Integrability and Multi-Soliton Solutions for a Variable-Coefficient Coupled Gross–Pitaevskii System for Atomic Matter Waves

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Under investigation in this paper is a variable-coefficient coupled Gross–Pitaevskii (GP) system, which is associated with the studies on atomic matter waves. Through the Painlevé analysis, we obtain the constraint on the variable coefficients, under which the system is integrable. The bilinear form and multi-soliton solutions are derived with the Hirota bilinear method and symbolic computation. We found that: (i) in the elastic collisions, an external potential can change the propagation of the soliton, and thus the density of the matter wave in the two-species Bose–Einstein condensate (BEC); (ii) in the shape-changing collision, the solitons can exchange energy among different species, leading to the change of soliton amplitudes. We also present the collisions among three solitons of atomic matter waves.

Key words: Variable-Coefficient Coupled Gross–Pitaevskii System; Painlevé Analysis; Atomic Matter Waves; Multi-Soliton Solutions; Soliton Collision.
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1. Introduction

Solitons, as one type of local excitations, have been discussed in such fields as condensed matter physics, plasma physics, nonlinear optics, and fluid dynamics [1 – 6]. Solitons arise in the nonlinear evolution equations (NLEEs) due to the balance between the dispersion and nonlinear effects, and can be used to explain some nonlinear mechanisms [1 – 3, 7 – 10]. For instance, the dynamic behaviour of Alfvén waves has been studied in the inhomogeneous plasma, which can be characterized with the variable-coefficient derivative nonlinear Schrödinger equation [7]. Soliton resonance phenomena, governed by the $(2+1)$-dimensional Boussinesq equation, have been studied in the propagation of gravity waves on a water surface [9].

Solitons have been found to exist in the Bose–Einstein condensates (BECs) [11 – 13]. There has been increasing interest on the atomic matter waves [14] since the experimental realization of the BEC in the vapour of alkali-metal atoms [11]. With the phase imprinting technique, [12] has created the dark soliton in a cigar-shaped BEC of ^{87}Rb, shown to maintain its shape under the balance between the kinetic energy and repulsive interatomic collisions. Trapping techniques have been developed to generate the two-species BECs, such as the overlapping condensate of ^{87}Rb in two spin states [15] and the BEC of ^{87}Rb and ^{41}K [16]. Two-species BECs can possess more phenomena than the one-species ones, like the vector solitons [13].

In this paper, via symbolic computation [17 – 20], we plan to investigate the propagation and collision of bright solitons in a two-species BEC with external potential and thermal cloud effects [21 – 23], which can be described with the three-dimensional coupled Gross–Pitaevskii (GP) equations [21]. As seen below, those bright solitons can exist in the region of varying scattering length, different from the dark solitons in [12]. If the transversal motion of the condensate is frozen as claimed in [21, 23], the radial freedom of the three-dimensional coupled GP equations mentioned above can be integrated out, and thus the quasi-one-dimensional coupled GP system can be obtained in the following form [21, 23]:

$$\begin{aligned}&\mathrm{i}\phi_{1,t}+a(t)\phi_{1,xx}+2b(t)(|\phi_1|^2+|\phi_2|^2)\phi_1\\&+V(x,t)\phi_1+\mathrm{i}c(t)\phi_1=0,\end{aligned} \tag{1a}$$

$$\begin{aligned}&\mathrm{i}\phi_{2,t}+a(t)\phi_{2,xx}+2b(t)(|\phi_1|^2+|\phi_2|^2)\phi_2\\&+V(x,t)\phi_2+\mathrm{i}c(t)\phi_2=0,\end{aligned} \tag{1b}$$

where t and x are respectively the dimensionless variables representing the time and space [21, 23], $\phi_j=\phi_j(x,t)$ denotes the macroscopic wave functions of the jth species $(j=1,2)$ [21, 23], $a(t)$ is the small perturbation in the BEC [21–23], $b(t)$ is the time-dependent atom–atom collision coefficient [21–23], $V(x,t)$ represents the external potential (including harmonic and linear potentials) [22–25], and $c(t)$ is related to the damping or feeding effect caused by the condensate thermal cloud [26–28].

Some bright and dark solitons have been derived for system (1) [22–24]. Without the influence of the thermal cloud effect $c(t)$, creation and collision of the dark solitons have been numerically simulated in a two-species BEC when the atom–atom collision effect $b(t)$ is stable [22]. Bright–bright [23] and dark–dark [23] solitons have been found in a two-species BEC under the balance between the tunable atom–atom collision effect $b(t)$ and external potential $V(x,t)$ [23]. Ref. [24] has presented two families of bright and dark solitons for a special case of (1) with $\phi_1=\phi_2$ and with $b(t)$ and $c(t)$ satisfying certain conditions in terms of $V(x,t)$.

On the other hand, the Painlevé analysis can be used for us to determine the integrable condition of the NLEEs [29], and the Hirota bilinear method, to derive the explicit multi-soliton solutions for the NLEEs [30]. Shape-changing collisions of the solitons have been discussed for the coupled NLEEs, for such a collision allows the energy exchange among different components, which exists as a distinct phenomenon of the coupled NLEEs, from the uncoupled ones [31–33].

To our knowledge, although some soliton solutions for (1) have been given with certain constraints [22–24], the integrable condition of (1) has not been analyzed yet. Shape-changing collisions of the solitons in the two-species BECs via (1) have not been observed, either. The main aim of the present paper will be to investigate (i) the integrable condition that permits the solitons in the two-species BEC via (1), and (ii) the shape-changing collisions of those solitons. The paper will proceed as follows: Section 2 will perform the Painlevé analysis to determine the integrable condition of (1). We will utilize the Hirota bilinear method to derive the multi-soliton solutions for (1) in Section 3. Then we will discuss in Section 4 the propagations and collisions of the bright soliton in the BEC, especially the shape-changing collisions. Section 5 will be our conclusions.

2. Painlevé Analysis of System (1)

Painlevé singularity-structure analysis can be used to identify the integrability of a given NLEE [34, 35]. According to [29], a NLEE is said to possess the Painlevé property if its solution is single valued about the movable singularity manifold [34].

