# Study of the Phan-Thien–Tanner Equation of Viscoelastic Blood Non-Newtonian Flow in a Pipe-Shaped Artery under an Emotion-Induced Pressure Gradient

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Z. Naturforsch. **67a,** 628 – 632 (2012) / DOI: 10.5560/ZNA.2012-0069

Received May 3, 2012 / revised July 14, 2012 / published online September 12, 2012

In this paper, a three-dimensional, unsteady state non-Newtonian fluid flow in a pipe-shaped artery of viscoelastic blood is considered in the presence of emotion-induced pressure gradient. The results have been expressed in terms of radial profiles of both axial velocity and viscosity and were presented numerically by using the shooting technique coupled with the Newtonian method and the Boubaker polynomials expansion scheme. The effects of some parameters on the dynamics are analyzed.

*Key words:* Boundary Conditions; Mass Transfer; Viscous Dissipation; Phan-Thien–Tanner (PTT) Equation; Viscosity.

## 1. Introduction

Blood viscoelasticity of steady state non-Newtonian fluids have been a subject of great interest for several decades. Such interest is motivated by several contributing factor to the viscoelastic behaviour of blood including plasma viscosity, plasma composition, temperature, the rate of flow or shear rate, pathological conditions and hematocrit etc. Considerable analysis of this non-Newtonian fluid principal has been carried out in [1 – 4].

A number of investigations on linear or nonlinear viscoelastic fluid model topic have been presented. For example Maxwell fluid model [5], Jeffery fluid model [6], Walters B fluid model [7], Oldroyd-B model [8], and many others. Among these a new linear viscoelastic model is the Phan-Thien–Tanner (PTT) model for which one can reasonably hope to obtain the analytic and numerical solutions. The PTT constitutive equation was derived using network theory [9, 10]. This model has advantages over other models because it includes not only shear viscosity and normal-stress differences but also an elongational parameter $e$. Therefore it reproduces many of the characteristics of the rheology of polymer solutions and non-Newtonian liquids. Recently, a number of researchers studied the PPT model considering different flow geometries, and assumptions were proposed in literature. The literature on the topic is quite extensive and hence can not be described here in detail. However, some most recent works of eminent researchers regarding the analytical solution of the PPT model for different geometries including channel and pipe flow, parallel plates, and concentric annulus with cylinders may be mentioned in [11 – 14]. Pinho and Olivieira [15], Coelho et al. [16 – 18], and Hashemabadi et al. [19] extended the PPT fluid model for heat transfer.

All the aforementioned studies do not discuss the numerical solutions for the PTT model equation. The objectives of this paper are three-fold: first, to model the PTT equation of viscoelastic blood steady-state in a pipe-shaped artery under an emotion-induced pressure gradient; second, to suggest the Boubaker polynomial expansion scheme (BPES) [20 – 28] first time for the PTT model, which primarily lies in its ability to avoid the unnecessary calculations of other analytical methods [29 – 40]; and third, to calculate the novel numerical solutions using the shooting method. To the best of our knowledge, it seems to me that no attempt is available in literature to solve the Phan-Thien–Tanner equation of viscoelastic blood steady-state in a pipe-shaped artery with the help of BPES and shooting method.

## 2. Formulation of the Nonlinear Phan-Thien–Tanner Problem

Consider the governing equation for the elastic part of viscoelastic blood unsteady-state flow obeying the Phan-Thien–Tanner law (Fig. 1).

In the presence of viscous dissipation, the laminar boundary layer equations are written as

$$\lambda\left(\frac{\partial\sigma_{\mathrm{el}}}{\partial t}+\overrightarrow{U}\cdot\overrightarrow{\nabla}\sigma_{\mathrm{el}}+g_{\mathrm{a}}\left(\overrightarrow{\nabla}\cdot\overrightarrow{U},\sigma_{\mathrm{el}}\right)\right)+f(\sigma_{\mathrm{el}})\sigma_{\mathrm{el}}=2\eta_{\mathrm{el}}D\left(\overrightarrow{U}\right),\quad \overrightarrow{U}=\begin{pmatrix}u_1\\u_2\\w\end{pmatrix},\tag{1}$$

where $\eta_{\mathrm{el}}$ is the fluid viscosity, $\lambda$ the relaxation time, $a$ the dimensionless material slip parameter, and $\overrightarrow{U}$ the lubricant velocity vector field. Further, we have the pressure $p$ and $\sigma_{\mathrm{el}}$ as extra-stress symmetric tensor. $g_{\mathrm{a}}$ is the bilinear application and $D(\overrightarrow{U})$ the symmetric part of the velocity gradient (rate of strain tensor).

