $\lambda_0 = 2.0$, $c = -0.01$, $\gamma = 0.1$, and $k = 50$.

soliton as a function of time at various oscillation frequencies and/or amplitudes of the harmonic potential. One can see from Figure 2a that with increasing oscillation frequency, the amplitude of the bright soliton oscillation decreases. This is because the frequency is above resonance, so there is little response. Furthermore, it is found that the period of the soliton oscillation decreases with increasing oscillation frequency of the potential. This is simply because the frequency of the soliton motion is equal to the frequency of the oscillating potential, which drives the soliton motion. Notice in Figure 2b that with increasing oscillation amplitude k of the harmonic potential, the oscillation am-

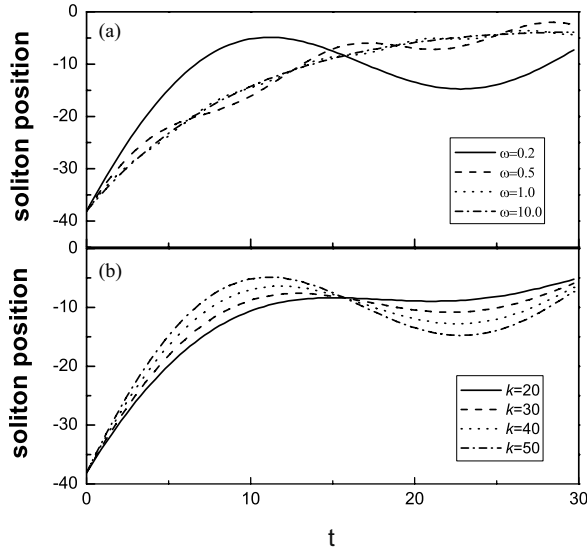


Fig. 2. Evolution of the center positions of the bright soliton. The other parameters are the same as in Figure 1.

plitude of the soliton increases. In fact, when the soliton is trapped in the harmonic potential, the soliton behaves like a classical particle which moves under the influence of the oscillating harmonic potential, and the soliton feels a force which comes from the oscillating harmonic potential. For convenience, we here define this force F as $F = -\partial V(x)/\partial x = 2\omega_1^2[-x + k\sin(\omega t)]/\omega_1^2$. For the case of $x < 0$, such as in Figure 2b, the force is increased with increasing oscillation amplitude k . Therefore, we can effectively control the moving trajectory of solitons by tuning the oscillation amplitude of the harmonic potential.

3.2. Oscillating Properties of two Bright Solitons

We further explore in Figure 3 the oscillating properties of two bright solitons under a periodically oscillating potential. We here choose $\omega = 10$ and $\omega = 0.1$ as typical examples for the fast and slow oscillating potential, respectively.

For a fast oscillating potential, one can see from Figure 3a that there exist two bright solitons at the initial time. With the time going on, the left bright soliton moves rightward while the right one moves leftward. Also, the amplitude of each soliton increases while their widths decrease. When the time further increases, their distance further decreases. At $t \approx 20$, the two bright solitons experience a head-on collision. This

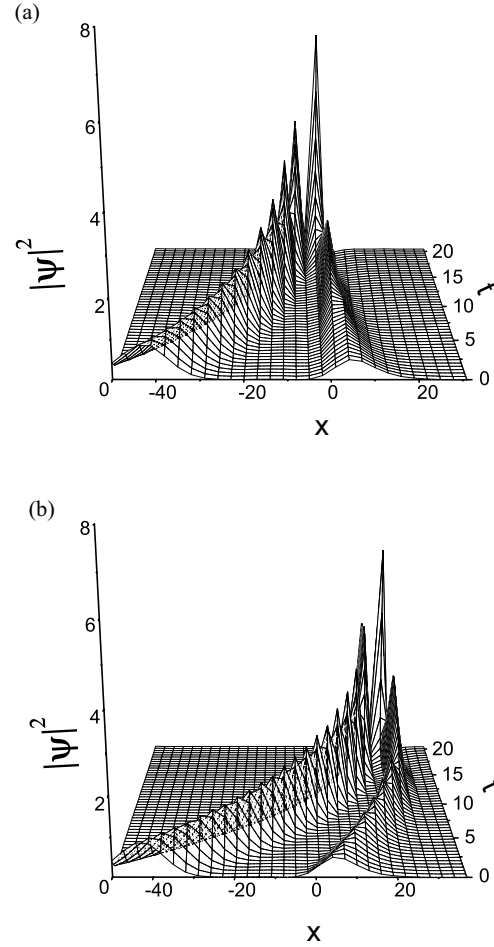


Fig. 3. Oscillating properties of two bright solitons in a BEC with a harmonic potential showing (a) a faster oscillation ($\omega = 10$), (b) a slower oscillation ($\omega = 0.1$). The other parameters used are $\lambda_{01} = 2.0$, $\lambda_{02} = 2.5$, and $c = -0.02$; further parameters are the same as in Figure 1.

phenomenon is similar to that of $k = 0$ in [36]. Therefore, the propagation properties of two bright solitons are also hardly affected by the fast oscillating potential.