To test the Painlevé property of (1), we will first rewrite (1) as

$$\begin{aligned}&\mathrm{i}\,\phi_{1,t}+a(t)\,\phi_{1,xx}+2b(t)(|\phi_1|^2+|\phi_2|^2)\phi_1\\&+V(x,t)\phi_1+\mathrm{i}c(t)\phi_1=0,\end{aligned} \tag{2a}$$

$$\begin{aligned}&-\mathrm{i}\,\varphi_{1,t}+a(t)\,\varphi_{1,xx}+2b(t)(|\varphi_1|^2+|\varphi_2|^2)\varphi_1\\&+V(x,t)\varphi_1-\mathrm{i}c(t)\varphi_1=0,\end{aligned} \tag{2b}$$

$$\begin{aligned}&\mathrm{i}\,\phi_{2,t}+a(t)\,\phi_{2,xx}+2b(t)(|\phi_2|^2+|\phi_1|^2)\phi_2\\&+V(x,t)\phi_2+\mathrm{i}c(t)\phi_2=0,\end{aligned} \tag{2c}$$

$$\begin{aligned}&-\mathrm{i}\,\varphi_{2,t}+a(t)\,\varphi_{2,xx}+2b(t)(|\varphi_2|^2+|\varphi_1|^2)\varphi_2\\&+V(x,t)\varphi_2-\mathrm{i}c(t)\varphi_2=0,\end{aligned} \tag{2d}$$

where φ_j is the complex conjugate of ϕ_j $(j=1,2)$. Then, we assume that the solutions for (2) in the generalized Laurent series expansions [34] are

$$\phi_1=\sum_{k=0}^{\infty}\phi_1^{(k)}(x,t)\,\psi^{-\alpha+k}(x,t),\quad \varphi_1=\sum_{k=0}^{\infty}\varphi_1^{(k)}(x,t)\,\psi^{-\beta+k}(x,t), \tag{3a}$$

$$\phi_2=\sum_{k=0}^{\infty}\phi_2^{(k)}(x,t)\,\psi^{-\gamma+k}(x,t),\quad \varphi_2=\sum_{k=0}^{\infty}\varphi_2^{(k)}(x,t)\,\psi^{-\delta+k}(x,t), \tag{3b}$$

where α, β, γ, and δ are the positive integers to be determined, $\phi_j^{(k)}(x,t)$ and $\varphi_j^{(k)}(x,t)$ are all analytic functions, $\psi(x,t)$ is an analytic function, and $\psi(x,t)=0$ defines a non-characteristic movable singularity manifold.

Substituting the leading-order behaviour of the forms $\phi_1\approx\phi_1^{(0)}\psi^{-\alpha}$, $\varphi_1\approx\varphi_1^{(0)}\psi^{-\beta}$, $\phi_2\approx\phi_2^{(0)}\psi^{-\gamma}$,

$\varphi_2 \approx \varphi_2^{(0)} \psi^{-\delta}$ into (2) and equating the dominant terms, we obtain the following results:

$$\alpha = \beta = \gamma = \delta = 1,$$
$$\phi_1^{(0)}(x,t) = \frac{-b(t)\phi_2^{(0)}(x,t)\varphi_2^{(0)}(x,t) - a(t)\psi_x^2(x,t)}{b(t)\varphi_1^{(0)}(x,t)}. \quad (4)$$

In order to determine the powers at which the arbitrary functions can enter into the series, called the resonances, substituting

$$\phi_j = \phi_j^{(0)}\psi^{-1} + \phi_j^{(k)}\psi^{-1+k}, \\ \varphi_j = \varphi_j^{(0)}\psi^{-1} + \varphi_j^{(k)}\psi^{-1+k}, \quad (5)$$

into (2), and making the term $\psi^{-3+j}(\phi_j^{(k)}, \varphi_j^{(k)})^{\mathrm{T}}$ vanish, we consider

$$A_{11}\phi_1^{(k)} + A_{12}\varphi_1^{(k)} + A_{13}\phi_2^{(k)} + A_{14}\varphi_2^{(k)} = F_1^{(k)}, \\ A_{21}\phi_1^{(k)} + A_{22}\varphi_1^{(k)} + A_{23}\phi_2^{(k)} + A_{24}\varphi_2^{(k)} = F_2^{(k)}, \\ A_{31}\phi_1^{(k)} + A_{32}\varphi_1^{(k)} + A_{33}\phi_2^{(k)} + A_{34}\varphi_2^{(k)} = F_3^{(k)}, \\ A_{41}\phi_1^{(k)} + A_{42}\varphi_1^{(k)} + A_{43}\phi_2^{(k)} + A_{44}\varphi_2^{(k)} = F_4^{(k)}, \quad (6)$$

where $F_m^{(k)} (m = 1,2,3,4)$ are the functions of $\phi_1^{(k-1)} \ldots \phi_1^{(0)}$, $\phi_2^{(k-1)} \ldots \phi_2^{(0)}$, $\varphi_1^{(k-1)} \ldots \varphi_1^{(0)}$, $\varphi_2^{(k-1)} \ldots \varphi_2^{(0)}$, ψ, x, and t. Making the determinant composed of these coefficients equal to 0, we have

$$\begin{vmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{vmatrix} = 0, \quad (7)$$

with

$$A_{11} = A_{22} = -2b(t)\phi_2^{(0)}\varphi_2^{(0)} + (k^2 - 3k - 2)a(t)\psi_x^2,$$
$$A_{21} = 2b(t)\left(\varphi_1^{(0)}\right)^2,$$
$$A_{12} = \frac{2b(t)\left(\phi_2^{(0)}\right)^2\left(\varphi_2^{(0)}\right)^2 + 4a(t)\phi_2^{(0)}\varphi_2^{(0)}\psi_x^2}{\left(\varphi_1^{(0)}\right)^2} + \frac{2a^2(t)\psi_x^4}{b(t)\left(\varphi_1^{(0)}\right)^2}, \quad A_{23} = A_{41} = 2b(t)\varphi_1^{(0)}\varphi_2^{(0)},$$
$$A_{13} = A_{42} = \frac{-2b(t)\phi_2^{(0)}\left(\varphi_2^{(0)}\right)^2 - 2a(t)\varphi_2^{(0)}\psi_x^2}{\varphi_1^{(0)}},$$
$$A_{24} = A_{31} = 2b(t)\varphi_1^{(0)}\phi_2^{(0)},$$
$$A_{14} = A_{32} = \frac{-2b(t)\left(\phi_2^{(0)}\right)^2\varphi_2^{(0)} - 2a(t)\phi_2^{(0)}\psi_x^2}{\varphi_1^{(0)}},$$
$$A_{34} = 2b(t)\left(\phi_2^{(0)}\right)^2, \quad A_{33} = A_{44} = 2b(t)\phi_2^{(0)}\varphi_2^{(0)} + (k^2 - 3k)a(t)\psi_x^2, \quad A_{43} = 2b(t)\left(\varphi_2^{(0)}\right)^2.$$