The bilinear application $g_{\mathrm{a}}$ is defined by

$$g_{\mathrm{a}}\left(\overrightarrow{\nabla}\cdot\overrightarrow{U},\sigma_{\mathrm{el}}\right)=\sigma_{\mathrm{el}}\cdot W\left(\overrightarrow{U}\right)-W\left(\overrightarrow{U}\right)\cdot\sigma_{\mathrm{el}}-a\left(\sigma_{\mathrm{el}}\cdot D\left(\overrightarrow{U}\right)+D\left(\overrightarrow{U}\right)\cdot\sigma_{\mathrm{el}}\right),\quad \overrightarrow{U}=\begin{pmatrix}u_1\\u_2\\w\end{pmatrix},\ |a|<1,\tag{2}$$

where $W\left(\overrightarrow{U}\right)$ is the skew-symmetric part of the velocity gradient (vorticity tensor).

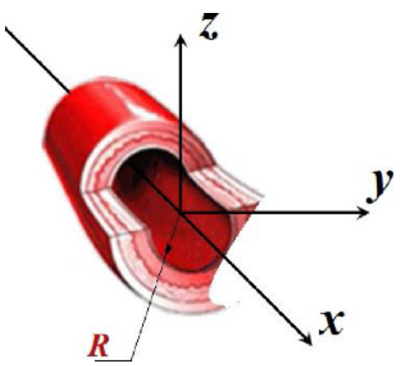


Fig. 1 (colour online). Outline of the studied model.

In the actual model, the function $f$ is chosen as

$$f(\sigma_{\mathrm{el}})=1+\frac{\kappa\lambda}{\eta_{\mathrm{el}}}h(\sigma_{\mathrm{el}})\tag{3}$$

where $h$ and $\kappa$ are a function and a constant related to the elongational behaviour of the model.

The behaviour of blood as viscoelastic non-Newtonian fluid is given by the balance of momentum equation:

$$\rho\left(\frac{\partial\overrightarrow{U}}{\partial t}+\overrightarrow{U}\cdot\nabla\overrightarrow{U}\right)+\nabla p=\operatorname{div}\left(\sigma_{\mathrm{el}}+2\eta_{\mathrm{visc}}D\left(\overrightarrow{U}\right)\right),\tag{4}$$

where $\rho$ is the fluid mass density, and $\eta_{\mathrm{visc}}$ corresponds to the Newtonian viscosity.

## 3. Dimensionless Formalization

In order to obtain numerical solutions, we transfer the problem (1)–(4) to a system of dimensionless-variables system by denoting new variables:

$$\varepsilon=\frac{2R}{L},\qquad \mathrm{De}=\frac{\lambda\varpi}{L},\qquad K=\frac{\kappa}{R},\qquad \mathrm{Re}=\frac{\rho\varpi L}{\eta_{\mathrm{el}}},\tag{5}$$

where $\varpi$ is the order of magnitude of the shear velocity, $\varepsilon$ the artery internal thickness ratio, De the Deborah number, $K$ the elongation number, Re the Reynolds number, and $r$ the retardation number.

Hence, and for the cited assumptions, the following system is obtained:

$$\begin{aligned}\mathrm{Re}\frac{\partial u_1}{\partial t}&=(1-r)\left[\frac{\partial^2u_1}{\partial x^2}+\frac{\partial^2u_1}{\partial y^2}+\varepsilon^{-2}\frac{\partial^2u_1}{\partial z^2}\right]-\varepsilon^{-2}\frac{\partial p}{\partial x},\\ \mathrm{Re}\frac{\partial u_2}{\mathrm{d}t}&=(1-r)\left[\frac{\partial^2u_2}{\partial x^2}+\frac{\partial^2u_2}{\partial y^2}+\varepsilon^{-2}\frac{\partial^2u_2}{\partial z^2}\right]-\varepsilon^{-2}\frac{\partial p}{\partial y},\\ \varepsilon\mathrm{Re}\frac{\partial w}{\mathrm{d}t}&=\varepsilon(1-r)\left[\frac{\partial^2w}{\partial x^2}+\frac{\partial^2w}{\partial y^2}+\varepsilon^{-2}\frac{\partial^2w}{\partial z^2}\right]-\varepsilon^{-2}\frac{\partial p}{\partial z}.\end{aligned}\tag{6}$$

An emotion-induced pressure gradient inside the pipe-shaped artery is simulated through the conditions

$$\begin{aligned}&\frac{\partial u_1}{\partial y}=\frac{\partial u_1}{\partial z}=0\,,\quad \frac{\partial u_2}{\partial y}=\frac{\partial u_2}{\partial z}=0\,,\\&\frac{\partial w}{\partial y}=\frac{\partial w}{\partial z}=0\,,\quad \frac{\partial p}{\partial y}=\frac{\partial p}{\partial z}=0\,,\\&\frac{\partial p}{\partial x}=\begin{cases}0\,,\ t\le 0\,,\\ \Phi=\text{const.}\,,\ t>0\,.\end{cases}\end{aligned}\tag{7}$$