For a slow oscillating potential, the dynamic properties of two bright solitons are shown in Figure 3b. When the time t increases from 0 to 15, the two solitons both move along the positive direction of the x -axis. While the time t increases from 15 to 20, the two solitons both move along the negative direction of the x -axis. This shows that the two solitons exhibit an oscillating behaviour. Meanwhile, the distance between the two solitons becomes smaller, the two solitons ex-

perience a ‘chase’ collision at $t \approx 20$. Obviously, both the head-on and ‘chase’ collision in Figure 3 can be controlled by the oscillating frequency of the harmonic potential.

The validity of the GP equation relies on the condition that the system is dilute and weakly interacting: $d|a_s(t)|^3 \ll 1$, where d is the average density of the condensate. In the real experiment with ^7Li atoms, the typical value of the atomic density is 10^{13} cm^{-3} . In our work, we consider that the absolute of the SL is $|a_s(t)|_{\max} = 30.6a_B$ with Bohr radius a_B so that $d|a_s(t)|^3 < 10^{-4} \ll 1$. Therefore, the GP equation is valid for the given parameters, and thus our results can be observed under the condition of the current experiments.

4. Conclusion

In summary, we present a family of single- and two-soliton solutions of BECs under a periodically oscillating harmonic potential by using Darboux transformation. It is found that a single bright soliton exhibits a nonperiodic oscillation for a slow oscillating

harmonic potential, while its propagation properties are hardly affected by the fast oscillating potential. And the moving trajectory of the soliton can be controlled by tuning the oscillation frequency and amplitude of the harmonic potential. Furthermore, for two bright solitons, a ‘chase’ collision takes place under a slow oscillating harmonic potential, while there occurs a head-on collision under a fast oscillating harmonic potential. The collisional behaviour can be controlled by the oscillation frequency of the harmonic potential. The results will stimulate experiments to manipulate solitons in BECs.

Acknowledgements

The work was supported by the National Natural Science Foundation of China (No. 51032002 and 11074212), the key Project of the National HighTechnology Research and Development Program (‘863’Program) of China (No. 2011AA050526), the Hunan Provincial Innovation Foundation for Postgraduate (Grant No. CX2010B254), and partially by PC-SIRT (IRT1080).