From (7), we have

$$(k+1)k^3(k-3)^3(k-4)a^4(t)\psi_x^8 = 0. \quad (8)$$

Solving (8), we obtain the resonances as

$$k = -1, 0, 0, 0, 3, 3, 3, 4. \quad (9)$$

The resonance at $k = -1$ naturally corresponds to the arbitrariness of the singular manifold, $k = 0$ represents the arbitrariness of the functions $\phi_j^{(0)}$ and $\varphi_j^{(0)}$, while $k = 3$ and $k = 4$ respectively correspond to arbitrary functions among $\phi_j^{(3)}$, $\varphi_j^{(3)}$, $\phi_j^{(4)}$, and $\varphi_j^{(4)}$ $(j = 1,2)$. By substituting (3) into (2) and collecting the coefficients of different powers of ψ_j, the existence of a sufficient number of arbitrary functions at each resonance value can be easily checked. From the compatibility condition at $k = 4$, we derive the following constraint on the variable coefficients:

$$V_{xx}(x,t) = 2\frac{c^2(t)}{a(t)} + \frac{c_t(t)}{a(t)} + \frac{a_{tt}(t) + 2a_t(t)c(t)}{2a^2(t)} - \frac{a_t^2(t)}{2a^3(t)} - \frac{a_t(t)b_t(t)}{2a^2(t)b(t)} - \frac{b_{tt}(t) + 4b_t(t)c(t)}{2a(t)b(t)} + \frac{b_t^2(t)}{a(t)b^2(t)}. \quad (10)$$

Thus, we can conclude that (1) is Painlevé integrable under constraint (10).

3. Bilinear Form and Multi-Solitons Solutions

The Hirota bilinear method is a tool for dealing with the soliton problems for integrable NLEEs [30]. Once we transform the original NLEE into its bilinear form, we can derive the soliton solutions directly via the perturbation technique [30].

Using (10), the transformations

$$\tau = \int a(t)\,\mathrm{d}t, \\ \phi_j(x,\tau) = \sqrt{\frac{a(\tau)}{b(\tau)}}u_j(x,\tau), \quad j = 1,2, \quad (11)$$

and the assumption $V(x,\tau) = a(\tau)\Omega(\tau)x^2 + a(\tau)\Pi(\tau)x + a(\tau)\Lambda(\tau)$, (1) can be rewritten as

$$\begin{aligned} &\mathrm{i}u_{j,\tau} + u_{j,xx} + 2\left(|u_j|^2 + |u_{3-j}|^2\right)u_j \\ &+ [\Omega(\tau)x^2 + \Pi(\tau)x + \Lambda(\tau) + \mathrm{i}C(\tau)]u_j = 0, \end{aligned} \tag{12}$$

where $\Pi(\tau)$ and $\Lambda(\tau)$ are both functions of τ, and

$$\Omega(\tau) = C^2(\tau) + \frac{1}{2}C_\tau(\tau),$$
$$C(\tau) = \frac{c(\tau)}{a(\tau)} + \frac{a_\tau(\tau)b(\tau) - a(\tau)b_\tau(\tau)}{2a(\tau)b(\tau)}.$$

Taking the transformations $u_j = \frac{g_j}{f}\,\mathrm{e}^{\mathrm{i}\frac{C(\tau)}{2}x^2}$, we can obtain the bilinear form of (12) as

$$\begin{aligned} &\left[\mathrm{i}D_\tau + D_x^2 + 2iC(\tau)xD_x + \Pi(\tau)x + \Lambda(\tau) \right.\\ &\left. + 2\mathrm{i}C(\tau)\right]g_j \cdot f = 0, \end{aligned} \tag{13a}$$
$$D_x^2 f\cdot f = 2\left(|g_j|^2 + |g_{3-j}|^2\right), \quad j = 1,2, \tag{13b}$$

where D is the Hirota bilinear operator [30] defined by

$$\begin{aligned} D_x^m D_\tau^n f\cdot g \equiv &\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m \\ &\cdot\left(\frac{\partial}{\partial \tau} - \frac{\partial}{\partial \tau'}\right)^n f(x,\tau)g(x',\tau')\Big|_{x=x',\tau=\tau'}. \end{aligned} \tag{14}$$

Multi-soliton solutions for (1) can be generated via solving (13) with the power-series expansions of g and f as

$$\begin{aligned} g_j &= \varepsilon g_{j,1} + \varepsilon^3 g_{j,3} + \varepsilon^5 g_{j,5} + \ldots, \\ f &= 1 + \varepsilon^2 f_2 + \varepsilon^4 f_4 + \varepsilon^6 f_6 + \ldots. \end{aligned} \tag{15}$$

