A Note on the Carathéodory Approximation Scheme for Stochastic Differential Equations under G-Brownian Motion

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In this note, the Carathéodory approximation scheme for vector valued stochastic differential equations under G-Brownian motion (G-SDEs) is introduced. It is shown that the Carathéodory approximate solutions converge to the unique solution of the G-SDEs. The existence and uniqueness theorem for G-SDEs is established by using the stated method.

Key words: Stochastic Differential Equations; G-Brownian Motion; Carathéodory's Approximation Scheme; Existence; Uniqueness.

Mathematics Subject Classification 2000: 60H10, 60H20, 60H35, 62L20

1. Introduction

The Carathéodory approximation scheme was introduced by the Greek mathematician named Constantine Carathéodory in the early part of 20th century for ordinary differential equations (Chapter 2 of [1]). Later this was extended by Bell and Mohammad to stochastic differential equations [2] and then by Mao [3, 4]. Generally, the solutions of stochastic differential equations (SDEs) do not have explicit expressions except the linear SDEs. We therefore look for the approximate solutions instead of the exact ones such as the Picard iterative approximate solutions etc. Practically, to compute $X^k(t)$ by the Picard approximation, one need to compute $X^0(t), X^1(t), \ldots, X^{k-1}(t)$, which involve a lot of calculations on Itô's integrals. However, by the Carathéodory approximation we directly compute $X^k(t)$ and do not need the above mentioned stepwise iterations, which is an admirable advantage as compared to the Picard approximation [5].

The theory of G-Brownian motion and the related Itô's calculus was introduced by Peng [6]. He developed the existence and uniqueness of solutions for stochastic differential equations under G-Brownian motion (G-SDEs) under the Lipschitz conditions via the contraction method [6, 7]. While by the Picard approximation the existence theory for G-SDEs was established by Gao [8] and then by Faizullah and Piao with the method of upper and lower solutions [9]. Also see [10]. In this paper, the Carathéodory approximation scheme for G-SDEs is entrenched. It is shown that under some suitable conditions the Carathéodory approximate solutions $X^k(t), k \geq 1$, converge to the unique solution $X(t)$ of the G-SDEs in the sense that

$$\lim_{k\to\infty} \tilde{\mathbb{E}}\left[\sup_{0\leq t\leq T}\left|X^k(t)-X(t)\right|^2\right]=0.$$

This paper is organized as follows. In Section 2, some mathematical groundwork is included. In Section 3, the Carathéodory approximation scheme for G-SDEs is introduced. In Section 4, some important results and the existence and uniqueness theorem for G-SDEs with the mentioned method are given.

2. Preliminaries

This section is devoted to some basic definitions and notions concerning the work of this paper [6 – 8, 11 – 13].

Let Ω be a (non-empty) basic space and $\mathcal{H}$ be a linear space of real valued functions defined on Ω such that any arbitrary constant $c \in \mathcal{H}$ and if $X \in \mathcal{H}$ then $|X| \in \mathcal{H}$. We consider that $\mathcal{H}$ is the space of random variables.

Definition 1. A functional $\tilde{\mathbb{E}} : \mathcal{H} \to \mathbb{R}$ is called sublinear expectation, if $\forall X, Y \in \mathcal{H}$, $c \in \mathbb{R}$, and $\lambda \geq 0$ the following properties are statified:

(i) Monotonicity: If $X \geq Y$ then $\tilde{\mathbb{E}}[X] \geq \tilde{\mathbb{E}}[Y]$.

(ii) Constant preserving: $\tilde{\mathbb{E}}[c] = c$.

(iii) Subadditivity: $\tilde{\mathbb{E}}[X+Y] \leq \tilde{\mathbb{E}}[X] + \tilde{\mathbb{E}}[Y]$ or $\tilde{\mathbb{E}}[X] - \tilde{\mathbb{E}}[Y] \leq \tilde{\mathbb{E}}[X-Y]$.

(iv) Positive homogeneity: $\tilde{\mathbb{E}}[\lambda X] = \lambda \tilde{\mathbb{E}}[X]$.