- [1] K. E. Strecker, G. B. Partridge, A. G. Truscott, and R. G. Hulet, *Nature* **417**, 150 (2002).
- [2] L. Khaykovich, F. Schreck, G. Ferrari, T. Bourdel, J. Cubizolles, L. D. Carr, Y. Castin, and C. Salomon, *Science* **296**, 1290 (2002).
- [3] B. Eiermann, T. Anker, M. Albiez, M. Taglieber, P. Treutlein, K. P. Marzlin, and M. K. Oberthaler, *Phys. Rev. Lett.* **92**, 230401 (2004).
- [4] S. Burger, K. Bongs, S. Dettmer, W. Ertmer, K. Sengstock, A. Sanpera, G. V. Shlyapnikov, and M. Lewenstein, *Phys. Rev. Lett.* **83**, 5198 (1999).
- [5] J. Denschlag, J. E. Simsarian, D. L. Feder, C. W. Clark, L. A. Collins, J. Cubizolles, L. Deng, E. W. Hagley, K. Helmerson, W. P. Reinhardt, S. L. Rolston, B. I. Schneider, and W. D. Phillips, *Science* **287**, 97 (2000).
- [6] B. P. Anderson, P. C. Haljan, C. A. Regal, D. L. Feder, L. A. Collins, C. W. Clark, and E. A. Cornell, *Phys. Rev. Lett.* **86**, 2926 (2001).
- [7] M. A. Hoefer, J. J. Chang, C. Hamner, and P. Engels, *Phys. Rev. A* **84**, 041605 (2011).
- [8] C. Hamner, J. J. Chang, P. Engels, and M. A. Hoefer, *Phys. Rev. Lett.* **106**, 065302 (2011).
- [9] T. W. Neely, E. C. Samson, A. S. Bradley, M. J. Davis, and B. P. Anderson, *Phys. Rev. Lett.* **104**, 160401 (2010).
- [10] S. Stellmer, C. Becker, P. Soltan-Panahi, E.-M. Richter, S. Dörscher, M. Baumert, J. Kronjäger, K. Bongs, and K. Sengstock, *Phys. Rev. Lett.* **101**, 120406 (2008).
- [11] A. Weller, J. P. Ronzheimer, C. Gross, J. Esteve, M. K. Oberthaler, D. J. Frantzeskakis, G. Theocharis, and P. G. Kevrekidis, *Phys. Rev. Lett.* **101**, 130401 (2008).
- [12] L. Shomroni, E. Lahoud, S. Levy, and J. Steinhauer, *Nat. Phys.* **5**, 193 (2009).
- [13] C. Becker, S. Stellmer, P. Soltan-Panahi, S. Dörscher, M. Baumert, E. Richter, J. Kronjäger, K. Bongs, and K. Sengstock, *Nat. Phys.* **4**, 496 (2008).
- [14] V. N. Serkin, A. Hasegawa, and T. L. Belyaeva, *Phys. Rev. A* **81**, 023610 (2010).
- [15] U. A. Khawaja, H. T. C. Stoof, R. G. Hulet, K. E. Strecker, and G. B. Partridge, *Phys. Rev. Lett.* **89**, 200404 (2002).
- [16] J. B. Beitia, V. M. Pérez-García, V. Vekslerchik, and V. V. Konotop, *Phys. Rev. Lett.* **100**, 164102 (2008).
- [17] F. K. Abdullaev and J. Garnier, *Phys. Rev. A* **77**, 023611 (2008).
- [18] R. Radha and V. R. Kumar, *Phys. Lett. A* **370**, 46 (2007).
- [19] P. K. Shukla and D. D. Tskhakaya, *Phys. Scripta* **107**, 259 (2004).

- [20] V. R. Kumar, R. Radha, and M. Wadati, *J. Phys. Soc. Jap.* **7**, 074005 (2010).
- [21] C. Raman, M. Köhl, R. Onofrio, D. S. Durfee, C. E. Kuklewicz, Z. Hadzibabic and W. Ketterle, *Phys. Rev. Lett.* **83**, 2502 (1999).
- [22] R. Onofrio, C. Raman, J. M. Vogels, J. R. Abo-Shaeer, A. P. Chikkatur, and W. Ketterle, *Phys. Rev. Lett.* **85**, 2228 (2000).
- [23] B. Jackson, J. F. McCann, and C. S. Adams, *Phys. Rev. A* **61**, 051603 (2000).
- [24] K. Fujimoto and M. Tsubota, *Phys. Rev. A* **82**, 043611 (2010).
- [25] K. Fujimoto and M. Tsubota, *Phys. Rev. A* **83**, 053609 (2011).
- [26] F. Dalfovo, S. Giorgini, L. P. Pitaevskii, and S. Stringari, *Rev. Mod. Phys.* **71**, 463 (1999).
- [27] A. J. Leggett, *Rev. Mod. Phys.* **73**, 307 (2001).
- [28] B. B. Baizakov, G. Filatrella, B. A. Malomed, and M. Salerno, *Phys. Rev. E* **71**, 036619 (2005).
- [29] S. Lnouye, M. R. Andrews, J. Stenger, H-J. Miesner, D. M. Stamper-Kurn, and W. Ketterle, *Nature* **392**, 151 (1998).
- [30] C. Chin, R. Grimm, P. Julienne, and E. Tiesinga, *Rev. Mod. Phys.* **82**, 1225 (2010).
- [31] Z. X. Liang, Z. D. Zhang, and W. M. Liu, *Phys. Rev. Lett.* **94**, 050402 (2005).
- [32] W. X. Zhang, D. L. Wang, Z. M. He, F. J. Wang, and J. W. Ding, *Phys. Lett. A* **372**, 4407 (2008).
- [33] Z. X. Liang, Z. D. Zhang, and W. M. Liu, *Mod. Phys. Lett. A* **21**, 383 (2006).
- [34] Z. D. Li, Q. Y. Li, L. Li, and W. M. Liu, *Phys. Rev. E* **76**, 026605 (2007).
- [35] U. A. Khawaja, *Phys. Lett. A* **373**, 2710 (2009).
- [36] X. F. Zhang, Q. Ying, J. F. Zhang, Z. X. Chen, and W. M. Liu, *Phys. Rev. A* **77**, 023613 (2008).