Taking

$$g_{j,1} = \sum_{k=1}^{N} l_j^{(k)}\sigma^{-1}\mathrm{e}^{\theta_k},$$
$$\xi_k = \sigma^{-1}[\eta_k + \mathrm{i}\int\sigma\Pi(\tau)\,\mathrm{d}\tau], \quad \sigma = \mathrm{e}^{2\int C(\tau)\,\mathrm{d}\tau},$$
$$\theta_k = \xi_k x + \mathrm{i}\int \xi_k^2\,\mathrm{d}\tau + \mathrm{i}\int\Lambda(\tau)d\tau + \omega_k,$$

with $l_j^{(k)}$, η_k, and ω_k $(k = 1,2,\ldots,N)$ as the arbitrary complex constants, we can obtain the multi-soliton solutions for (1). For $N = 1$, we choose

$$g_{j,1} = l_j^{(1)}\sigma^{-1}\mathrm{e}^{\theta_1} \tag{16}$$

and have

$$f_2 = \mathrm{e}^{\theta_1+\theta_1^*+\varpi}, \quad \mathrm{e}^{\varpi} = \frac{|l_1^{(1)}|^2 + |l_2^{(1)}|^2}{(\eta_1+\eta_1^*)^2},$$
$$g_{j,m} = 0, \quad m = 3,5,7,\ldots, \quad f_n = 0, \quad n = 4,6,8,\ldots.$$

Then the one-soliton solutions can be written as

$$\begin{aligned} \phi_j = &\frac{l_j^{(1)}\sigma^{-1}}{2}\sqrt{\frac{a(\tau)}{b(\tau)}} \\ &\cdot\exp\left\{\frac{1}{2}\left[\theta_1 - \theta_1^* - \varpi + \mathrm{i}C(\tau)x^2\right]\right\} \\ &\cdot\operatorname{sech}\left[\frac{1}{2}(\theta_1 + \theta_1^* + \varpi)\right], \quad j = 1,2. \end{aligned} \tag{17}$$

For $N = 2$, we take

$$g_{j,1} = \sigma^{-1}\left(l_j^{(1)}\mathrm{e}^{\theta_1} + l_j^{(2)}\mathrm{e}^{\theta_2}\right) \tag{18}$$

and obtain the two-soliton solutions as

$$\phi_j = \sqrt{\frac{a(\tau)}{b(\tau)}}\frac{(g_{j,1}+g_{j,3})}{(1+f_2+f_4)}\,\mathrm{e}^{\mathrm{i}C(\tau)x^2}, \quad j = 1,2, \tag{19}$$

with

$$\begin{aligned} f_2 = &M_{11}\mathrm{e}^{\theta_1+\theta_1^*} + M_{12}\mathrm{e}^{\theta_1+\theta_2^*} + M_{21}\mathrm{e}^{\theta_2+\theta_1^*} \\ &+ M_{22}\mathrm{e}^{\theta_2+\theta_2^*}, \end{aligned}$$
$$g_{j,3} = L_{j,121}\mathrm{e}^{\theta_1+\theta_2+\theta_1^*} + L_{j,122}\mathrm{e}^{\theta_1+\theta_2+\theta_2^*},$$
$$f_4 = G_{1212}\mathrm{e}^{\theta_1+\theta_2+\theta_1^*+\theta_2^*},$$
$$g_{j,m} = 0, \quad m = 3,5,7,\ldots, \quad f_n = 0, \quad n = 4,6,8,\ldots,$$
$$M_{dh} = \frac{l_1^{(d)}l_1^{(h)*} + l_2^{(d)}l_2^{(h)*}}{(\eta_d+\eta_h^*)^2}, \quad d,h = 1,2,$$
$$E_{sv} = (\eta_s + \eta_v^*)M_{sv}, \quad s,v = 1,2,$$
$$L_{j,121} = \frac{\sigma^{-1}(\eta_1-\eta_2)}{(\eta_1+\eta_1^*)(\eta_2+\eta_1^*)}\begin{vmatrix} l_j^{(1)} & E_{11} \\ l_j^{(2)} & E_{21}\end{vmatrix},$$
$$L_{j,122} = \frac{\sigma^{-1}(\eta_1-\eta_2)}{(\eta_1+\eta_2^*)(\eta_2+\eta_2^*)}\begin{vmatrix} l_j^{(1)} & E_{12} \\ l_j^{(2)} & E_{22}\end{vmatrix},$$
$$\begin{aligned} G_{1212} = &\frac{(\eta_1-\eta_2)(\eta_1^*-\eta_2^*)}{(\eta_1+\eta_1^*)(\eta_1+\eta_2^*)(\eta_2+\eta_1^*)(\eta_2+\eta_2^*)} \\ &\cdot\begin{vmatrix} E_{11} & E_{12} \\ E_{21} & E_{22}\end{vmatrix}. \end{aligned}$$

Similarly, the three-soliton solutions can be expressed as

$$\phi_j = \sqrt{\frac{a(\tau)}{b(\tau)}}\frac{(g_{j,1}+g_{j,3}+g_{j,5})}{(1+f_2+f_4+f_6)}\,\mathrm{e}^{\mathrm{i}C(\tau)x^2}, \quad j=1,2, \tag{20}$$

with

$$g_{j,1} = \sigma^{-1}\left(l_j^{(1)}\,\mathrm{e}^{\theta_1} + l_j^{(2)}\,\mathrm{e}^{\theta_2} + l_j^{(3)}\,\mathrm{e}^{\theta_3}\right),$$

$$f_2 = M_{11}\,\mathrm{e}^{\theta_1+\theta_1^*} + M_{12}\,\mathrm{e}^{\theta_1+\theta_2^*} + M_{13}\,\mathrm{e}^{\theta_1+\theta_3^*} + M_{21}\,\mathrm{e}^{\theta_2+\theta_1^*} + M_{22}\,\mathrm{e}^{\theta_2+\theta_2^*} + M_{23}\,\mathrm{e}^{\theta_2+\theta_3^*} + M_{31}\,\mathrm{e}^{\theta_3+\theta_1^*} + M_{32}\,\mathrm{e}^{\theta_3+\theta_2^*} + M_{33}\,\mathrm{e}^{\theta_3+\theta_3^*},$$

$$g_{j,3} = L_{j,121}\,\mathrm{e}^{\theta_1+\theta_2+\theta_1^*} + L_{j,122}\,\mathrm{e}^{\theta_1+\theta_2+\theta_2^*} + L_{j,123}\,\mathrm{e}^{\theta_1+\theta_2+\theta_3^*} + L_{j,131}\,\mathrm{e}^{\theta_1+\theta_3+\theta_1^*} + L_{j,132}\,\mathrm{e}^{\theta_1+\theta_3+\theta_2^*} + L_{j,133}\,\mathrm{e}^{\theta_1+\theta_3+\theta_3^*} + L_{j,231}\,\mathrm{e}^{\theta_2+\theta_3+\theta_1^*} + L_{j,232}\,\mathrm{e}^{\theta_2+\theta_3+\theta_2^*} + L_{j,233}\,\mathrm{e}^{\theta_2+\theta_3+\theta_3^*},$$