The triple $(\Omega, \mathcal{H}, \tilde{\mathbb{E}})$ is called a sublinear expectation space. Consider the space of random variables $\mathcal{H}$ such that if $X_1, X_2, \ldots, X_n \in \mathcal{H}$ then $\varphi(X_1, X_2, \ldots, X_n) \in \mathcal{H}$ for each $\varphi \in \mathbb{C}_{\text{l.Lip}}(\mathbb{R}^n)$, where $\mathbb{C}_{\text{l.Lip}}(\mathbb{R}^n)$ is the space of linear functions φ defined as

$$\mathbb{C}_{\text{l.Lip}}(\mathbb{R}^n) = \Big\{ \varphi : \mathbb{R}^n \to \mathbb{R} | \exists C \in \mathbb{R}^+, \ m \in \mathbb{N} \ \text{s.t.} \ |\varphi(x) - \varphi(y)| \leq C\,(1 + |x|^m + |y|^m)\,|x-y| \Big\}$$

for $x, y \in \mathbb{R}^n$.

Definition 2. An n-dimensional random vector $Y = (Y_1, Y_2, \ldots, Y_n)$ is said to be independent from an m-dimensional random vector $X = (X_1, X_2, \ldots, X_m)$ if

$$\tilde{\mathbb{E}}[\varphi(X,Y)] = \tilde{\mathbb{E}}\left[\tilde{\mathbb{E}}[\varphi(x,Y)]_{x=X}\right] \quad \forall \varphi \in \mathbb{C}_{\text{l.Lip}}(\mathbb{R}^m \times \mathbb{R}^n) .$$

Definition 3. Two n-dimensional random vectors X and $\hat{X}$ defined, respectively, on the sublinear expectation spaces $(\tilde{\Omega}, \tilde{\mathcal{H}}, \tilde{\mathbb{E}})$ and $(\hat{\Omega}, \hat{\mathcal{H}}, \hat{\mathbb{E}})$ are said to be identically distributed, denoted by $X \sim \hat{X}$ if

$$\tilde{\mathbb{E}}[\varphi(X)] = \hat{\mathbb{E}}[\varphi(\hat{X})] \quad \forall \varphi \in \mathbb{C}_{\text{l.Lip}}(\mathbb{R}^n) .$$

Definition 4. Let $(\Omega, \mathcal{H}, \tilde{\mathbb{E}})$ be a sublinear expectation space and $X \in \mathcal{H}$ with

$$\bar{\sigma}^2 = \tilde{\mathbb{E}}[X^2], \ \underline{\sigma}^2 = -\tilde{\mathbb{E}}[-X^2] .$$

Then X is said to be G-distributed or $\mathcal{N}(0; [\bar{\sigma}^2, \underline{\sigma}^2])$-distributed if $\forall a, b \geq 0$, we have

$$aX + bY \sim \sqrt{a^2 + b^2} X ,$$

for each $Y \in \mathcal{H}$ which is independent to X and $Y \sim X$.

2.1. G-expectation and G-Brownian Motion

Let $\Omega = C_0(\mathbb{R}^+)$, that is, the space of all $\mathbb{R}$-valued continuous paths $(w_t)_{t \in \mathbb{R}^+}$ with $w_0 = 0$ equipped with the distance

$$\rho(w^1, w^2) = \sum_{k=1}^{\infty} \frac{1}{2^k} \left(\max_{t \in [0,k]} |w_t^1 - w_t^2| \wedge 1 \right) ,$$

and consider the canonical process $B_t(w) = w_t$ for $t \in [0, \infty)$, $w \in \Omega$, then for each fixed $T \in [0, \infty)$, we have

$$\mathrm{L}_{\text{ip}}(\Omega_T) = \{ \varphi(B_{t_1}, B_{t_2}, \ldots, B_{t_n}) : t_1, \ldots, t_n \in [0, T] , \varphi \in \mathbb{C}_{\text{l.Lip}}(\mathbb{R}^n) , n \in \mathbb{N} \} ,$$

where $\mathrm{L}_{\text{ip}}(\Omega_t) \subseteq \mathrm{L}_{\text{ip}}(\Omega_T)$ for $t \leq T$ and $\mathrm{L}_{\text{ip}}(\Omega) = \cup_{m=1}^{\infty} \mathrm{L}_{\text{ip}}(\Omega_m)$.

Consider a sequence $\{\xi_i\}_{i=1}^{\infty}$ of n-dimensional random vectors on a sublinear expectation space $(\hat{\Omega}, \hat{\mathcal{H}}_p, \hat{\mathbb{E}})$ such that ξ_{i+1} is independent of $(\xi_1, \xi_2, \ldots, \xi_i)$ for each $i = 1, 2, \ldots, n-1$, and ξ_i is G-normally distributed. Then a sublinear expectation $\tilde{\mathbb{E}}[.]$ defined on $\mathrm{L}_{\text{ip}}(\Omega)$ is introduced as follows.

