Soliton Solutions, Conservation Laws, and Reductions of Certain Classes of Nonlinear Wave Equations

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In this paper, the soliton solutions and the corresponding conservation laws of a few nonlinear wave equations will be obtained. The Hunter–Saxton equation, the improved Korteweg–de Vries equation, and other such equations will be considered. The Lie symmetry approach will be utilized to extract the conserved densities of these equations. The soliton solutions will be used to obtain the conserved quantities of these equations.

Key words: Lie Symmetries; Conservation Laws; Double Reduction.

1. Introduction

The theory of nonlinear waves is a very important area of research in the field of applied mathematics and theoretical physics. Nonlinear waves appear in the areas of quantum mechanics, nonlinear optics, fluid dynamics, plasma physics, mathematical biology, and several other areas. They are studied diligently in all of these contexts. There are several aspects of these waves that are touched upon in the research conducted in this area. A couple of important issues are the integrability aspects that are important to move forward in this area of these nonlinear wave equations and the corresponding conservation laws.

The analytical study leads to the integrability issues of these equations which consequently extracts exact nonlinear wave solutions. Another obligation to the analysis of these equations is the conservation laws that can be obtained by the multiplier approach using the Lie symmetry analysis. This is a very useful technique that reveals several hidden conservation laws. These laws describe the physics of the waves in a profound manner and are therefore very well appreciated.

2. Improved Korteweg–de Vries Equation

There are several nonlinear evolution equations that govern various physical situations [1 – 20]. In order to study the dynamics of shalow water waves, the improved Korteweg–de Vries (KdV) equation is the one that models it best. With power law nonlinearity, this equation is given by [2, 6, 11]

$$u_t + au^n u_x + bu_{xxt} + \beta u_{xxx} = 0\,, \quad n \neq 0, -1, -2\,. \tag{1}$$

The first term represents the evolution term, while the seconde term is the nonlinear term. The two dispersion terms are due to the coefficients b and β, where the coefficient b accounts for the improved KdV equation. If however $b = 0$, then (1) collapses to the regular KdV equation. The parameter n dictates the power law nonlinearity.

The Lie point symmetry generators that leave (1) invariant are the translations $X_1 = \partial_x$ and $X_2 = \partial_t$. For $n = 1$, we obtain the additional symmetry

$$X_3 = at\partial_x + \frac{ab}{\beta} t\partial_x + \left(1 - \frac{ab}{\beta} u\right)\partial_u\,. \tag{2}$$

The search is going to be a solitary wave solution for (1). Thus, we use the ansatz

$$u(x,t) = A\,\mathrm{sech}^p\,\tau\,, \tag{3}$$

where

$$\tau = B(x - vt)\,. \tag{4}$$

A is the soliton amplitude, B the inverse width of the soliton, and v the velocity of the soliton. The value of the unknown index p will fall out during the course of derivation of the solution of the equation. Thus, substituting (3) into (1) gives

$$\begin{aligned} &pv\,\mathrm{sech}^p\,\tau - apA^n\,\mathrm{sech}^{(n+1)p}\,\tau \\ &+bp^3B^2v\,\mathrm{sech}^p\,\tau - bvp(p+1)(p+2)B^2\,\mathrm{sech}^{p+2}\,\tau \\ &-\beta p^3B^2\,\mathrm{sech}^p\,\tau + \beta p(p+1)(p+2)B^2\,\mathrm{sech}^{p+2}\,\tau \\ &=0. \end{aligned} \tag{5}$$

By the aid of balancing principle, equating the exponents $(n+1)p$ and $p+2$ implies

$$(n+1)p = p+2 \tag{6}$$

i.e.,

$$p = \frac{2}{n}. \tag{7}$$

Now from (5), setting the coefficients of the linearly independent functions $\mathrm{sech}^{p+j}\,\tau$ for $j = 0,2$ to zero, yields

$$v = \frac{4B^2}{n^2}(\beta - b) \tag{8}$$

and

$$B = n\sqrt{\frac{aA^n}{2(n+1)(n+2)(\beta - bv)}} \tag{9}$$

which leads to the constraint condition

$$a(\beta - bv) > 0. \tag{10}$$

Hence, the one-soliton solution to the improved KdV equation is given by

$$u(x,t) = A\,\mathrm{sech}^{\frac{2}{n}}\left[B(x-vt)\right], \tag{11}$$

where the amplitude A and the inverse width B are related as in (9), and the velocity of the soliton is given by (8). This poses a constraint condition that is given by (10) which must stay valid in order for the soliton solution to exist.

