A Novel Analytical Implementation of Nonlinear Volterra Integral Equations

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This article aims at preferring a new and viable algorithm, specifically a two-step homotopy perturbation transform algorithm (TSHPTA). This novel technique is a feasible way in finding exact solutions with a small amount of calculations. As a simple but typical example, it demonstrates the strength and the great potential of the two-step homotopy perturbation transform method to solve nonlinear Volterra-type integral equations efficiently. The results reveal that the proposed scheme is suitable for the nonlinear Volterra equations.

Key words: Two Step Homotopy Perturbation Transforms Algorithm; Nonlinear Volterra Integral Equations.

1. Introduction

Modelling numerous problems of engineering, applied physics, and other disciplines leads to linear and nonlinear Volterra integral equations of the second kind. The Volterra integral equations were initiated by Vito Volterra and then analysed by Traian Lalescu in 1908.

Several schemes such as the finite difference method, finite element method, and the shooting method have been applied separately to grip the Volterra integral equations numerically [1 – 4]. Though, there exist various difficulties for example a mesh improvement, a stable condition, and selection of small or large parameters etc.

To prevent these complexities, different analytical methods such as Adomian decomposition methods [5, 6], homotopy analysis method [7, 8], homotopy perturbation method [9, 10], variational iterative method [11, 12], Laplace decomposition method [13 – 16], variational iterative decomposition method [17, 18], modified Laplace decomposition method [19], and two-step Laplace decomposition method [20 – 22] were introduced for solving linear and nonlinear problems in many fields.

Enthused and motivated by the current study in this area, we now propose a fairly new algorithm, the two-step homotopy perturbation transform algorithm, for solving Volterra-type integral equations which is a combination of Laplace transforms and standard homotopy perturbation method. We use He's polynomials for the nonlinear parts in our understudy problems [23].

The fundamental inspiration of this article is to suggest a mathematical approach without perturbation, restrictive assumptions or linearization. The two-step homotopy perturbation transform algorithm which accurately computes the solution within two step and is of great interest to applied sciences. The most important improvement of the method is that it can be applied directly to integral equations. Moreover, the proposed method is capable of greatly reducing the size of computational work. Some examples are given to measure the efficiency and the worth of the recommended algorithm.

The present paper is organized as follows: In Section 2, the TSHPTA for nonlinear Volterra integral equations is presented. The numerical applications of TSHPTA are given in Section 3. In Section 4, the conclusion is briefly discussed.

2. Two-Step Homotopy Perturbation Transforms Algorithm

The nonlinear Volterra-type integral equations are of the form [1]

$$w(x) = r(x) + \Omega \int_0^x G(x,t) w^j(t)\,\mathrm{d}t\,, \tag{1}$$

where $r(x)$ is a non-homogeneous term, Ω a parameter, $G(x,t)$ the kernel of the equation, and $w^j(t)$ is the nonlinear term. Taking the Laplace transform of both sides of the equation yields

$$£[w(x)] = £[r(x)] + £\left[\Omega \int_0^x G(x,t) w^j(t)\,\mathrm{d}t\right]. \tag{2}$$

Operating the inverse Laplace transform of both sides of (2), we get

$$w(x) = r(x) + £^{-1}\left[£\left[\Omega \int_0^x G(x,t) w^j(t)\,\mathrm{d}t\right]\right]. \tag{3}$$

In order to use the homotopy perturbation method, we have to represent our solution in a convergence control parameter dependent infinite series of the form

$$w = \sum_{j=0}^{\infty} q^j w_j\,. \tag{4}$$

The nonlinear term $w^j(t)$ can be decomposed as He's polynomial given as follows:

$$w^j(t) = \sum_{j=0}^{\infty} q^j H_j\,, \tag{5}$$

where H_j is the He's polynomial [1], and $q \in [0,1]$ is an embedding parameter. Making use of (4)–(5) in (3), the solution can be written as

$$\sum_{j=0}^{\infty} q^j w_j(x) = r(x) + q£^{-1}\left[£\left[\Omega \int_0^x G(x,t) H_j\,\mathrm{d}t\right]\right]. \tag{6}$$

