New Positon, Negaton, and Complexiton Solutions for a Coupled Korteweg–de Vries – Modified Korteweg–de Vries System

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New positon, negaton, and complexiton solutions of a coupled Korteweg–de Vries – modified Korteweg–de Vries system are constructed directly from the zero seed solution by means of Darboux transformation with different constant selection. The positon, negaton, and complexiton solutions are given out analytically and graphically.

Key words: Positon; Negaton; Complexiton; Darboux Transformation.
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1. Introduction

On the exact solutions of integrable models, there is a new classification way recently based on the property of associated spectral parameters [1]. Negatons, related to the negative spectral parameter, are usually expressed by hyperbolic functions, and positons are expressed by means of trigonometric functions related to the positive spectral parameters. The so-called complexiton, which is expressed by combinations of trigonometric and hyperbolic functions, is related to the complex spectral parameters. For many important integrable systems such as the Korteweg–de Vries (KdV) equation, there is no nonsingular positon. But negatons can be both singular and nonsingular. Especially, the nonsingular negatons are just the solitons which have been studied extensively. The known complexitons for $(1+1)$-dimensional integrable systems are singular. But some types of the analytical positons and complexitons in $(2+1)$- and $(3+1)$-dimensions can be easily obtained because of the existence of arbitrary functions in their expressions of exact functions [2, 3].

For other integrable systems, the complexiton solutions have been presented by different approaches. For example, using the Casoratian technique, the authors of [4] constructed the complexiton solutions to the Toda lattice equation through the Casoratian formulation. The new rational solutions, solitons, positons, negatons, and complexiton solutions for the KdV equation are given by the Wronskian formula with the help of its bilinear form [5]. Ma provided the complexiton solutions of the KdV equation and the Toda lattice equation through the Wronskian and Casoratian techniques [6]. The authors of [7, 8] presented the positons, negatons, and complexitons and their interaction solutions for the Boussinesq equation through its Wronskian determinant.

On the other hand, the coupled integrable systems, which have attracted more and more attention from the mathematicians and physicists, have also been studied extensively since the first coupled KdV system was put forward by Hirota and Satsuma. Since then, many other coupled KdV systems are constructed such as the Ito system, the Nutku–Oğuz model, and so on. Recently, new positon, negaton, and complexiton solutions for the two types of the coupled KdV system are presented by means of the Darboux transformation [9, 10]. The analytical positon, negaton, and complexiton solutions for the coupled modified KdV (mKdV) system are given out directly in [11] from the zero seed solution by means of Darboux transformation. For different integrable systems, these new positon, negaton, and complexiton solutions are analytical or singular.

It is known that some generalized KdV–mKdV system have solitary wave solutions and explicit solutions in terms of Jacobi elliptic functions [12, 13]. A special coupled KdV–mKdV system is written as

$$f_t+\frac{1}{2}f_{xxx}+\frac{3}{2}(uf)_x-\frac{3}{4}f_xf^2=0,$$
$$u_t-\frac{1}{4}u_{xxx}-\frac{3}{2}uu_x+3vv_x+\frac{3}{4}u_xf^2-\frac{3}{2}(f_xv)_x=0, \quad (1)$$
$$v_t+\frac{1}{2}v_{xxx}+\frac{3}{2}uv_x-\frac{3}{4}(vf^2)_x+\frac{3}{4}u_{xx}f+\frac{3}{2}u_xf_x=0.$$

The Lax pair and Darboux transformation are given in [14] and the system (1) exhibits two integrable reductions, Hirota–Satsuma coupled KdV equation and a two components KdV–mKdV system. Some analytical and numerical solutions for system (1) have been introduced by the compound elementary Darboux transformation and a numerical method in [15], and the convergence of the scheme is also proved and tested by numerical simulation.