For $0 = t_0 < t_1 < \ldots < t_n < \infty$, $\varphi \in \mathbb{C}_{\text{l.Lip}}(\mathbb{R}^n)$ and each

$$X = \varphi\left(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}\right) \in \mathrm{L}_{\text{ip}}(\Omega) ,$$
$$\tilde{\mathbb{E}}\left[\varphi\left(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}\right)\right] = \hat{\mathbb{E}}\left[\varphi\left(\sqrt{t_1 - t_0}\,\xi_1, \ldots, \sqrt{t_n - t_{n-1}}\,\xi_n\right)\right] .$$

Definition 5. The sublinear expectation $\tilde{\mathbb{E}} : \mathrm{L}_{\text{ip}}(\Omega) \to \mathbb{R}$ defined above is called a G-expectation and the corresponding canonical process $\{B_t, t \geq 0\}$ is called a G-Brownian motion.

The completion of $\mathrm{L}_{\text{ip}}(\Omega)$ under the norm $\|X\|_p = (\tilde{\mathbb{E}}[|X|^p])^{1/p}$ for $p \geq 1$ is denoted by $L_G^p(\Omega)$ and $L_G^p(\Omega_t) \subseteq L_G^p(\Omega_T) \subseteq L_G^p(\Omega)$ for $0 \leq t \leq T < \infty$. The filtration generated by the canonical process $\{B_t, t \geq 0\}$ is denoted by $\mathcal{F}_t = \sigma\{B_s, 0 \leq s \leq t\}$, $\mathcal{F} = \{\mathcal{F}_t\}_{t \geq 0}$.

2.2. Itô's Integral of G-Brownian Motion

For any $T \in \mathbb{R}^+$, a finite ordered subset $\pi_T = \{t_0, t_1, \ldots, t_N\}$ such that $0 = t_0 < t_1 < \ldots < t_N = T$ is a partition of $[0, T]$ and

$$\mu(\pi_T) = \max\{|t_{i+1} - t_i| : i = 0, 1, \ldots, N-1\} .$$

A sequence of partitions of $[0,T]$ is denoted by $\pi_T^N = \{t_0^N, t_1^N, \dots, t_N^N\}$ such that $\lim_{N\to\infty} \mu(\pi_T^N) = 0$.

Consider the following simple process: Let $p \geq 1$ be fixed. For a given partition $\pi_T = \{t_0, t_1, \dots, t_N\}$ of $[0,T]$,

$$\eta_t(w) = \sum_{i=0}^{N-1} \xi_i(w) I_{[t_i, t_{i+1}]}(t), \tag{1}$$

where $\xi_i \in L_G^p(\Omega_{t_i})$, $i = 0,1,\dots,N-1$. The collection containing the above type of processes is denoted by $M_G^{p,0}(0,T)$. The completion of $M_G^{p,0}(0,T)$ under the norm $\|\eta\| = \left\{\int_0^T \tilde{\mathbb{E}}[|\eta_v|^p] \, \mathrm{d}v\right\}^{1/p}$ is denoted by $M_G^p(0,T)$ and for $1 \leq p \leq q$, $M_G^p(0,T) \supset M_G^q(0,T)$.

Definition 6. For each $\eta_t \in M_G^{2,0}(0,T)$, Itô's integral of G-Brownian motion is defined by

$$I(\eta) = \int_0^T \eta_v \, \mathrm{d}B_v = \sum_{i=0}^{N-1} \xi_i \left(B_{t_{i+1}} - B_{t_i}\right).$$

Definition 7. An increasing continuous process $\{\langle B\rangle_t, t \geq 0\}$ with $\langle B\rangle_0 = 0$, defined by

$$\langle B\rangle_t = B_t^2 - 2\int_0^t B_v \, \mathrm{d}B_v,$$

is called the quadratic variation process of G-Brownian motion.