The two Figures 1a and 1b show the profile of a one-soliton solution of the improved KdV equation with $n = 1$ and $n = 2$, respectively. Also in both cases, the parameter values chosen are $a = b = \beta = 1$.

2.1. Conservation Laws

In order to determine conserved densities and fluxes, we resort to the invariance and multiplier approach based on the well known result that the Euler–Lagrange operator annihilates a total divergence (see [8]). Firstly, if (T,S) is a conserved vector corresponding to a conservation law, then

$$D_tT + D_xS = 0$$

along the solutions of the differential equation $E(t,x,u,u_{(1)},u_{(2)},\ldots) = 0$, where $u_{(i)}$ represents all the possible ith derivatives of u.

Moreover, if there exists a nontrivial differential function f, called a 'multiplier', such that

$$\mathcal{E}_u[fE] = 0,$$

then fE is a total divergence, i.e.,

$$fE = D_tT^t + D_xT^x,$$

for some (conserved) vector (T,S), and $\mathcal{E}_u$ is the respective Euler–Lagrange operator. Thus, a knowledge

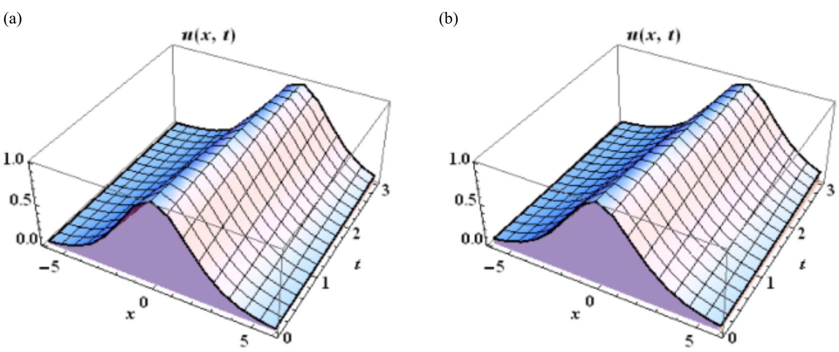

Fig. 1 (colour online). (a) Profile of one-soliton solution $(n = 1)$; (b) Profile of one-soliton solution $(n = 2)$.

of each multiplier f leads to a conserved vector determined by, inter alia, a homotopy operator. See details and references in [8, 9].

For (1), it turns out that multipliers up to the second order in derivatives are given by f_i with corresponding conserved densities T_i, $i=1,2,3$.

(i) $f_1 = 1:$

$$T_1 = u + \frac{b}{3}u_{xx}\,.$$

(ii) $f_2 = u:$

$$T_2 = \frac{1}{6}\left(3u^2 - bu_x^2 + 2buu_{xx}\right).$$

(iii) $f_3 = u_{xt} + \frac{\beta}{b}u_{xx} + \frac{a}{b(n+1)}u^{n+1}:$

$$\begin{aligned} T_3 = {} & \frac{3a^2}{n+1}u^{2n+2} + \frac{ab(-1+n^2)}{n+2}u^n u_t u_x \\ & + \frac{ab(11+7n)}{n+2}u^{n+1}u_{xt} + 6a\beta u^{n+1}u_{xx} \\ & + \frac{(n+1)}{2}\Big(3bu_t^2 + 4b^2u_{xt}^2 - 2b^2u_xu_{xtt} \\ & + 10b\beta u_{xt}u_{xx} + 6\beta^2u_{xx}^2 - 2b\beta u_xu_{xxt} \\ & + 6\beta u_tu_x + b^2u_tu_{xxt} + b\beta u_tu_{xxx} \\ & - 3buu_{tt} - 6\beta uu_{xt} + b^2uu_{xxtt} + b\beta uu_{xxxt}\Big). \end{aligned}$$