From (6), He's polynomial can be constructed by a number of ways; but at the moment, we employ the following recursive formulation:

$$H_j(w_0,\ldots,w_j) = \frac{1}{j!}\frac{\partial^j}{\partial q^j}\left[N\left(\sum_{i=0}^{\infty} q^j w_j\right)\right]_{q=0}, \quad j = 0,1,2,\ldots\,. \tag{7}$$

By comparing like powers of the embedding parameter, our recursive relation is given by

$$\begin{aligned}
q^0 &: w_0 = r(x)\,,\\
q^1 &: w_1 = £^{-1}\left[£\left[\Omega \int_0^x G(x,t) H_0\,\mathrm{d}t\right]\right],\\
q^2 &: w_2 = £^{-1}\left[£\left[\Omega \int_0^x G(x,t) H_1\,\mathrm{d}t\right]\right],\\
q^3 &: w_3 = £^{-1}\left[£\left[\Omega \int_0^x G(x,t) H_2\,\mathrm{d}t\right]\right],\\
q^4 &: w_4 = £^{-1}\left[£\left[\Omega \int_0^x G(x,t) H_3\,\mathrm{d}t\right]\right],\\
&\vdots
\end{aligned} \tag{8}$$

where $r(x)$ represents the non-homogeneous term. Now we illustrate TSHTA after applying the inverse operator and represented $r(x)$ by an additional function ξ as follows:

$$\xi = r(x)\,. \tag{9}$$

We decomposed our non-homogeneous function into a finite number of functions as

$$\xi = r_0(x) + r_1(x) + r_2(x) + \ldots + r_j(x)\,. \tag{10}$$

Next step is to find a function or combination of functions that satisfy the original equation; we take that function as an initial approximation and put the rest of the part in the first-order solution. On the other hand, if no such function is found in the non-homogeneous part, we continue with the standard homotopy perturbation method (HPM) as given below:

$$w_{j+1}(x) = q£^{-1}\left[£\left[\Omega \int_0^x G(x,t) H_j\,\mathrm{d}t\right]\right], \quad j \geq 0\,. \tag{11}$$

3. Case Studies

In this segment, two examples are given in order to show the usefulness of TSHPTA. Also we calculate the L_2-norm and the relative error for each example.

3.1. Example 1

Consider a nonlinear Volterra integral equation [2, 3] of the form

$$w(x) = x + \frac{x^5}{5} - \int_0^x t w^3(t)\,\mathrm{d}t\,. \tag{12}$$

Applying the Laplace transform algorithm, we have

$$\pounds\,[w(x)] = \frac{1}{s^2} + \frac{5!}{s^6} - \pounds\left[\int_0^x t w^3(t)\,\mathrm{d}t\right]. \quad (13)$$

Applying the inverse Laplace transform to (13), we get

$$w(x) = x + \frac{x^5}{5} - \pounds^{-1}\left[\pounds\left[\int_0^x t w^3(t)\,\mathrm{d}t\right]\right]. \quad (14)$$

The homotopy perturbation method (HPM) [6, 7] assumes a series solution of the function $w(x)$ given by

$$w = \sum_{j=0}^{\infty} q^j w_j(x). \quad (15)$$

Using (15) into (14) yields

$$\sum_{j=0}^{\infty} q^j w_j(x) = x + \frac{x^5}{5} - q\pounds^{-1}\left[\pounds\left[\int_0^x t H_j(t)\right]\right], \quad (16)$$

where H_j is He's polynomial [23], that represents the nonlinear term. Therefore, the He's polynomial is given as

$$\sum_{j=0}^{\infty} H_{1j}(t) = w^3(t). \quad (17)$$

The few components of the He's polynomial are given as follows:

$$\begin{aligned}
H_{10}(w_0) &= w_0^3(t),\\
H_{11}(w_0, w_1) &= 3w_0^2(t) w_1(t),\\
H_{12}(w_0, w_1, w_2) &= 3w_0^2(t) w_2(t) + 3w_0(t) w_1^2(t), \quad (18)\\
&\vdots\\
H_{1j}(w_0, w_1, w_2, \ldots, w_j) &= \sum_{i=0}^{j}\sum_{s=0}^{i} w_{j-i}(t) w_{i-s}(t) w_s(t).
\end{aligned}$$