In the remaining part of this article, we try to construct the new positon, negaton, and complexiton solutions of the coupled KdV–mKdV system (1) by means of the Darboux transformation, which is one of the most powerful tools to construct the exact and explicit solutions of nonlinear evolution equations. In Section 2, the Darboux transformation for the coupled KdV–mKdV system is briefly introduced from the reduction of the second-order differential equation with 2×2 matrix coefficients. In Section 3, new positon, negaton, and complexiton solutions for the coupled KdV–mKdV system are given out directly from Darboux transformation with constant seed solutions, and these exact solutions are singular or analytical depending on different constant selections. The discussions and summary are presented in Section 4.

2. Darboux Transformation for the Coupled KdV–mKdV System

In this section, the Darboux transformation for the coupled KdV–mKdV system in [15] is introduced firstly and briefly. Consider two differential equations of the second order with spectral parameter λ and 2×2 matrix coefficients

$$\Phi_{xx}+F\Phi_x+U\Phi=\lambda\sigma_3\Phi, \quad (2)$$
$$\Phi_t=\Phi_{xxx}+B\Phi_x+C\Phi, \quad (3)$$

where the vector $\Phi=(\varphi_1,\varphi_2)^{\mathrm{T}}$ and the matrix potentials are $U=\{u_{ij}\}$, $F=\{f_{ij}, f_{ii}=0\}$, $i=1,2$, while $\sigma_3=\mathrm{diag}(1,-1)$ is the Pauli matrix. And two coefficients B and C are determined by

$$B=\frac{3}{2}\mathrm{diag}\,U+\frac{3}{2}F_x-\frac{3}{4}F^2,$$
$$C=\frac{3}{2}U_x-\frac{3}{4}\mathrm{diag}\,U_x-\frac{3}{4}(f_{12}u_{21}+f_{21}u_{12})I$$
$$+\frac{3}{8}(f_{12x}f_{21}-f_{12}f_{21x})\sigma_3+\frac{3}{4}(u_{11}-u_{22})\sigma_3F,$$

with I being unit matrix.

If the new wave function for system (2) and (3) is in the form

$$\Phi[1]=P\Phi_x+K\Phi,$$

where $P=\begin{pmatrix}1&0\\0&0\end{pmatrix}$ is a projection operator and $K=\begin{pmatrix}k_{11}&k_{12}\\k_{21}&1\end{pmatrix}$, then the Darboux transformation of the new wave functions for system (2) and (3) is as follows:

$$\begin{aligned}
&\varphi_1[1]=\varphi_{1x}+k_{11}\varphi_1+k_{12}\varphi_2,\\
&\varphi_2[1]=\varphi_2+k_{21}\varphi_1,\\
&\psi_1[1]=\psi_{3x}+k_{11}\psi_3+k_{12}\psi_4,\\
&\psi_2[1]=\psi_4+k_{21}\psi_3,\\
&k_{11}=-\left(\psi_{1x}+\frac{1}{2}f_{12}\psi_2\right)/\psi_1,\\
&k_{12}=f_{12}/2,\\
&k_{21}=-\psi_2/\psi_1,
\end{aligned} \quad (4)$$

and the new potentials are found to have the expressions

$$\begin{aligned}
&f_{12}[1]=u_{12}+f_{12}k_{11},\\
&f_{21}[1]=-2k_{21},\\
&u_{11}[1]=u_{11}-2k_{11x}-f_{12}[1]k_{21}-f_{21}k_{12},\\
&u_{12}[1]=u_{12x}-k_{12xx}+k_{11}u_{12}-k_{12}(u_{11}[1]+u_{22}),\\
&u_{21}[1]=f_{21}-2k_{21x}-f_{21}[1]k_{11},\\
&u_{22}[1]=u_{22}-k_{21}u_{12}-u_{21}[1]k_{12}-f_{21}[1]k_{12x},
\end{aligned} \quad (5)$$

where $(\psi_1,\psi_2)^{\mathrm{T}}$ and $(\psi_3,\psi_4)^{\mathrm{T}}$ are two solutions of (2) and (3) corresponding to different values of the spectral parameter.