The respective conserved quantities are

$$\begin{aligned} I_1 &= \int_{-\infty}^{\infty}\left(u + \frac{b}{3}u_{xx}\right)\mathrm{d}x = \int_{-\infty}^{\infty}u\,\mathrm{d}x \\ &= \frac{A}{B}\frac{\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{n}\right)}, \end{aligned} \tag{12}$$

$$\begin{aligned} I_2 &= \int_{-\infty}^{\infty}\left\{\frac{1}{2}u^2 - \frac{b}{6}(u_x)^2 + \frac{b}{3}uu_{xx}\right\}\mathrm{d}x \\ &= \frac{1}{2}\int_{-\infty}^{\infty}\left\{u^2 - b(u_x)^2\right\}\mathrm{d}x \\ &= \frac{A^2}{2n^2(n+4)B}\Big\{n^2(n+4) - 4b(n+4)B^2 \\ &\quad + 16bA^2B^2\Big\}\frac{\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{2}{n}+\frac{1}{2}\right)}, \end{aligned} \tag{13}$$

and

$$\begin{aligned} I_3 = {} & \int_{-\infty}^{\infty}\Bigg[\frac{3a^2}{n+1}u^{2n+2} + \frac{ab(n^2-1)}{n+2}u^nu_tu_x \\ & + \frac{ab(11+7n)}{n+2}u^{n+1}u_{xt} + 6a\beta u^{n+1}u_{xx} \\ & + \frac{(n+1)}{2}\Big(3bu_t^2 + 4b^2u_{xt}^2 - 2b^2u_xu_{xtt} \\ & + 10b\beta u_{xt}u_{xx} + 6\beta^2u_{xx}^2 - 2b\beta u_xu_{xxt} \\ & + 6\beta u_tu_x + b^2u_tu_{xxt} + b\beta u_tu_{xxx} - 3buu_{tt} \\ & - 6\beta uu_{xt} + b^2uu_{xxtt} + b\beta uu_{xxxt}\Big)\Bigg]\mathrm{d}x \\ = {} & \frac{4A^2}{n^4(n+1)(n+2)(n+4)(3n+4)B} \\ & \cdot\Big[6a^2A^{2n}n^4(n+2)^2 - 24a\beta A^nB^2n^3(n+1)^2(n+2) \\ & + 80b\beta vB^4(n+1)^3(2n+3)(n+2) \\ & + 4abA^nB^2n^3(n+1)^2\Big(A^n(1-n) + v(7n+11)\Big) \\ & + 3vB^2n^3(bv-2\beta)(n+1)^2(n+2)(3n+4) \\ & + 4B^4n^2\left(3b^2v^2 + 3\beta^2 - bv\beta\right)(n+1)^2 \\ & \cdot(2n+3)(n+2)\Big]\frac{\Gamma(\frac{2}{n})\Gamma(\frac{1}{2})}{\Gamma(\frac{2}{n}+\frac{1}{2})}. \end{aligned} \tag{14}$$

These integrals are evaluated from the one-soliton solution (11) that is derived in the previous section. In fact, each of these conserved quantities refer to specific physical quantities. They are the mass, energy, and the Hamiltonian of the soliton, respectively.

Note: it can be shown that the action of the symmetries on the multipliers satisfy

$$X_jf_i = 0, \quad j=1,2, \quad i=1,2,3, \tag{15}$$

and for $n=1$, we have the additional relationships

$$\begin{aligned} & X_3f_1 = 0, \quad X_3f_2 = f_1 - \frac{ab}{r}\beta f_2, \\ & X_3f_3 = \frac{a}{b}f_2 - \frac{2ab}{\beta}f_3. \end{aligned} \tag{16}$$

3. Other Classes of Nonlinear Wave Equations

In this section, we analyse another class of nonlinear wave equations related to the above equation but with greater generality, viz., the partial differntial equation (PDE)

$$\begin{aligned} & au_t - 2m(u)u_x + u_{txx} + 2u_xu_{xx} + uu_{xxx} \\ & + ku_{xxx} = 0, \quad m(u), a, k \neq 0. \end{aligned} \tag{17}$$

In [10], the authors study various cases of (17) construing it as one that is ‘lying mid-way between the periodic Hunter–Saxton and Camassa–Holm equations, and which describes evolution of rotators in liquid crystals with external magnetic field and selfinteraction'. We would like to bring the paper and the references cited therein to the attention of the reader.

As in the previous section, we determine the conservation laws and symmetries of the equations. However, here, we show how one can obtain reductions and exact solutions via a particular procedure we refer to as 'double reduction' [2, 9].

3.1. Symmetries, Conservation Laws and Double Reductions

Firstly, we present some of the preliminaries dealing with symmetries and double reductions of PDEs.

A function $f(x,u,u_{(1)},\dots,u_{(k)})$ of a finite number of variables is called a differential function of order k. $u_{(1)},u_{(2)},\dots,u_{(k)}$ denote the collections of all first, second, …, kth-order partial derivatives, that is, $u_i^\alpha = D_i(u^\alpha), u_{ij}^\alpha = D_jD_i(u^\alpha),\dots$, respectively, with the total differentiation operator with respect to x^i given by

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \cdots \qquad (18)$$

in which the summation convention is used whenever appropriate.