3. The Carathéodory Approximation Scheme for G-SDEs

We now consider the following stochastic differential equation under G-Brownian motion (G-SDE):

$$\begin{aligned} X(t) = X_0 &+ \int_0^t f(v, X(v)) \, \mathrm{d}v \\ &+ \int_0^t g(v, X(v)) \, \mathrm{d}\langle B\rangle(v) \\ &+ \int_0^t h(v, X(v)) \, \mathrm{d}B(v), \quad t \in [0,T], \end{aligned} \tag{2}$$

where the initial condition $X_0 \in \mathbb{R}^n$ is a given constant, the coefficients $f(t,x), g(t,x), h(t,x) \in M_G^2(0,T;\mathbb{R}^n)$ [7], and $\{\langle B\rangle(t), t \geq 0\}$ is the quadratic variation process of one-dimensional (only for simplicity) G-Brownian motion $\{B(t), t \geq 0\}$. A process X_t belongs to the above mentioned space satisfying the G-SDE (2) is said to be its solution. The Carathéodory approximation scheme for G-SDE (2) is given as follows. For any integer $k \geq 1$, we define $X^k(t) = X_0$ on $t \in [-1,0]$ and

$$\begin{aligned} X^k(t) = X_0 &+ \int_0^t f\left(v, X^k\left(v - \frac{1}{k}\right)\right) \mathrm{d}v \\ &+ \int_0^t g\left(v, X^k\left(v - \frac{1}{k}\right)\right) \mathrm{d}\langle B\rangle(v) \\ &+ \int_0^t h\left(v, X^k\left(v - \frac{1}{k}\right)\right) \mathrm{d}B(v) \quad \text{on } t \in (0,T]. \end{aligned} \tag{3}$$

Also we have to note that $X^k(t)$ can be computed step by step on the intervals $[0, \frac{1}{k})$, $[\frac{1}{k}, \frac{2}{n})$, …with the procedure given below. For $t \in [0, \frac{1}{k})$,

$$\begin{aligned} X^k(t) = X_0 &+ \int_0^t f(v, X_0) \, \mathrm{d}v \\ &+ \int_0^t g(v, X_0) \, \mathrm{d}\langle B\rangle(v) + \int_0^t h(v, X_0) \, \mathrm{d}B(v), \end{aligned}$$

then for $t \in [\frac{1}{k}, \frac{2}{k})$,

$$\begin{aligned} X^k(t) = X^k\left(\frac{1}{k}\right) &+ \int_{\frac{1}{k}}^t f\left(v, X^k\left(v - \frac{1}{k}\right)\right) \mathrm{d}v \\ &+ \int_{\frac{1}{k}}^t g\left(v, X^k\left(v - \frac{1}{k}\right)\right) \mathrm{d}\langle B\rangle(v) \\ &+ \int_{\frac{1}{k}}^t h\left(v, X^k\left(v - \frac{1}{k}\right)\right) \mathrm{d}B(v), \end{aligned}$$

and so on. We make the following assumptions.

Suppose that $\hat{K}$ and $\tilde{K}$ are positive constants such that the following two conditions hold:

(i) Lipschitz condition: For all $x, y \in \mathbb{R}^n$ and $t \in [0,T]$

$$|J(t,y) - J(t,x)|^2 \leq \hat{K}|y - x|^2, \tag{4}$$

where $J = f, g$, and h, respectively.

(ii) Linear growth condition: For all $x \in \mathbb{R}^n$ and $t \in [0,T]$

$$|J(t,x)|^2 \leq \tilde{K}(1 + |x|^2), \tag{5}$$

where $J = f, g$, and h, respectively.

By the definition of $X^k(t)$, Burkholder–Davis–Gundy (BDG) inequalities [8] and above assumptions it is easy to see that the sequence $\{X^k(t)\}_{t\in[0,T]}$ for each $k \geq 1$ is well defined in $M_G^2(0,T;\mathbb{R}^n)$.

4. Main Results

The following lemma is very important; it will be used in the forthcoming results.

Lemma 1. *Suppose that the linear growth condition (5) holds. Then we have*

(i) For all $k \geq 1$,

$$\sup_{0\leq t\leq T} \tilde{\mathbb{E}}\left[\left|X^k(t)\right|^2\right] \leq C, \tag{6}$$

where $C = (1+4\tilde{\mathbb{E}}[|X_0|^2])\,\mathrm{e}^{4\tilde{K}K_1T}$, $K_1 = (TC_1 + C_2T + C_3)$, *and* C_1, C_2, C_3 *are arbitrary positive constants.*

(ii) For all $k \geq 1$ *and* $0 \leq s < t \leq T$,

$$\tilde{\mathbb{E}}\left[\left|X^k(t) - X^k(s)\right|^2\right] \leq K_2(t-s), \tag{7}$$

where $K_2 = 3\tilde{K}K_1(1+C)$.