This gives the simplified system

$$\begin{aligned}&\mathrm{Re}\frac{\partial u_1}{\mathrm{d}t}=(1-r)\left[\frac{\partial^2 u_1}{\partial x^2}\right]-\varepsilon^{-2}\Phi_{t>0}\,,\\&\mathrm{Re}\frac{\mathrm{d}u_2}{\mathrm{d}t}=(1-r)\left[\frac{\partial^2 u_2}{\partial x^2}\right],\\&\varepsilon\mathrm{Re}\frac{\mathrm{d}w}{\mathrm{d}t}=\varepsilon\,(1-r)\left[\frac{\partial^2 w}{\partial x^2}\right].\end{aligned}\tag{8}$$

## 4. Boubaker Polynomials Expansion Scheme Solution

The Boubaker polynomials expansion scheme (BPES) [20–28] solutions start from assigning, for example for $u_1(x,t)$ (transversal velocity), the following expression:

$$\begin{aligned}u_1(x,t)=&\left(\frac{1}{2M}\sum_{j=1}^{M}\xi_j\times \mathrm{B}_{4j}(\varpi_j x)\right)\\&\cdot\left(\frac{1}{2M}\sum_{j=1}^{M}\xi_j'\times \mathrm{B}_{4j}(\varpi_j t)\right),\end{aligned}\tag{9}$$

where $B_{4j}$ are $4j$-order Boubaker polynomials, $\varpi_j$ are $B_{4j}$ minimal positive roots, $M$ is a prefixed integer, and $\xi_j,\xi_j'\big|_{j=1\ldots M}$ are unknown pondering real coefficients. Using (9), the boundary conditions are verified automatically with reference to the Boubaker polynomials properties:

$$\begin{aligned}&\sum_{q=1}^{N}B_{4q}(x)\Big|_{x=0}=-2N\neq 0\,,\\&\sum_{q=1}^{N}B_{4q}(x)\Big|_{x=\alpha_q}=0\,,\end{aligned}\tag{10}$$

and

$$\begin{aligned}&\sum_{q=1}^{N}\frac{\mathrm{d}B_{4q}(x)}{\mathrm{d}x}\Big|_{x=0}=0\,,\\&\sum_{q=1}^{N}\frac{\mathrm{d}B_{4q}(x)}{\mathrm{d}x}\Big|_{x=\alpha_q}=\sum_{q=1}^{N}H_q\,,\\&\sum_{q=1}^{N}\frac{\mathrm{d}^2B_{4q}(x)}{\mathrm{d}x^2}\Big|_{x=0}=\frac{8}{3}\left(N(N^2-1)\right)\,,\\&\sum_{q=1}^{N}\frac{\mathrm{d}^2B_{4q}(x)}{\mathrm{d}x^2}\Big|_{x=\alpha_q}=\sum_{q=1}^{N}G_q\,,\end{aligned}\tag{11}$$

with

$$\begin{aligned}H_n&=B_{4n}'(\alpha_n)\\&=\left(\frac{4\alpha_n[2-\alpha_n^2]\times\sum_{q=1}^{n}B_{4q}^2(\alpha_n)}{B_{4(n+1)}(\alpha_n)}+4\alpha_n^3\right)\end{aligned}\tag{12}$$

and

$$\begin{aligned}G_q&=\frac{\mathrm{d}^2B_{4q}(x)}{\mathrm{d}x^2}\Big|_{x=\alpha_q}\\&=\frac{3\alpha_q(4q\alpha_q^2+12q-2)H_q}{(\alpha_q^2-1)(12q\alpha_q^2+4q-2)}\\&\quad-\frac{8q(24q^2\alpha_q^2+8q^2-3q+4)}{(\alpha_q^2-1)(12q\alpha_q^2+4q-2)}\,.\end{aligned}\tag{13}$$

The final solution is derived by introducing (9) and its equivalents in system (8) and calculating the coefficients $\xi_{j,\text{sol.}}\big|_{j=1\ldots M}$ and $\xi_{j,\text{sol.}}'\big|_{j=1\ldots M}$ which minimize the functional determinant $\Omega$

$$\Omega=\left|\begin{aligned}&\mathrm{Re}\left(\frac{1}{2M}\sum_{j=1}^{M}\xi_j\times B_{4j}(\varpi_j x)\right)\\&\cdot\left(\frac{1}{2M}\sum_{j=1}^{M}\xi_j'\times\frac{\mathrm{d}B_{4j}(\varpi_j t}{\mathrm{d}t})\right)\\&-(1-r)\left(\frac{1}{2M}\sum_{j=1}^{M}\xi_j\times\frac{\mathrm{d}^2B_{4j}(\varpi_j x}{\mathrm{d}x^2}\right)\\&\cdot\left(\frac{1}{2M}\sum_{j=1}^{M}\xi_j'\times B_{4j}(\varpi_j t))\right)-\varepsilon^{-2}\Phi\end{aligned}\right|\,.\tag{14}$$