Consider a kth-order system of PDEs of n independent variables $x=(x^1,x^2,\dots,x^n)$ and m dependent variables $u=(u^1,u^2,\dots,u^m)$:

$$E^\mu(x,u,u_{(1)},\dots,u_{(k)})=0,\quad \mu=1,\dots,\tilde{m}. \qquad (19)$$

The Lie–Bäcklund or generalised operator is given by

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha},\quad \xi^i,\eta^\alpha \in \mathcal{A}, \qquad (20)$$

where $\mathcal{A}$ is the universal vector space of differential functions. The operator (20) is an abbreviated form of the infinite formal sum

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1} \zeta^\alpha_{i_1 i_2\dots i_s} \frac{\partial}{\partial u^\alpha_{i_1 i_2 \dots i_s}}, \qquad (21)$$

where the additional coefficients are determined uniquely by the prolongation formulae

$$\zeta_i^\alpha = D_i(W^\alpha) + \xi^j u_{ij}^\alpha,$$
$$\zeta^\alpha_{i_1\dots i_s} = D_{i_1}\dots D_{i_s}(W^\alpha) + \xi^j u^\alpha_{ji_1\dots i_s},\quad s>1. \qquad (22)$$

In (22), W^α is the Lie characteristic function

$$W^\alpha = \eta^\alpha - \xi^j u_j^\alpha. \qquad (23)$$

A current vector $T=(T^1,\dots,T^n)$ is conserved if it satisfies

$$D_iT^i = 0 \qquad (24)$$

along the solutions of (19).

Definition 1. [14] A Lie–Bäcklund symmetry generator X of the form (21) is associated with a conserved vector T of the system (19) if X and T satisfy the relations

$$X(T^i) + T^iD_k(\xi^k) - T^kD_k(\xi^i) = 0,\quad i=1,\dots,n. \qquad (25)$$

Theorem 1. *[15, 16] Suppose that X is any Lie–Bäcklund symmetry of (19) and T^i, $i=1,\dots,n$, are the components of the conserved vector of (19). Then*

$$T^{*i} = [T^i, X] = X(T^i) + T^iD_j\xi^j - T^jD_j\xi^i,\quad i=1,\dots,n, \qquad (26)$$

constitute the components of a conserved vector of (19), i.e., $D_iT^{*i}\,|_{(19)} = 0$.

Theorem 2. *[13] Suppose that $D_iT^i=0$ is a conservation law of the PDE system (19). Then under a contact transformation, there exist functions $\tilde{T}^i$ such that $J\,D_iT^i = \tilde{D}_i\tilde{T}^i$, where $\tilde{T}^i$ is given as*

$$\begin{pmatrix}\tilde{T}^1\\ \tilde{T}^2\\ \vdots\\ \tilde{T}^n\end{pmatrix} = J(A^{-1})^{\mathrm{T}} \begin{pmatrix}T^1\\ T^2\\ \vdots\\ T^n\end{pmatrix},\qquad J\begin{pmatrix}T^1\\ T^2\\ \vdots\\ T^n\end{pmatrix} = A^{\mathrm{T}} \begin{pmatrix}\tilde{T}^1\\ \tilde{T}^2\\ \vdots\\ \tilde{T}^n\end{pmatrix} \qquad (27)$$

in which

$$A = \begin{pmatrix} \tilde{D}_1x_1 & \tilde{D}_1x_2 & \cdots & \tilde{D}_1x_n\\ \tilde{D}_2x_1 & \tilde{D}_2x_2 & \cdots & \tilde{D}_2x_n\\ \vdots & \vdots & \vdots & \vdots\\ \tilde{D}_nx_1 & \tilde{D}_nx_2 & \cdots & \tilde{D}_nx_n \end{pmatrix},\qquad A^{-1} = \begin{pmatrix} D_1\tilde{x}_1 & D_1\tilde{x}_2 & \cdots & D_1\tilde{x}_n\\ D_2\tilde{x}_1 & D_2\tilde{x}_2 & \cdots & D_2\tilde{x}_n\\ \vdots & \vdots & \vdots & \vdots\\ D_n\tilde{x}_1 & D_n\tilde{x}_2 & \cdots & D_n\tilde{x}_n \end{pmatrix} \qquad (28)$$

and $J = \det(A)$.