Proof. To prove (i), from (3) it follows that for $t \in [0,T]$,

$$|X^k(t)|^2 \leq 4|X_0|^2 + 4\left|\int_0^t f\left(v, X^k\left(v-\frac{1}{k}\right)\right)\mathrm{d}v\right|^2 + 4\left|\int_0^t g\left(v, X^k\left(v-\frac{1}{k}\right)\right)\mathrm{d}\langle B\rangle(v)\right|^2 + 4\left|\int_0^t h\left(v, X^k\left(v-\frac{1}{k}\right)\right)\mathrm{d}B(v)\right|^2.$$

Taking G-expectation, using the BDG inequalities [8] and the linear growth condition (5), we have

$$\begin{aligned}
&\sup_{0\leq u\leq t} \tilde{\mathbb{E}}\left[\left|X^k(u)\right|^2\right] \\
&\leq 4\tilde{\mathbb{E}}\left[|X_0|^2\right] + 4\tilde{K}(TC_1 + C_2T + C_3) \cdot \int_0^t \left(1 + \tilde{\mathbb{E}}\left[\left|X^k\left(v-\frac{1}{k}\right)\right|^2\right]\right)\mathrm{d}v \\
&\leq 4\tilde{\mathbb{E}}\left[|X_0|^2\right] + 4\tilde{K}K_1 \cdot \int_0^t \left(1 + \sup_{0\leq u\leq v} \tilde{\mathbb{E}}\left[\left|X^k(u)\right|^2\right]\right)\mathrm{d}v
\end{aligned}$$

for all $t \in [0,T]$. Then the well-known Gronwall inequality yields

$$1 + \sup_{0\leq u\leq t} \tilde{\mathbb{E}}\left[\left|X^k(u)\right|^2\right] \leq \left(1 + 4\tilde{\mathbb{E}}\left[|X_0|^2\right]\right)\mathrm{e}^{4\tilde{K}K_1t}.$$

Consequently,

$$\sup_{0\leq t\leq T} \tilde{\mathbb{E}}\left[\left|X^k(t)\right|^2\right] \leq \left(1 + 4\tilde{\mathbb{E}}\left[|X_0|^2\right]\right)\mathrm{e}^{4\tilde{K}K_1T} = C$$

which is the required result (6).

To prove (ii), it is easy to see that for any $k \geq 1$ and $0 \leq s < t \leq T$,

$$X^k(t) - X^k(s) = \int_s^t f\left(v, X^k\left(v-\frac{1}{k}\right)\right)\mathrm{d}v + \int_s^t g\left(v, X^k\left(v-\frac{1}{k}\right)\right)\mathrm{d}\langle B\rangle(v) + \int_s^t h\left(v, X^k\left(v-\frac{1}{k}\right)\right)\mathrm{d}B(v).$$

Hence by using the BDG inequalities [8] and the linear growth condition (5), we have

$$\begin{aligned}
&\tilde{\mathbb{E}}\left[\sup_{s\leq r<u\leq t}\left|X^k(u) - X^k(r)\right|^2\right] \\
&\leq 3\tilde{\mathbb{E}}\left[\sup_{s\leq r<u\leq t}\left|\int_r^u f\left(v, X^k\left(v-\frac{1}{k}\right)\right)\mathrm{d}v\right|^2\right] \\
&\quad + 3\tilde{\mathbb{E}}\left[\sup_{s\leq r<u\leq t}\left|\int_r^u g\left(v, X^k\left(v-\frac{1}{k}\right)\right)\mathrm{d}\langle B\rangle(v)\right|^2\right] \\
&\quad + 3\tilde{\mathbb{E}}\left[\sup_{s\leq r<u\leq t}\left|\int_r^u h\left(v, X^k\left(v-\frac{1}{k}\right)\right)\mathrm{d}B(v)\right|^2\right] \\
&\leq 3C_1(t-s)\int_s^t \tilde{\mathbb{E}}\left[\left|f\left(v, X^k\left(v-\frac{1}{k}\right)\right)\right|^2\right]\mathrm{d}v \\
&\quad + 3C_2(t-s)\int_s^t \tilde{\mathbb{E}}\left[\left|g\left(v, X^k\left(v-\frac{1}{k}\right)\right)\right|^2\right]\mathrm{d}v \\
&\quad + 3C_3\int_s^t \tilde{\mathbb{E}}\left[\left|h\left(v, X^k\left(v-\frac{1}{k}\right)\right)\right|^2\right]\mathrm{d}v \\
&\leq 3\tilde{K}\left(C_1(t-s) + C_2(t-s) + C_3\right) \cdot \int_s^t \left(1 + \tilde{\mathbb{E}}\left[\left|X^k\left(v-\frac{1}{k}\right)\right|^2\right]\right)\mathrm{d}v \\
&\leq 3\tilde{K}\left(C_1T + C_2T + C_3\right) \cdot \int_s^t \left(1 + \tilde{\mathbb{E}}\left[\left|X^k\left(v-\frac{1}{k}\right)\right|^2\right]\right)\mathrm{d}v \\
&= 3\tilde{K}K_1(t-s) + 3\tilde{K}K_1\int_s^t \tilde{\mathbb{E}}\left[\left|X^k\left(v-\frac{1}{k}\right)\right|^2\right]\mathrm{d}v,
\end{aligned}$$