Imposing the reduction $f_{12}=f_{21}=f$, $u_{11}=u_{22}=u$, $u_{12}=u_{21}=v$, one can verify directly that (2) and (3) are just the Lax pair of the coupled KdV–mKdV system, and the first step Darboux transformation for the

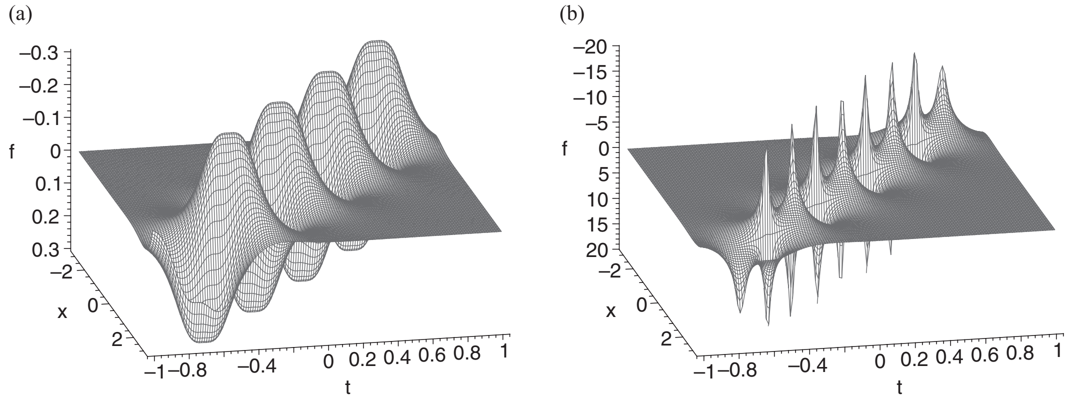

Fig. 1. Analytical positon solution for potential f with constants (13). (a) $k_1 = \frac{2}{27} < 1$; (b) $k_2 = 0.99$.

coupled KdV–mKdV system (1) is nothing else but (4) and (5).

In other words, the new potential functions of the system (1) with trivial zero initial solution for simplicity have the forms

$$f = \frac{2(\phi_1\phi_{2x} - \phi_2\phi_{1x})}{\phi_1^2 - \phi_2^2}, \tag{6}$$

$$u = \left(\frac{(\phi_1^2)_x - (\phi_2^2)_x}{\phi_1^2 - \phi_2^2}\right)_x + 2\left(\frac{\phi_1\phi_{2x} - \phi_2\phi_{1x}}{\phi_1^2 - \phi_2^2}\right)^2, \tag{7}$$

$$v = 2\left(\frac{\phi_1\phi_{2x} - \phi_2\phi_{1x}}{\phi_1^2 - \phi_2^2}\right)_x + \frac{[\phi_1\phi_{2x} - \phi_2\phi_{1x}][(\phi_1^2)_x - (\phi_2^2)_x]}{[\phi_1^2 - \phi_2^2]^2}, \tag{8}$$

where ϕ_1, ϕ_2 are solutions of (2) and (3) with zero initial potential

$$\phi_1 = c_1\,\mathrm{e}^{\sqrt{\lambda}x + \sqrt{\lambda^3}t} + c_2\,\mathrm{e}^{-(\sqrt{\lambda}x + \sqrt{\lambda^3}t)}, \tag{9}$$

$$\phi_2 = d_1\,\mathrm{e}^{(\sqrt{\lambda}x - \sqrt{\lambda^3}t)\mathrm{i}} + d_2\,\mathrm{e}^{-(\sqrt{\lambda}x - \sqrt{\lambda^3}t)\mathrm{i}}, \tag{10}$$

with c_1, c_2, d_1, d_2 being arbitrary constants and $\mathrm{i}^2 = -1$. Then we can obtain the new positon, negaton, and complexiton solutions for the coupled KdV–mKdV system from (6)–(8) by different spectral parameter selections in (9) and (10). It is interesting that the new exact and explicit solutions obtained from the first-step Darboux transformation (6)–(8) are singular or analytical depending on the arbitrary constants c_1, c_2, d_1, d_2. The detailed discussions are given out in the next section.