$$f_4 = G_{1212}\,\mathrm{e}^{\theta_1+\theta_2+\theta_1^*+\theta_2^*} + G_{1213}\,\mathrm{e}^{\theta_1+\theta_2+\theta_1^*+\theta_3^*} + G_{1223}\,\mathrm{e}^{\theta_1+\theta_2+\theta_2^*+\theta_3^*} + G_{1312}\,\mathrm{e}^{\theta_1+\theta_3+\theta_1^*+\theta_2^*} + G_{1313}\,\mathrm{e}^{\theta_1+\theta_3+\theta_1^*+\theta_3^*} + G_{1323}\,\mathrm{e}^{\theta_1+\theta_3+\theta_2^*+\theta_3^*} + G_{2312}\,\mathrm{e}^{\theta_2+\theta_3+\theta_1^*+\theta_2^*} + G_{2313}\,\mathrm{e}^{\theta_2+\theta_3+\theta_1^*+\theta_3^*} + G_{2323}\,\mathrm{e}^{\theta_2+\theta_3+\theta_2^*+\theta_3^*},$$

$$g_{j,5} = H_{j,12}\,\mathrm{e}^{\theta_1+\theta_2+\theta_3+\theta_1^*+\theta_2^*} + H_{j,13}\,\mathrm{e}^{\theta_1+\theta_2+\theta_3+\theta_1^*+\theta_3^*} + H_{j,23}\,\mathrm{e}^{\theta_1+\theta_2+\theta_3+\theta_2^*+\theta_3^*},$$

$$f_6 = K_{123}\,\mathrm{e}^{\theta_1+\theta_2+\theta_3+\theta_1^*+\theta_2^*+\theta_3^*},$$

$$g_{j,m}=0,\ m=7,9,\ldots,\quad f_n=0,\ n=8,10,\ldots.$$

The coefficients of the three-soliton solutions are given as

$$M_{dh} = \frac{l_1^{(d)}l_1^{(h)*}+l_2^{(d)}l_2^{(h)*}}{(\eta_d+\eta_h^*)^2},\ E_{ds} = (\eta_d+\eta_s^*)M_{ds},$$

$$L_{j,dhs} = \frac{\sigma^{-1}(\eta_d-\eta_h)}{(\eta_d+\eta_s^*)(\eta_h+\eta_s^*)}\begin{vmatrix} l_j^{(d)} & E_{ds} \\ l_j^{(h)} & E_{hs}\end{vmatrix},$$

$$G_{dhsv} = \frac{(\eta_d-\eta_h)(\eta_s^*-\eta_v^*)}{(\eta_d+\eta_s^*)(\eta_d+\eta_v^*)(\eta_h+\eta_s^*)(\eta_h+\eta_v^*)} \cdot \begin{vmatrix} E_{ds} & E_{dv} \\ E_{hs} & E_{hv}\end{vmatrix},\ (d,h,s,v=1,2,3),$$

$$H_{j,12} = \frac{\sigma^{-1}(\eta_1-\eta_2)(\eta_1-\eta_3)(\eta_2-\eta_3)(\eta_1^*-\eta_2^*)}{(\eta_1+\eta_1^*)(\eta_1+\eta_2^*)(\eta_2+\eta_1^*)(\eta_2+\eta_2^*)(\eta_3+\eta_1^*)(\eta_3^*+\eta_2^*)}\begin{vmatrix} l_j^{(1)} & E_{11} & E_{12} \\ l_j^{(2)} & E_{21} & E_{22} \\ l_j^{(3)} & E_{31} & E_{32}\end{vmatrix},$$

$$H_{j,13} = \frac{\sigma^{-1}(\eta_1-\eta_2)(\eta_1-\eta_3)(\eta_2-\eta_3)(\eta_1^*-\eta_3^*)}{(\eta_1+\eta_1^*)(\eta_1+\eta_3^*)(\eta_2+\eta_1^*)(\eta_2+\eta_3^*)(\eta_3+\eta_1^*)(\eta_3^*+\eta_3^*)}\begin{vmatrix} l_j^{(1)} & E_{11} & E_{13} \\ l_j^{(2)} & E_{21} & E_{23} \\ l_j^{(3)} & E_{31} & E_{33}\end{vmatrix},$$

$$H_{j,23} = \frac{\sigma^{-1}(\eta_1-\eta_2)(\eta_1-\eta_3)(\eta_2-\eta_3)(\eta_2^*-\eta_3^*)}{(\eta_1+\eta_2^*)(\eta_1+\eta_3^*)(\eta_2+\eta_2^*)(\eta_2+\eta_3^*)(\eta_3+\eta_2^*)(\eta_3^*+\eta_3^*)}\begin{vmatrix} l_j^{(1)} & E_{12} & E_{13} \\ l_j^{(2)} & E_{22} & E_{23} \\ l_j^{(3)} & E_{32} & E_{33}\end{vmatrix},$$

$$K_{123} = \frac{(\eta_1-\eta_2)(\eta_1-\eta_3)(\eta_2-\eta_3)(\eta_1^*-\eta_2^*)}{(\eta_1+\eta_1^*)(\eta_1+\eta_2^*)(\eta_1+\eta_3^*)(\eta_2+\eta_1^*)(\eta_2+\eta_2^*)(\eta_2+\eta_3^*)} \cdot \frac{(\eta_1^*-\eta_3^*)(\eta_2^*-\eta_3^*)}{(\eta_3+\eta_1^*)(\eta_3+\eta_2^*)(\eta_3+\eta_3^*)}\begin{vmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33}\end{vmatrix}.$$