by using (6), we have

$$\tilde{\mathbb{E}}\left[\left|X^k(t)-X^k(s)\right|^2\right] \leq 3\tilde{K}K_1(t-s)+3\tilde{K}K_1C(t-s)$$
$$= 3\tilde{K}K_1(1+C)(t-s)$$
$$= K_2(t-s)\,,$$

which is the required result (7). □

Theorem 1. *Suppose that the respective Lipschitz and linear growth conditions (4) and (5) hold. Also assume that $X(t)$ is the unique solution of the G-SDE (2). Then*

$$\tilde{\mathbb{E}}\left[\sup_{0\leq t\leq T}\left|X^k(t)-X(t)\right|^2\right] \leq \frac{K_3}{k}\,, \tag{8}$$

where $K_3 = 6\hat{K}K_1K_2\,\mathrm{e}^{6\hat{K}K_1T}$.

Proof. Using the BDG inequalities [8], the Lipschitz condition (4) and the inequality (7), we have

$$\begin{aligned}
&\tilde{\mathbb{E}}\left[\sup_{0\leq u\leq t}\left|X^k(u)-X(u)\right|^2\right]\\
&\leq 3\hat{K}K_1\int_0^t \tilde{\mathbb{E}}\left[\left|X^k\left(v-\frac{1}{k}\right)-X(v)\right|^2\right]\mathrm{d}v\\
&\leq 6\hat{K}K_1\int_0^t \tilde{\mathbb{E}}\left[\left|X^k(v)-X^k\left(v-\frac{1}{k}\right)\right|^2\right]\mathrm{d}v\\
&\quad+6\hat{K}K_1\int_0^t \tilde{\mathbb{E}}\left[\left|X^k(v)-X(v)\right|^2\right]\mathrm{d}v\\
&\leq 6\hat{K}K_1K_2\frac{1}{k}\\
&\quad+6\hat{K}K_1\int_0^t \tilde{\mathbb{E}}\left[\sup_{0\leq u\leq v}\left|X^k(u)-X(u)\right|^2\right]\mathrm{d}v\,.
\end{aligned}$$

Then the Gronwall inequality gives

$$\tilde{\mathbb{E}}\left[\sup_{0\leq u\leq t}\left|X^k(u)-X(u)\right|^2\right] \leq 6\hat{K}K_1K_2\,\mathrm{e}^{6\hat{K}K_1T}\frac{1}{k}\,.$$

Consequently,

$$\tilde{\mathbb{E}}\left[\sup_{0\leq t\leq T}\left|X^k(t)-X(t)\right|^2\right] \leq \frac{K_3}{k}\,,$$

where $K_3 = 6\hat{K}K_1K_2\,\mathrm{e}^{6\hat{K}K_1T}$. □

Remark 1. From the above inequality (8), it is obvious to see that

$$\lim_{k\to\infty}\tilde{\mathbb{E}}\left[\sup_{0\leq t\leq T}\left|X^k(t)-X(t)\right|^2\right] = 0\,,$$

that is, the Carathéodory approximate solutions $X^k(t)$ converge to the unique solution $X(t)$ of the G-SDE (2).