3. Positon, Negaton, and Complexiton Solutions

We will construct new positon, negaton and complexiton solutions for the coupled KdV–mKdV system from its Darboux transformation with zero constant seed solution by selecting the spectral parameters as real or complex number in the wave functions (9) and (10).

3.1. Positon Solutions, $\lambda > 0$

Based on the first-step Darboux transformation (6)–(8) with the two wave functions (9) and (10), one can obtain the positon solutions by fixing the spectral parameter $\lambda > 0$. It should be pointed out that the amplitude of the positon is different depending on the selections of the four arbitrary constants c_1, c_2, d_1, d_2.

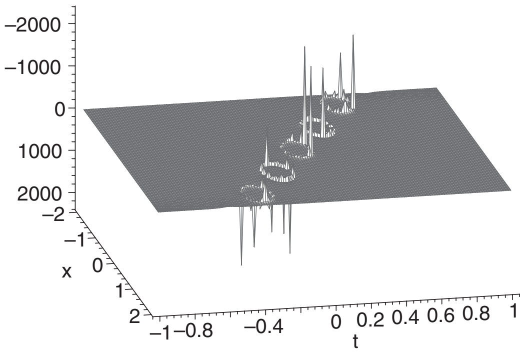

Fig. 2. Analytical positon solution for potential f of the coupled KdV–mKdV system with constants (13) and $k_3 = \frac{3}{2} > 1$.

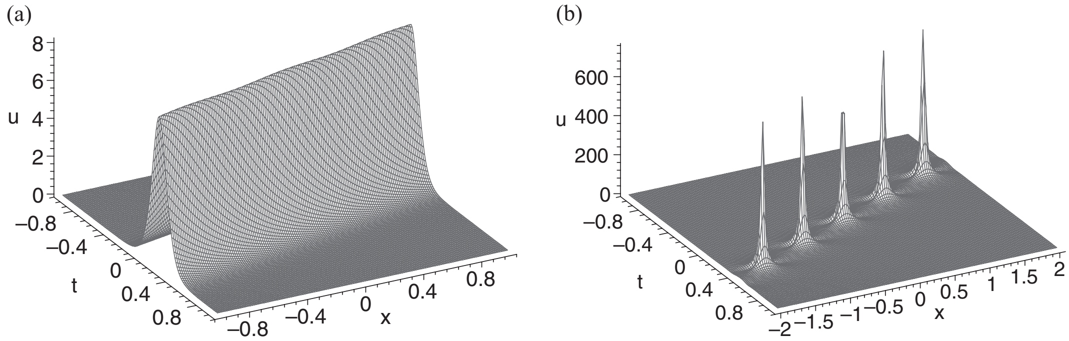

Fig. 3. Analytical positon solution for potential u with constants (13). (a) $k_1 = \frac{2}{27} < 1$; (b) $k_2 = 0.99$.

Fixing the constants as

$$c_1 = c_2 = c, \ \ d_1 = d_2 = ck,$$

then the wave functions are in the form

$$\phi_1 = c\left[\mathrm{e}^{\sqrt{\lambda}x+\sqrt{\lambda^3}t} + \mathrm{e}^{-(\sqrt{\lambda}x+\sqrt{\lambda^3}t)}\right], \qquad (11)$$

$$\phi_2 = ck\left[\mathrm{e}^{(\sqrt{\lambda}x-\sqrt{\lambda^3}t)\mathrm{i}} + \mathrm{e}^{-(\sqrt{\lambda}x-\sqrt{\lambda^3}t)\mathrm{i}}\right]. \qquad (12)$$

Selecting the constants as

$$\lambda = 4, \ \ c = \frac{3}{4}, \qquad (13)$$

and $k_1 = \frac{2}{27}$, $k_2 = 0.99$, $k_3 = \frac{3}{2}$, we can construct the analytical positon solutions from the Darboux transformation (6) for the potential f of the coupled KdV–mKdV system which are given in Figures 1 and 2.