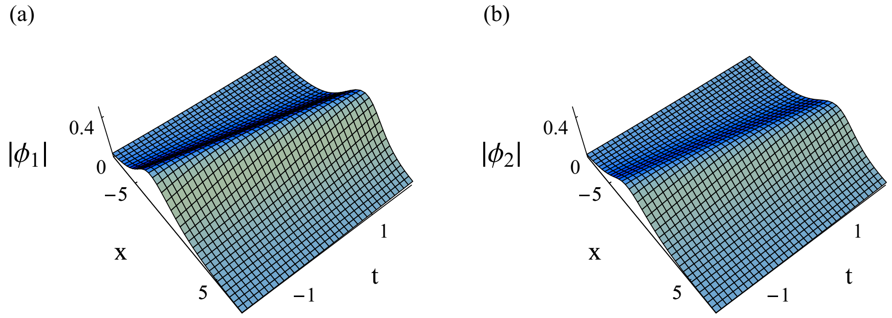

Fig. 1 (colour online). One soliton via (17) with the parameters $a(t) = 1$, $b(t) = 1$, $c(t) = 0$, $\eta_1 = 0.5 + 0.5\mathrm{i}$, $V(x,t) = \sin(t) + 1$, $\omega_1 = 0$, $l_1^{(1)} = \sqrt{0.6}$, $l_2^{(1)} = \sqrt{0.3}$.

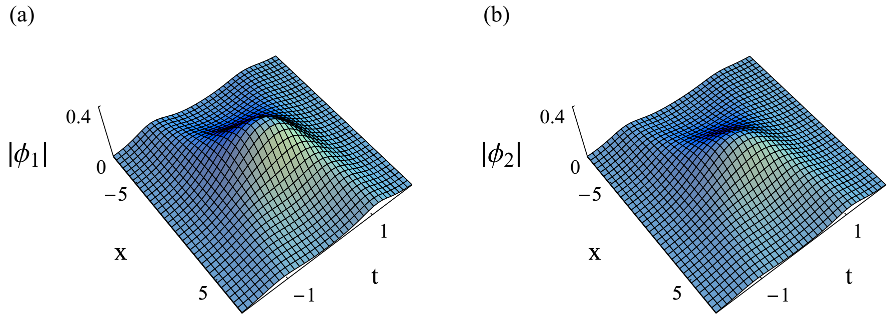

Fig. 2 (colour online). One soliton via (17) with the parameters $a(t) = 1$, $b(t) = 1$, $c(t) = t$, $\eta_1 = 0.5 + 0.5\mathrm{i}$, $V(x,t) = (t^2 + 0.5)x^2 + \sin(t) + 1$, $\omega_1 = 0$, $l_1^{(1)} = \sqrt{0.6}$, $l_2^{(1)} = \sqrt{0.3}$.

4. Propagations and Collisions of Solitons in BECs

In this section, we will study the propagations and collisions of solitons in a BEC and analyze the influence of the parameters $a(t)$, $b(t)$, and $c(t)$.

Based on (17), soliton propagations with $c(t) = 0$ and $c(t) = t$ are respectively shown in Figures 1 and 2. Figure 1 shows that the soliton can stably propagate with $c(t) = 0$. In Figure 2, when $c(t) = t$, the soliton width will become narrower when the amplitude becomes higher, while the width will widen when it reaches the narrowest place, and the amplitude begins to drop at the same time.

When the values of $a(t)$ and $b(t)$ are both constants, for simplicity, we take $a(t) = b(t) = 1$. Based on (19), we find that the parameters $l_j^{(k)} (j,k = 1,2)$ have a direct connection with the collision patterns: elastic and inelastic collisions. When $l_1^{(1)} : l_2^{(1)} = l_1^{(2)} : l_2^{(2)}$, the collision is elastic, which means that there is no energy exchange between the two solitons. When $l_1^{(1)} : l_2^{(1)} \neq l_1^{(2)} : l_2^{(2)}$, the collision is inelastic, which means that there exists an energy exchange between two solitons. That is, the shape-changing collisions arise in this case.

With the parameters chosen as $V(x,t) = \sin(t) + 1$, $c(t) = 0$, $\eta_1 = 0.5 + 0.5\mathrm{i}$, $\eta_2 = 0.6 - 0.5\mathrm{i}$, $\omega_1 = \omega_2 = 0$,

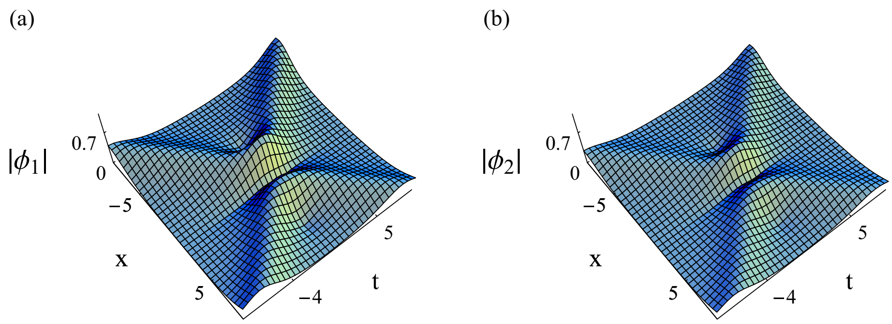

Fig. 3 (colour online). Elastic collisions between two solitons via (19) with the parameters $a(t) = 1$, $b(t) = 1$, $c(t) = 0$, $V(x,t) = \sin(t) + 1$, $\eta_1 = 0.5 + 0.5\mathrm{i}$, $\eta_2 = 0.6 - 0.5\mathrm{i}$, $\omega_1 = \omega_2 = 0$, $l_1^{(1)} = l_1^{(2)} = \sqrt{0.6}$, $l_2^{(1)} = l_2^{(2)} = \sqrt{0.3}$.