Theorem 3. *[13] (**fundamental theorem on double reduction**) Suppose that $D_iT^i = 0$ is a conservation law of the PDE system (19). Then under a similarity transformation of a symmetry X of the form (21) for the PDE, there exist functions $\tilde{T}^i$ such that X is still a symmetry for the PDE, satisfying $\tilde{D}_i\tilde{T}^i = 0$ and*

$$\begin{pmatrix} X\tilde{T}^1 \\ X\tilde{T}^2 \\ \vdots \\ X\tilde{T}^n \end{pmatrix} = J(A^{-1})^{\mathrm{T}} \begin{pmatrix} [T^1,X] \\ [T^2,X] \\ \vdots \\ [T^n,X] \end{pmatrix}, \qquad (29)$$

where

$$A = \begin{pmatrix} \tilde{D}_1x_1 & \tilde{D}_1x_2 & \cdots & \tilde{D}_1x_n \\ \tilde{D}_2x_1 & \tilde{D}_2x_2 & \cdots & \tilde{D}_2x_n \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{D}_nx_1 & \tilde{D}_nx_2 & \cdots & \tilde{D}_nx_n \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} D_1\tilde{x}_1 & D_1\tilde{x}_2 & \cdots & D_1\tilde{x}_n \\ D_2\tilde{x}_1 & D_2\tilde{x}_2 & \cdots & D_2\tilde{x}_n \\ \vdots & \vdots & \vdots & \vdots \\ D_n\tilde{x}_1 & D_n\tilde{x}_2 & \cdots & D_n\tilde{x}_n \end{pmatrix} \qquad (30)$$

and $J = \det(A)$.

Our original system is equivalent to

$$\mathrm{sys}_1 = \begin{cases} f_1{}^1E^1 + f_1{}^2E^2 = 0, \\ f_1{}^1E^1 - f_1{}^2E^2 = 0. \end{cases} \qquad (31)$$

This system can be rewritten as

$$D_tT^t + D_xT^x = 0, \quad f_1^1E^1 - f_1^2E^2 = 0. \qquad (32)$$

3.1.1. Equation 1

As a first case, we list the conserved densities of

$$u_t + au^nu_x + bu_{txx} + cu_xu_{xx} + \alpha uu_{xxx} + \beta u_{xxx} = 0. \qquad (33)$$

When $\alpha = c = 0$, then (33) reduces to the improved KdV equation that was studied in the previous section.
(i) For all c, $f = 1$ and

$$T = u + \frac{b}{3}u_{xx}.$$

(ii) For $c = 2\alpha$, $f = u$ leads to another density

$$T = \frac{1}{6}(3u^2 - bu_x^2 + 2buu_{xx}).$$

(iii) When $c = 2$, $\alpha = 1$, $b = 1$, $f = u_{xt} + \frac{1}{2}u_x^2 + (u + \beta)u_{xx} + \frac{a}{n+1}u^{n+1}$ yields conserved densities. For example, when $n = 10$, the density is given by

$$\begin{aligned} T = \frac{1}{792}\big(&6au^{12} - 99au^{10}u_x{}^2 - 9au^{11}u_{xx} \\ &+ 66\big(3u_tu_x + 2u_x{}^2u_{xx} + 2u_{xx}(u_{xt} + \beta u_{xx}) \\ &+ u_x(u_{xxt} + \beta u_{xxx})\big) + 44u^2(6u_{xx} - u_{xxxx}) \\ &+ 22u\big(6u_x{}^2 + 9u_{xt} + 18\beta u_{xx} - 4u_xu_{xxx} \\ &- 3u_{xxxt} - 3\beta u_{xxxx}\big)\big). \end{aligned}$$

3.1.2. Equation 2

For the equation

$$au_t - 2m(u)u_x + u_{txx} + 2u_xu_{xx} + uu_{xxx} + ku_{xxx} = 0, \quad m(u), a, k \neq 0, \qquad (34)$$

which appears in the study of shallow water waves in lake or ocean shores, we separate the results into two cases for the parameter a and list some multipliers with corresponding conserved vectors.