Now, we give a very general and main result. In the above theorem it was supposed that the G-SDE (2) has a unique solution. In the next theorem, without making this supposition, we use complectly the Carathéodory approximation scheme to obtain the existence and uniqueness of solutions for the G-SDE (2).

Theorem 2. *Under the hypothesis (4) and (5), the sequence $\{X^k, k\geq 1\}$ defined by (3) is a Cauchy sequence in $M_G^2(0,T;\mathbb{R}^n)$ and converges to a limit $X(t)$ which is a unique solution of the G-SDE (2).*

Proof. To show that $\{X^k, k\geq 1\}$ is a Cauchy sequence, let $l>k$, then for $t\in[0,T]$,

$$\begin{aligned}
\left|X^l(t)-X^k(t)\right|^2 &\leq 3\left|\int_0^t\left[f\left(v,X^l\left(v-\frac{1}{l}\right)\right)\right.\right.\\
&\quad\left.\left.-f\left(v,X^k\left(v-\frac{1}{k}\right)\right)\right]\mathrm{d}v\right|^2\\
&\quad+3\left|\int_0^t\left[g\left(v,X^l\left(v-\frac{1}{l}\right)\right)\right.\right.\\
&\quad\left.\left.-g\left(v,X^k\left(v-\frac{1}{k}\right)\right)\right]\mathrm{d}\langle B\rangle(v)\right|^2\\
&\quad+3\left|\int_0^t\left[h\left(v,X^l\left(v-\frac{1}{l}\right)\right)\right.\right.\\
&\quad\left.\left.-h\left(v,X^k\left(v-\frac{1}{k}\right)\right)\right]\mathrm{d}B(v)\right|^2.
\end{aligned}$$

By using the BDG inequalities [8], Lipschitz condition (4), and Lemma 1, we have

$$\begin{aligned}
&\sup_{0\leq t\leq T}\tilde{\mathbb{E}}\left[\left|X^l(t)-X^m(t)\right|^2\right] \leq 3\hat{K}\left(C_1T+C_2T+C_3\right)\\
&\quad\cdot\int_0^t\tilde{\mathbb{E}}\left[\left|X^l\left(v-\frac{1}{l}\right)-X^k\left(v-\frac{1}{k}\right)\right|^2\right]\mathrm{d}v\\
&\leq 6\hat{K}K_1\int_0^t\tilde{\mathbb{E}}\left[\left|X^l\left(v-\frac{1}{l}\right)-X^k\left(v-\frac{1}{l}\right)\right|^2\right]\mathrm{d}v\\
&\quad+6\hat{K}K_1\int_0^t\tilde{\mathbb{E}}\left[\left|X^k\left(v-\frac{1}{l}\right)-X^k\left(v-\frac{1}{k}\right)\right|^2\right]\mathrm{d}v\\
&\leq 6\hat{K}K_1\int_0^t\sup_{0\leq u\leq v}\tilde{\mathbb{E}}\left[\left|X^l(u)-X^k(u)\right|^2\right]\mathrm{d}v\\
&\quad+6\hat{K}K_1K_2T\left(\frac{1}{k}-\frac{1}{l}\right).
\end{aligned}$$

Hence, by Gronwall's inequality,

$$\sup_{0\le t\le T} \tilde{\mathbb{E}}\left[\left|X^l(t)-X^k(t)\right|^2\right] \le K_4\,\mathrm{e}^{6\hat{K}K_1T}\left(\frac{1}{k}-\frac{1}{l}\right), \quad (9)$$

where $K_4 = 6\hat{K}K_1K_2T$. From (9) it is clear that the sequence $\{X^k(t), k \ge 1\}$ is a Cauchy sequence in $M_G^2(0,T;\mathbb{R}^n)$ and denote its limits by $X(t)$. Letting $l \to \infty$ in (9) yields

$$\sup_{0\le t\le T} \tilde{\mathbb{E}}\left[\left|X^k(t)-X(t)\right|^2\right] \le K_4\,\mathrm{e}^{6\hat{K}K_1T}\frac{1}{k}. \quad (10)$$