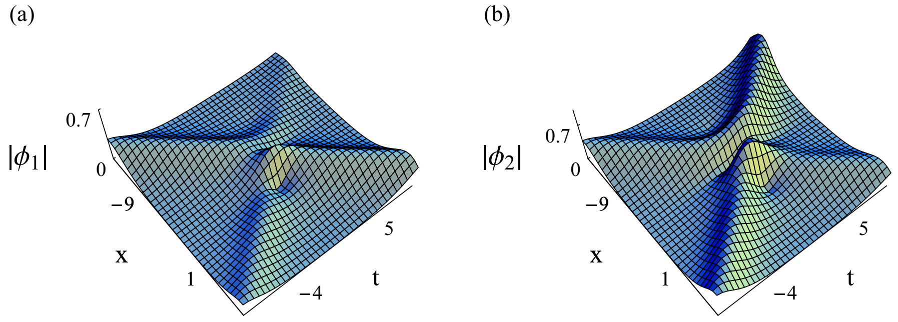

Fig. 4 (colour online). Inelastic collisions between two solitons via (19) with the parameters $a(t) = 1$, $b(t) = 1$, $c(t) = 0$, $V(x,t) = \sin(t) + 1$, $\eta_1 = 0.5 + 0.5\mathrm{i}$, $\eta_2 = 0.6 - 0.5\mathrm{i}$, $\omega_1 = \omega_2 = 0$, $l_1^{(1)} = 6$, $l_1^{(2)} = 1$, $l_2^{(1)} = l_2^{(2)} = 8$.

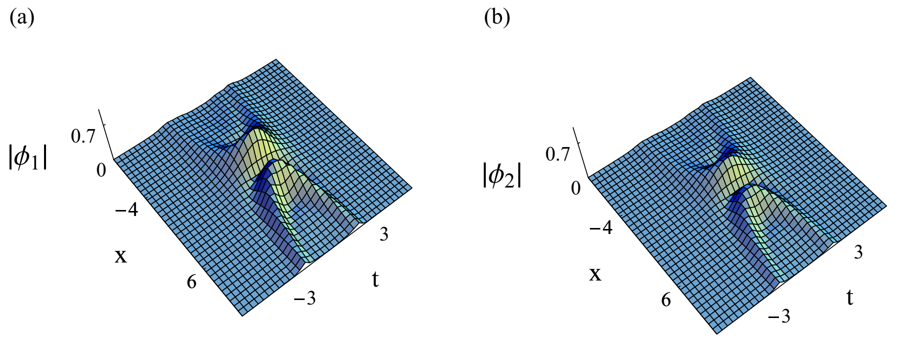

Fig. 5 (colour online). Two solitons via (19) with the parameters $a(t) = 1$, $b(t) = 1$, $c(t) = 0.35t + 0.005$, $V(x,t) = (0.1225t^2 + 0.175)x^2 + \sin(t) + 1$, $\eta_1 = 0.5 + 0.5\mathrm{i}$, $\eta_2 = 0.6 - 0.5\mathrm{i}$, $\omega_1 = \omega_2 = 0$, $l_1^{(1)} = l_1^{(2)} = \sqrt{0.6}$, $l_2^{(1)} = l_2^{(2)} = \sqrt{0.3}$.

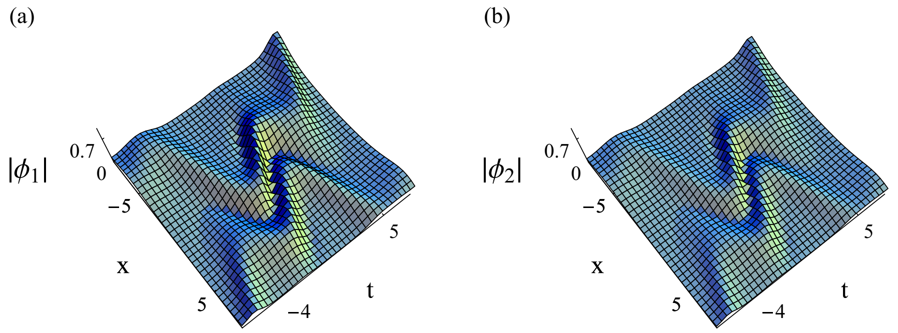

Fig. 6 (colour online). Elastic collisions between two solitons via (19) with the parameters $a(t) = t^2$, $b(t) = 3^{1/3}t^2$, $c(t) = 0$, $V(x,t) = t^2[\sin(\frac{t^3}{3})x + \sin(\frac{t^3}{3}) + 1]$, $\eta_1 = 0.5 + 0.5\mathrm{i}$, $\eta_2 = 0.6 - 0.5\mathrm{i}$, $\omega_1 = \omega_2 = 0$, $l_1^{(1)} = l_1^{(2)} = \sqrt{0.6}$, $l_2^{(1)} = l_2^{(2)} = \sqrt{0.3}$.

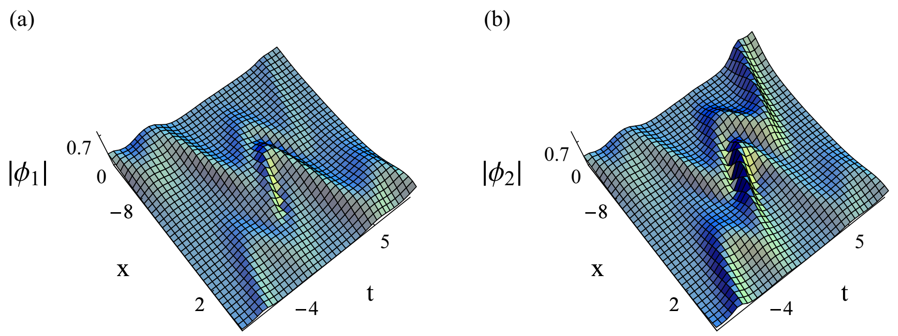

Fig. 7 (colour online). Inelastic collisions between two solitons via (19) with the parameters $a(t) = t^2$, $b(t) = 3^{1/3}t^2$, $c(t) = 0$, $V(x,t) = t^2[\sin(\frac{t^3}{3})x + \sin(\frac{t^3}{3}) + 1]$, $\eta_1 = 0.5 + 0.5\mathrm{i}$, $\eta_2 = 0.6 - 0.5\mathrm{i}$, $\omega_1 = \omega_2 = 0$, $l_1^{(1)} = 6$, $l_1^{(2)} = 1$, $l_2^{(1)} = l_2^{(2)} = 6$.