The principal Lie algebra of Lie point symmetries is $\langle \partial_t, \partial_x \rangle$. The case $m = u$ admits an additional generator

$$X = \frac{2+a}{k}t\partial_t + 2t\partial_x - \frac{ak + (2+a)u}{k}\partial_u.$$

In the enumeration below, we choose $m = \cos u$ for illustrative purposes.
I. $a \neq 0$.

$$f = a_1 + a_2u + a_3\left(\frac{2u_{tx} + 2uu_{xx} + 2ku_{xx} + u_x^2}{2} - 2\int m(u)\,\mathrm{d}u\right).$$

(i) $f_1 = 1$:

$$T = au + \frac{u_{xx}}{3},$$

$$S = -2\sin u + \frac{u_x{}^2}{2} + \frac{2u_{xt}}{3} + ku_{xx} + uu_{xx}.$$

(ii) $f_2 = u$:

$$T = \frac{1}{6}\left(3au^2 - u_x{}^2 + 2uu_{xx}\right),$$

$$S = 2-2\cos u - \frac{1}{3}u_t u_x - \frac{1}{2}k{u_x}^2 + u^2 u_{xx} + u\left(-2\sin u + \frac{2u_{xt}}{3} + ku_{xx}\right).$$

(iii) $f_3 = \frac{1}{2}(2u_{tx} + 2uu_{xx} + 2ku_{xx} + u_x^2) - 2\int m(u)\,\mathrm{d}u$:

$$\begin{aligned}T = \frac{1}{36u^2}\big(&-36(-1+\cos u){u_x}^2 - 36u\big(\sin u{u_x}^2 \\ &-(-1+\cos u)u_{xx}\big) + 3u^2\big(3au_t u_x \\ &+ 2{u_x}^2(2\cos u + u_{xx}) + 2(12a(-1+\cos u) \\ &+ (2\sin u + u_{xt})u_{xx} + k{u_{xx}}^2) + u_x(u_{xxt} \\ &+ ku_{xxx})\big) + 2u^4(6au_{xx} - u_{xxxx}) \\ &+ u^3\big(6a{u_x}^2 + 9au_{xt} + 18aku_{xx} - 4u_x u_{xxx} \\ &- 3u_{xxxt} - 3ku_{xxxx}\big)\big),\end{aligned}$$

$$\begin{aligned}S = \frac{1}{72u^2}\big(&72(-1+\cos u)u_t u_x + 72u(\sin u u_t u_x \\ &-(-1+\cos u)u_{xt}) + 3u^2\big(6a{u_t}^2 + 3{u_x}^4 \\ &+ {u_x}^2(-24\sin u + 8u_{xt} + 12ku_{xx}) \\ &+ 4\big(2{u_{xt}}^2 + 3(-2\sin u + ku_{xx})^2 \\ &+ u_{xt}(-14\sin u + 5ku_{xx})\big) - 4u_x(u_{xtt} + ku_{xxt}) \\ &+ 2u_t\big((6ak - 4\cos u)u_x + u_{xxt} + ku_{xxx}\big)\big) \\ &+ 4u^4(-6au_{xt} + 9{u_{xx}}^2 + u_{xxxt}) \\ &+ u^3\big(-18au_{tt} + 8u_t(3au_x + u_{xxx}) \\ &+ 6\big(-6u_{xt}(ak - 2u_{xx}) \\ &+ 6(-4\sin u + {u_x}^2)u_{xx} + 12k{u_{xx}}^2 + u_{xxtt} \\ &+ ku_{xxxt}\big)\big)\big).\end{aligned}$$

II. $a = 0$.
Equation (17) becomes

$$-2m(u)u_x + u_{txx} + 2u_x u_{xx} + uu_{xxx} + ku_{xxx} = 0, \quad m(u), k \neq 0 \tag{35}$$

whose principal Lie algebra of Lie point symmetries is $\langle \partial_t, \partial_x \rangle$.

The following cases admit an additional generator:

1. $m = u$: $X_1 = -kt\partial_x - t\partial_t + u\partial_u$.
2. $m = u^\beta$: $X_2 = \left(\frac{2kt}{\beta-1} + x\right)\partial_x + \frac{1+\beta}{\beta-1}t\partial_t - \frac{2u\partial_u}{\beta-1}$.
3. $m = e^u$: $X_3 = (-2t+x)\partial_x + t\partial_t - 2\partial_u$.

Thus, $m = u, u^\beta, e^u$ are special cases.