The solution, for i. e., $u_1(x,t)$, is consequently

$$\begin{aligned}u_1(x,t)=&\left(\frac{1}{2M}\sum_{j=1}^{M}\xi_{j,\text{sol.}}\times B_{4j}(\varpi_j x)\right)\\&\cdot\left(\frac{1}{2M}\sum_{j=1}^{M}\xi_{j,\text{sol.}}'\times B_{4j}(\varpi_j t)\right).\end{aligned}\tag{15}$$

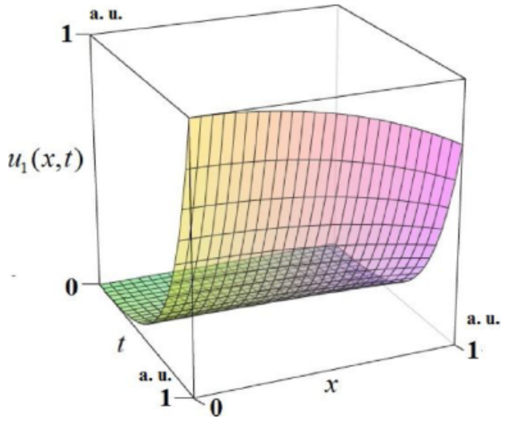


Fig. 2 (colour online). Transversal velocity profile for $r = 0.25$.

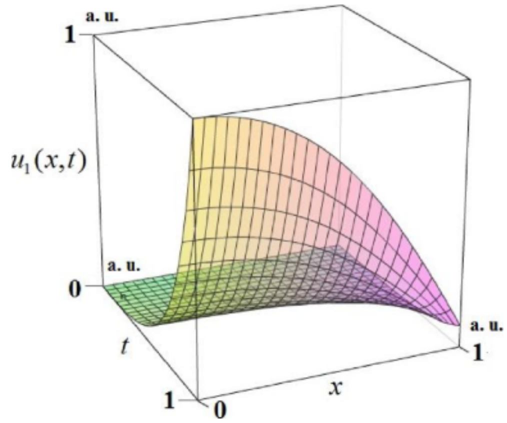


Fig. 4 (colour online). Transversal velocity profile for $r = 0.75$.

## 5. Numerical Results and Plots

Plots of the solution obtained with the present method are presented in Figure 2.

It can be noticed that the velocity profile is time-increasing as expected. Nevertheless, the spatial profile is decreasing after a significant delay. This feature is in concordance with the results of Akyildiz and Bellout [41], Zhang and Li [42], Georgiou and Vlassopoulos [43], Bair [44], Bair and Khonsari [45] who recorded that there was a reciprocal inhibition of the flow speed along the artery for increasing values of $r$.

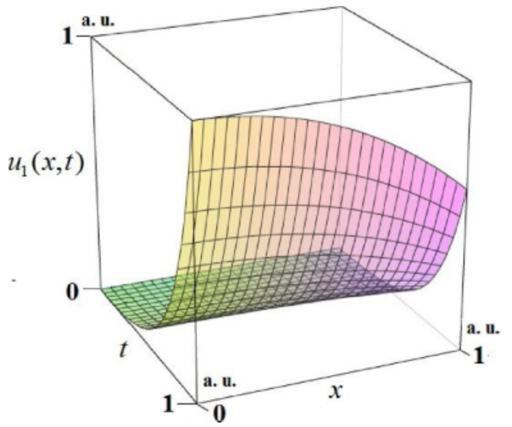


Fig. 3 (colour online). Transversal velocity profile for $r = 0.5$.

For discussing this intrigue, an additional set of plots, for $r = 0.50$ and $r = 0.75$, has been performed (Figs. 3 and 4). The decreasing behaviour is confirmed through the deceasing slope monitored in these figures.

## 6. Conclusions

The Phan-Thien–Tanner (PTT) fluid flow equations are derived in this paper. The results have been expressed in terms of transversal velocity profiles. The nonlinear equations are solved numerically by using the shooting technique coupled with the Newtonian method and the Boubaker polynomials expansion scheme (BPES). The results are presented graphically and the effects of the parameters are discussed. The numerical results for the PTT fluid in a pipe-shaped artery by means of BPES was not available in the literature until now. Such kind of numerical results have never been reported by using shooting technique coupled with Newtonian method and Boubaker Polynomials Expansion Scheme.

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