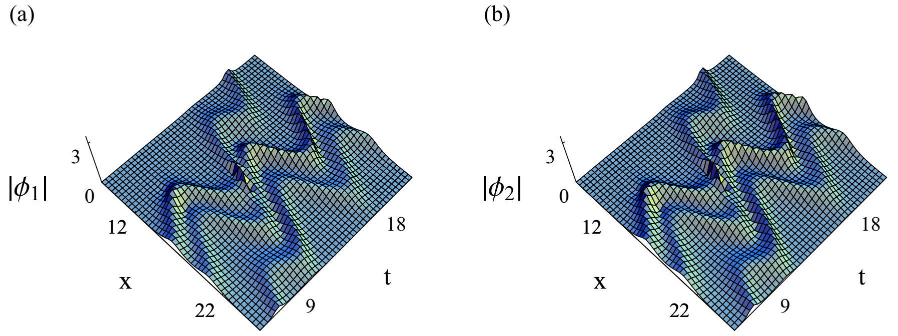

Fig. 8 (colour online). Elastic collisions among three solitons via (20) with the parameters $a(t)=t^2$, $b(t)=3^{1/3}t^2$, $c(t)=0$, $V(x,t)=t^2[\sin(\frac{t^3}{3})x+\sin(\frac{t^3}{3})]$, $\eta_1=1$, $\eta_2=1.5$, $\eta_3=1-0.6\mathrm{i}$, $\omega_1=-15$, $\omega_2=-22$, $\omega_3=-30$, $l_j^{(k)}=1\,(j=1,2;\,k=1,2,3)$.

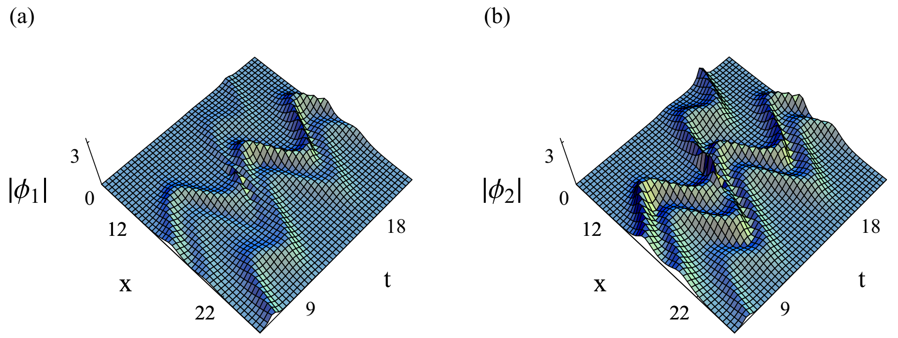

Fig. 9 (colour online). Inelastic collisions among three solitons via (20) with the parameters $a(t)=t^2$, $b(t)=3^{1/3}t^2$, $c(t)=0$, $V(x,t)=t^2[\sin(\frac{t^3}{3})x+\sin(\frac{t^3}{3})]$, $\eta_1=1$, $\eta_2=1.5$, $\eta_3=1-0.6\mathrm{i}$, $\omega_1=-15$, $\omega_2=-22$, $\omega_3=-30$, $l_1^{(1)}=\sqrt{0.6}$, $l_1^{(2)}=l_2^{(1)}=1$, $l_1^{(3)}=\sqrt{0.3}$, $l_2^{(2)}=l_2^{(3)}=2$.

$l_1^{(1)}=l_1^{(2)}=\sqrt{0.6}$, $l_1^{(1)}=l_1^{(2)}=\sqrt{0.3}$, the elastic collisions between two solitons are shown in Figure 3, and we can see that there is no change in the amplitudes of the solitons. Figure 4 reveals how the energy exchange occurs between the two solitons, where the energy transfers from the soliton in the vertical direction in component ϕ_1 to that in ϕ_2 after collision. Such a collision scenario is different from that in Figure 3. Figure 5 presents two solitons with $c(t)=0.35t+0.005$; the width and amplitude of solitons are changed.

When the values of $a(t)$ and $b(t)$ are variable, such as $a(t)=t^2$ and $b(t)=\sqrt[3]{3}t^2$, we analyze the elastic and inelastic collisions, respectively, shown in Figures 6 and 7. We can see that the two solitons propagate forward in the form of a curve. Similar to Figure 3, there is no energy exchange between the solitons in Figure 6. As shown in Figure 7, the amplitude of the soliton in the vertical direction in component ϕ_1 decreases, while that in ϕ_2 increases after the collision. Then the shape-changing collisions can be observed in Figure 7.

In the following, we pay attention to the collisions among three solitons. Soliton collisions among N $(N>3)$ soliton excitations would be more complicated than those between two excitations. According to (20), if $l_1^{(1)}:l_2^{(1)}=l_1^{(2)}:l_2^{(2)}=l_1^{(3)}:l_2^{(3)}$, as seen in Figure 8, the collisions among three solitons are elastic which means there is no energy exchange among three solitons. If $l_1^{(1)}:l_2^{(1)}\neq l_1^{(2)}:l_2^{(2)}$, or $l_1^{(2)}:l_2^{(2)}\neq l_1^{(3)}:l_2^{(3)}$, or $l_1^{(1)}:l_2^{(1)}\neq l_1^{(3)}:l_2^{(3)}$, inelastic collisions appear as in Figure 9. That is, there are some shape-changing collisions with energy exchange among three solitons.

5. Conclusions

Via symbolic computation, we have investigated system (1), a variable-coefficient coupled GP system, which can describe the matter-wave solitons in BECs with small perturbation $a(t)$, atom–atom collision effect $b(t)$, external potential $V(x,t)$, and damping or feeding effect $c(t)$. We have found the integrable condition for (1), i.e., constraint (10). Based on bilinear form (13), soliton solutions (17), (19), and (20) have been derived through the Hirota bilinear method. Moreover, we have studied the propagations and collisions of solitons, e.g., elastic and inelastic collisions of two solitons. We have found that: (i) in the elastic collisions, the external potential $V(x,t)$ can change the propagation of the soliton and thus the density of the matter wave in the two-species BEC (see Figs. 3 and 5); (ii) in the shape-changing collision, the solitons can exchange energy among different species, leading to the change of soliton amplitude (see Figs. 4 and 6). We have also obtained the collisions of three solitons of atomic matter waves (see Figs. 7 and 8).

Acknowledgements

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