Similarly, when the constants are selected as (13) and $k_1 = \frac{2}{27}$, $k_2 = 0.99$, $k_3 = \frac{11}{10}$, the analytical or singular positon solutions for the potentials u and v are constructed directly from the Darboux transformation (7) and (8). The detailed structures are shown in Figures 3–6.

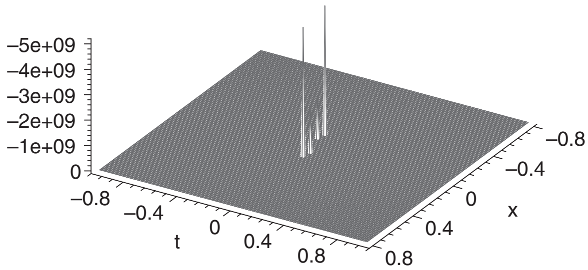

Fig. 4. Singular positon solution for potential u with constants (13) and $k_3 = \frac{11}{10} > 1$.

It is seen that when the coefficient $k \leq 1$, the positon solutions for the potentials $\{f, u, v\}$ of the coupled KdV–mKdV system are nonsingular, but for $k > 1$ poles appear, the positon solutions for u and v have singularities.

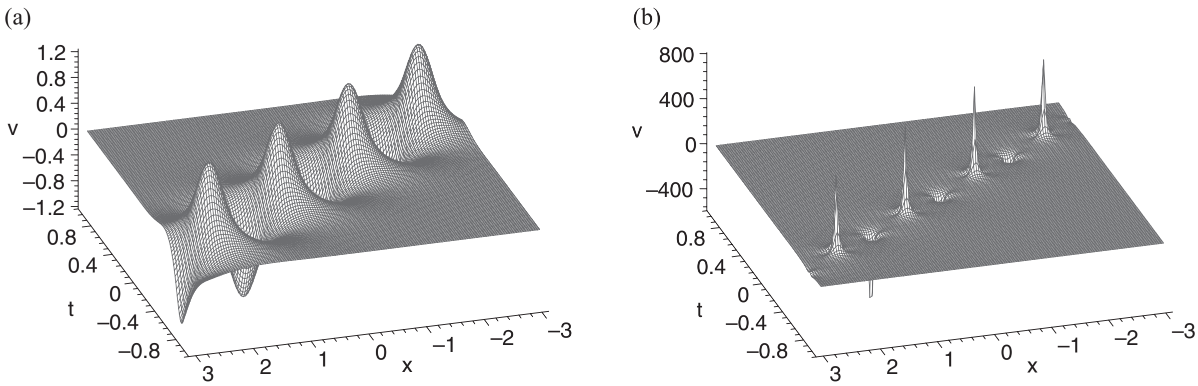

Fig. 5. Analytical positon solution for potential v with constants (13). (a) $k_1 = \frac{2}{27} < 1$; (b) $k_2 = 0.99$.

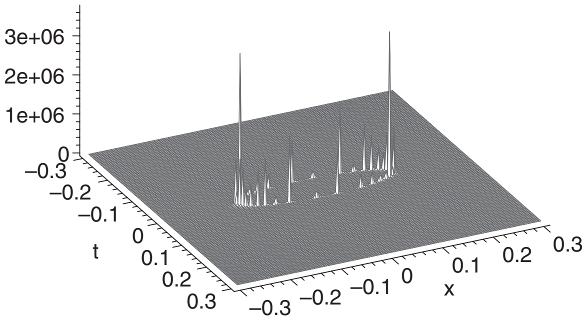

Fig. 6. Singular positon solution for potential v with constants (13) and $k_3 = \frac{11}{10} > 1$.