From (16), our required recursive relation is given as

$$\sum_{j=0}^{\infty} q^j w_j(x) = x + \frac{x^5}{5} - q\pounds^{-1}\left[\pounds\left[\int_0^x t H_{1j}\,\mathrm{d}t\right]\right], \quad j \geq 0. \quad (19)$$

On comparing the like powers of q, we have

$$q^0 : w_0(x) = x + \frac{x^5}{5}, \quad (20)$$

$$\begin{aligned}
q^1 : w_1(x) &= -\pounds^{-1}\left[\pounds\left[\int_0^x t H_0\,\mathrm{d}t\right]\right] \quad (21)\\
&= -\frac{1}{5}x^5 - \frac{1}{15}x^9 - \frac{3}{325}x^{13} - \frac{1}{2125}x^{17},\\
&\vdots
\end{aligned}$$

3.2. The Two-Step Homotopy Perturbation Transforms Algorithm for Example 1

In order to apply this new algorithm, we first decompose the non-homogeneous part into a set of functions:

$$r(x) = r_0(x) + r_1(x), \quad (22)$$

where

$$r_0(x) = x, \quad r_1(x) = \frac{x^5}{5}. \quad (23)$$

It can be seen that r_1 does not satisfy (12). Therefore we select $w_0 = r_0$ and verify that r_0 satisfy (12). So the exact solution is obtained within two steps as follows:

$$q^0 : w_0(x) = x, \quad (24)$$

$$\begin{aligned}
q^1 : w_1(x) &= \frac{x^5}{5} - \pounds^{-1}\left[\pounds\left[\int_0^x t H_{10}(t)\,\mathrm{d}t\right]\right] \quad (25)\\
&= 0,\\
&\vdots
\end{aligned}$$

$$q^j : w_j(x) = 0, \quad j \geq 1. \quad (26)$$

Consequently, the solution by TSHPTA is

$$w = \lim_{q \to 1} w_j(x) = x. \quad (27)$$

3.3. Example 2

Consider another nonlinear Volterra-type integral equation of the form [3]

$$w(x) = \mathrm{e}^x - \frac{1}{3}x\mathrm{e}^{3x} + \frac{1}{3}x + \int_0^x x w^3(t)\,\mathrm{d}t. \quad (28)$$

Applying the Laplace transform on both sides of (28) yields

$$\pounds[w(x)] = \frac{1}{s-1} - \frac{1}{3(s-3)^2} + \frac{1}{3s^2} + \pounds\left[\int_0^x xw^3(t)\,\mathrm{d}t\right]. \tag{29}$$

Applying the inverse Laplace transform to (29), we get

$$w(x) = \mathrm{e}^x - \frac{1}{3}x\mathrm{e}^{3x} + \frac{1}{3}x + \pounds^{-1}\left[\pounds\left[\int_0^x xw^3(t)\,\mathrm{d}t\right]\right]. \tag{30}$$

Applying same steps as in the first example, we get our recursive relation as follows:

$$\sum_{j=0}^{\infty} q^j w_j(x) = \mathrm{e}^x - \frac{1}{3}x\mathrm{e}^{3x} + \frac{1}{3}x + \pounds^{-1}\left[\pounds\left[\int_0^x xH_{2j}(t)\,\mathrm{d}t\right]\right], \tag{31}$$

where H_{2j} is the He's polynomial that represents a nonlinear term. The few components of the He's polynomial are given as

$$\begin{aligned} H_{20} &= w_0^3(t), \\ H_{21}(w_0, w_1) &= 3w_0^2(t)w_1(t), \\ H_{22}(w_0, w_1, w_2) &= 3w_0^2(t)w_2(t) + 3w_0(t)w_1^2(t), \\ &\vdots \\ H_{2j}(w_0, w_1, w_2, \ldots, w_j) &= \sum_{i=0}^{j}\sum_{l=0}^{k} w_{j-k}(t)w_{k-l}(t)w_l(t). \end{aligned} \tag{32}$$