In the enumeration below, we choose $m = u$ for illustrative purposes. In case (iii), we just list the conserved density.

$$f = b_1 u + G_1(t) + G_2\left(t, -\int 2m(u)\,\mathrm{d}u + u_{xt} + uu_{xx} + ku_{xx} + \frac{1}{2}u_x^2\right):$$

(i) $f_1 = g(t)$:

$$T = \frac{1}{3}g(t)u_{xx},$$
$$S = \frac{1}{6}(-2g' u_x + g(t)(-6u^2 + 3{u_x}^2 + 4u_{xt} + 6ku_{xx} + 6uu_{xx})).$$

(ii) $f_2 = u$:

$$T = \frac{1}{6}(-{u_x}^2 + 2uu_{xx}),$$
$$S = \frac{1}{6}(-4u^3 - u_x(2u_t + 3ku_x) + 6u^2 u_{xx} + u(4u_{xt} + 6ku_{xx})).$$

(iii) $f_3 = -\int 2m(u)\,\mathrm{d}u + u_{xt} + uu_{xx} + ku_{xx} + \frac{1}{2}u_x^2$:

$$T = \frac{1}{36}(6{u_x}^2 u_{xx} + 6u_{xt}u_{xx} + 6k{u_{xx}}^2 + u_x(3u_{xxt} + (3k-4u)u_{xxx}) - 3uu_{xxxt} - 3kuu_{xxxx} - 2u^2 u_{xxxx}).$$

4. Illustration: A Double Reduction of Equation (35)

We perform the double reduction procedure for case (i) using the symmetry generator X_1 and denote the conserved vector (T,S) by $T_1 = (T_1^t, T_1^x)$.

Without loss of generality, we choose $g(t) = t$.

We first show that X_1 is associated with T_1.

Using (25) for $i = 1,2$, we obtain

$$\begin{pmatrix} T_1^{*t} \\ T_1^{*x} \end{pmatrix} = X_1^{[2]}\begin{pmatrix} T_1^t \\ T_1^x \end{pmatrix} - \begin{pmatrix} -1 & 0 \\ -k & 0 \end{pmatrix}\begin{pmatrix} T_1^t \\ T_1^x \end{pmatrix} - \begin{pmatrix} T_1^t \\ T_1^x \end{pmatrix} = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix},$$

where

$$U_1 = -\frac{1}{3}tu_{xx} + \frac{1}{3}tu_{xx}$$

and

$$U_2 = tu^2 - \frac{1}{2}tu_x^2 - \frac{2}{3}tu_{xt} - ktu_{xx} - tuu_{xx} - 2tu^2 + tuu_{xx} - \frac{1}{3}u_x + tu_x^2 + ktu_{xx} + tuu_{xx} + \frac{4}{3}tu_{xt} + \frac{2}{3}ktu_{xx} + \frac{1}{3}ktu_{xx} + \frac{1}{3}u_x + tu^2 - \frac{1}{2}tu_x^2 - \frac{2}{3}tu_{xt} - ktu_{xx} - tuu_{xx}.$$

This computation shows that

$$U_1 = 0 = U_2,$$

where the prolongation of X_1 is given by

$$X_1^{[2]} = -t\frac{\partial}{\partial t} - kt\frac{\partial}{\partial x} + u\frac{\partial}{\partial u} + u_x\frac{\partial}{\partial u_x} + u_{xx}\frac{\partial}{\partial u_{xx}} + (2u_{xt} + ku_{xx})\frac{\partial}{\partial u_{xt}}.$$

Thus, X_1 is associated with T_1.

We can get a reduced conserved form for the equation

$$fE = 0 \tag{36}$$

since X_1 is an associated symmetry of the conserved vector T_1.

We transform the generator X_1 to its canonical form $Y = \frac{\partial}{\partial s}$, where we assume that this generator is of the form $Y = 0\frac{\partial}{\partial r} + \frac{\partial}{\partial s} + 0\frac{\partial}{\partial w}$.

From $X_1(r) = 0$, $X_1(s) = 1$, and $X_1(w) = 0$, we obtain

$$\frac{\mathrm{d}t}{-t} = \frac{\mathrm{d}x}{-kt} = \frac{\mathrm{d}u}{u} = \frac{\mathrm{d}r}{0} = \frac{\mathrm{d}s}{1} = \frac{\mathrm{d}w}{0}. \tag{37}$$

The invariants of X_1 from (37) are given by

$$a_1 = kt - x,\ a_2 = tu,\ a_3 = r,\ a_4 = s + \ln t,\ a_5 = w, \tag{38}$$

where a_3, a_4, and a_5 are arbitrary functions all dependent on a_1 and a_2.

By choosing $a_3 = a_1$, $a_4 = 0$, and $a_5 = a_2$, we obtain the canonical coordinates

$$r = kt - x,\ s = -\ln t,\ w = tu, \tag{39}$$

where $w = w(r)$, since $Y = \frac{\partial}{\partial s}$.