Next, we have to show that $X(t)$ satisfies the G-SDE (2). Let $t \in [0,T]$, then by the BDG inequalities [8], the Lipschitz condition (4), and Lemma 1, we obtain

$$\begin{aligned}
&\tilde{\mathbb{E}}\left[\left|\int_0^t\left[f\left(v,X^k\left(v-\frac{1}{k}\right)\right)-f\big(v,X(v)\big)\right]\mathrm{d}v\right|^2\right]\\
&\quad+\tilde{\mathbb{E}}\left[\left|\int_0^t\left[g\left(v,X^k\left(v-\frac{1}{k}\right)\right)\right.\right.\right.\\
&\quad\left.\left.\left.-g\big(v,X(v)\big)\right]\mathrm{d}\langle B\rangle(v)\right|^2\right]\\
&\quad+\tilde{\mathbb{E}}\left[\left|\int_0^t\left[h\left(v,X^k\left(v-\frac{1}{k}\right)\right)\right.\right.\right.\\
&\quad\left.\left.\left.-h\big(v,X(v)\big)\right]\mathrm{d}B(v)\right|^2\right]\\
&\le \hat{K}K_1\int_0^t\tilde{\mathbb{E}}\left[\left|X^k\left(v-\frac{1}{k}\right)-X(v)\right|^2\right]\mathrm{d}v\\
&\le 2\hat{K}K_1\int_0^t\tilde{\mathbb{E}}\left[\left|X^k(v)-X^k\left(v-\frac{1}{k}\right)\right|^2\right]\mathrm{d}v\\
&\quad+2\hat{K}K_1\int_0^t\tilde{\mathbb{E}}\left[\left|X^k(v)-X(v)\right|^2\right]\mathrm{d}v\\
&\le 2\hat{K}K_1T\frac{1}{k}+2\hat{K}K_1\int_0^t\tilde{\mathbb{E}}\left[\left|X^k(v)-X(v)\right|^2\right]\mathrm{d}v\\
&\le 2\hat{K}K_1T\frac{1}{k}+2\hat{K}K_1\int_0^t K_4\,\mathrm{e}^{6\hat{K}K_1T}\frac{1}{k}\,\mathrm{d}v\\
&= 2\hat{K}K_1T\frac{1}{k}+2\hat{K}K_1K_4T\,\mathrm{e}^{6\hat{K}K_1T}\frac{1}{k}\\
&= K_5\frac{1}{k}\to 0 \text{ as } k\to\infty,
\end{aligned}$$

where $K_5 = 2\hat{K}K_1T(1+K_4\,\mathrm{e}^{6\hat{K}K_1T})$. Thus by taking limits as $k \to \infty$ in (3), we get $X(t)$ as a solution of (2). To show the uniqueness of solutions, contrary suppose that $X(t)$ and $Y(t)$ are two solutions of the G-SDE (2). Then by the BDG inequalities [8] and Lipschitz condition (4) etc., one can obtain in a similar fashion as above

$$\sup_{0\le t\le T}\tilde{\mathbb{E}}\left[|X(t)-Y(t)|^2\right]=0,$$

which yields $X(t) = Y(t)$ for $t \in [0,T]$. □

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[1] E. A. Coddington and N. Levinson, Theory of Ordinary Differentail Equations, McGraw-Hill, New York, Toronto, London 1955.
[2] D. R. Bell and S. E. A. Mohammed, Stoch. Stoch. Rep. **4**, 293 (1989).
[3] X. Mao, Numer. Funct. Anal. Optimiz. **12**, 525 (1991).
[4] X. Mao, Stoch. Anal. Appl. **3**, 355 (1994).
[5] X. Mao, Stochastic Differentail Equations and their Applications, Horwood Publishing, Chichester, England 1997.
[6] S. Peng, G-Expectation, G-Brownian Motion and Related Stochastic Calculus of Itô's Type, Stochastic Analysis and Applications, 541–567, Abel Symp., 2, Springer, Berlin 2007.
[7] S. Peng, Stoch. Proc. Appl. **12**, 2223 (2008).
[8] F. Gao, Stoch. Proc. Appl. **10**, 3356 (2009).
[9] F. Faizullah and D. Piao, Int. J. Phy. Sci. **3**, 432 (2012).
[10] Y. Ren and L. Hu, Stat. Prob. Lett. **81**, 580 (2011).
[11] L. Denis, M. Hu, and S. Peng, Potential Anal. **34**, 139 (2011).
[12] F. Faizullah and U. Rahman, Int. J. Curr. Res. Rev. **4**, 76 (2012).
[13] X. Li and S. Peng, Stoch. Proc. Appl. **121** 1492 (2011).