From (31), our required recursive relation is

$$\sum_{j=0}^{\infty} q^j w_j(x) = \mathrm{e}^x - \frac{1}{3}x\mathrm{e}^{3x} + \frac{1}{3}x + q\pounds^{-1}\left[\pounds\left[\int_0^x tH_{2j}(t)\,\mathrm{d}t\right]\right]. \tag{33}$$

For the first few components of $w_j(x)$ by using the recursive relation (33), we have

$$\begin{aligned} q^0 &: w_0(x) = \mathrm{e}^x - \frac{1}{3}x\mathrm{e}^{3x} + \frac{1}{3}x, \\ q^1 &: w_1(x) = \pounds^{-1}\left[\pounds\left[\int_0^x tH_{02}\,\mathrm{d}t\right]\right] \\ &= \frac{2}{3}\mathrm{e}^x x - \frac{1}{4}\mathrm{e}^{2x}x + \frac{83}{243}\mathrm{e}^{3x}x + \ldots \\ &\vdots \end{aligned} \tag{34}$$

We conclude, applying HPM meets a large amount of difficulties in integration, and moreover there are no noise term phenomena in w_0.

3.4. The Two-Step Homotopy Perturbation Transforms Algorithm for Example 2

By means of TSHPTA, we decompose the function $r(x)$ as follows:

$$r(x) = r_0(x) + r_1(x) + r_2(x), \tag{35}$$

where

$$r_0(x) = \mathrm{e}^x, \quad r_1(x) = -\frac{1}{3}x\mathrm{e}^{3x}, \quad r_2(x) = \frac{1}{3}x. \tag{36}$$

r_i $(i = 0, 1, 2)$ does not satisfy (28). Now we choose $w_0 = r_0$ as a initial guess and confirm that w_0 satisfies (28); so the exact solution will be obtained immediately:

$$q^0 : w_0(x) = \mathrm{e}^x, \tag{37}$$

$$q^1 : w_1(x) = -\frac{1}{3}x\mathrm{e}^{3x} + \frac{1}{3}x + \pounds^{-1}\left[\pounds\left[\int_0^x xH_{20}(t)\,\mathrm{d}t\right]\right] = 0, \tag{38}$$

$$\vdots$$

$$q^j : w_{j+1}(x) = 0, \quad j \geq 1. \tag{39}$$

Table 1. Comparison of different errors for standard HPM and TSHPTA.

x	L_2-norm	Relative error
0.4	1.88738×10^{-15}	4.71845×10^{-15}
0.5	1.23875×10^{-12}	2.47757×10^{-12}
0.6	2.61376×10^{-10}	4.35626×10^{-10}
0.7	2.53944×10^{-8}	3.62776×10^{-8}
0.8	1.43495×10^{-6}	1.79368×10^{-6}
0.9	0.0000552025	0.00162015

Table 2. Comparison of different errors for standard HPM and TSHPTA.

x	L_2-norm	Relative error
0.01	1.35002×10^{-4}	1.33659×10^{-4}
0.02	5.46695×10^{-4}	5.35871×10^{-4}
0.03	1.24513×10^{-3}	1.20833×10^{-3}
0.04	2.24034×10^{-3}	3.62776×10^{-8}
0.05	3.54237×10^{-3}	1.79368×10^{-6}
0.06	5.16115×10^{-3}	0.00162015

Therefore, the solution by TSHPTA is

$$w = \lim_{q \to 1} w_j(x) = e^x . \tag{40}$$

4. Conclusion

In the present work, we employed the two-step homotopy perturbation transform method for solving nonlinear Volterra-type integral equations. The findings in two illustrated examples clearly show the consistency and accuracy of the proposed algorithm. Our newly suggested method is applied straightforwardly without using restrictive assumptions on the nonlinear part. This proposed method will serve to be a milestone for research in this particular area.

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