From (39), the inverse canonical coordinates are given by

$$t = \mathrm{e}^{-s},\ x = k\mathrm{e}^{-s} - r,\ u = w\mathrm{e}^{s}. \tag{40}$$

From (28), we compute A and $(A^{-1})^{\mathrm{T}}$:

$$A = \begin{pmatrix} D_r t & D_r x \\ D_s t & D_s x \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -\mathrm{e}^{-s} & -k\mathrm{e}^{-s} \end{pmatrix}$$

and

$$(A^{-1})^{\mathrm{T}} = \begin{pmatrix} D_t r & D_x r \\ D_t s & D_x s \end{pmatrix} = \begin{pmatrix} k & -1 \\ -\mathrm{e}^{s} & 0 \end{pmatrix},$$

where $J = \det(A) = -\mathrm{e}^{-s}$.

The partial derivatives of u from (40) are given by

$$u_x = -w_r\mathrm{e}^s,\ u_{xt} = \mathrm{e}^s(-kw_{rr} + w_r\mathrm{e}^s),\ u_{xx} = w_{rr}\mathrm{e}^s,\ u_{xxt} = \mathrm{e}^s(kw_{rrr} - w_{rr}\mathrm{e}^s),\ u_{xxx} = -w_{rrr}\mathrm{e}^s. \tag{41}$$

We now apply the formula from (27) to obtain the reduced conserved form

$$\begin{pmatrix} T_1^r \\ T_1^s \end{pmatrix} = J(A^{-1})^{\mathrm{T}}\begin{pmatrix} T_1^t \\ T_1^x \end{pmatrix}. \tag{42}$$

By substituting (40) and (41) into (42), we obtain

$$T_1^r = w_r - w^2 + \frac{1}{2}w_r^2 + ww_{rr},\quad T_1^s = \frac{1}{3}w_{rr}, \tag{43}$$

where the reduced conserved form is also given by

$$D_r T_1^r = 0. \tag{44}$$

From (44), we have $T_1^r = k_1$, i.e.,

$$w_r - w^2 + \frac{1}{2}w_r^2 + ww_{rr} = k_1, \tag{45}$$

where k_1 is a constant.

Differentiating (45) implicitly with respect to r results in

$$w_{rr} - 2ww_r + 2w_r w_{rr} + ww_{rrr} = 0. \tag{46}$$

After transforming (36) using (40) and (41), this results in (46).

We analyse (45) for $k_1 = 0$, i.e.,

$$w_r - w^2 + \frac{1}{2}w_r^2 + ww_{rr} = 0. \quad (47)$$

Since $\frac{\partial}{\partial r}$ is a Lie symmetry generator of (47), we have the zero, first-order, and second-order invariants given by

$$\alpha = w, \ \beta = w_r, \ \frac{d\beta}{d\alpha} = \frac{w_{rr}}{w_r}. \quad (48)$$

Substituting (48) into (47) results in the first-order ordinary differential equation

$$\frac{d\beta}{d\alpha} = \frac{\alpha}{\beta} - \frac{1}{\alpha} + \frac{\beta}{2\alpha}. \quad (49)$$

Solving (49) leads to a solution for w in (47) and hence a solution for u in (35).

5. Conclusions

In this paper, a couple of nonlinear wave equations were studied. The improved KdV equation with power law nonlinearity was studied by the aid of the ansatz method, and a solitary wave solution was established along with a constraint condition that must hold for the existence of the solitary wave. The conserved quantities for this equation are established by the multiplier method. Then, using the one-soliton solution, the conserved quantities are formulated. Finally, a numerical simulation is given for this equation. Subsequently, another nonlinear wave equation was studied that stands 'mid-way' between the Hunter–Saxton equation and the Camassa–Holm equation. For this equation, symmetries are established and the technique of double reduction was applied to extract the conservation laws. These results are going to be very useful in further future studies.

Later, these results are going to be extended to extract further solutions, if possible, to these equations. They are the shock waves, cnoidal waves, snoidal waves, singular solitary waves, peakons, stumpons, cuspons, covatons, kinks–antikinks, and several others [19]. These variety of nonlinear waves will give a further insight into these wave equations. Furthermore, perturbation terms will be added to obtain a better understanding of the physical situation the system models. Those perturbed equation will be modelled by the variety of integration tools that are available in the modern times. These results will all be reported in future publications.

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