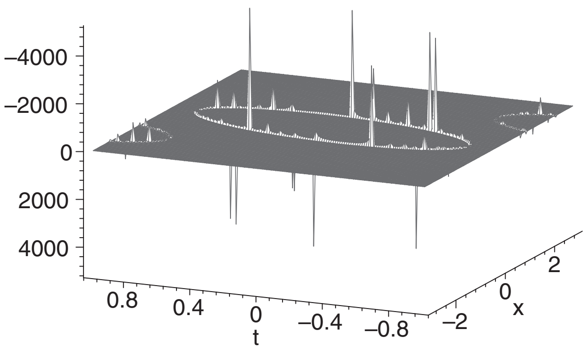

Fig. 7. Analytical negaton solution for potential f with constants (14) and $k_1 = \frac{1}{2} < 1$.

3.2. Negaton Solutions, $\lambda < 0$

In order to construct the negaton solutions for the coupled KdV–mKdV system, one only need to fix the spectral parameter $\lambda < 0$. In the similar way, we obtain the singular or nonsingular negaton solutions of the coupled KdV–mKdV system. When the constants are chosen as

$$\lambda = -1, \ c = \frac{3}{4}, \tag{14}$$

and $k_1 = \frac{1}{2}$, $k_2 = 1$, $k_3 = \frac{11}{10}$, the negaton solutions for the potentials $\{f, u, v\}$ can be obtained easily from (6) – (8). Here we only give out the detailed structures for the potential f and omit the ones for potentials u and v for simplicity because they can be constructed in the same way. From Figures 7 and 8, we can see that the negaton solutions are different to the positon solutions. That is to say, when $k < 1$, the negaton solutions are nonsingular with big amplitude, but the amplitude will become much smaller for $k \geq 1$.

3.3. Complexiton Solutions

Complexiton solutions for many integrable systems, which correspond to the complex parameters in the Lax pair, have been found for KdV equation, Toda lattice equation, and many coupled integrable systems by means of different approaches [4 – 11]. Most of the known complexiton solutions for nonlinear integrable systems are singular except for the coupled KdV system and the coupled mKdV system, and it is meaningful and interesting to find much more analytical complexiton solutions for the nonlinear integrable system. Fixing the spectral parameter $\lambda = \mathrm{i}$ in (9) and (10) and selecting the constants as

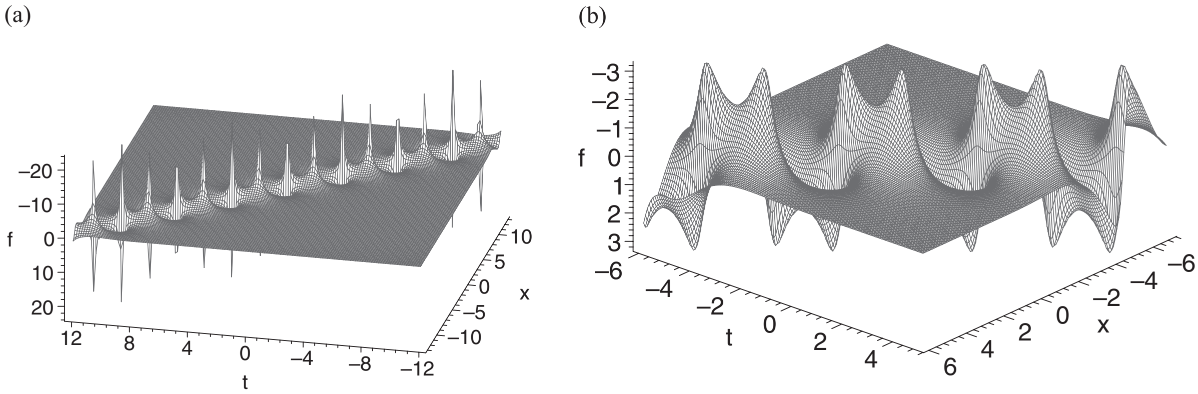

Fig. 8. Analytical negaton solution for potential f with constants (14). (a) $k_2 = 1$; (b) $k_3 = \frac{11}{10} > 1$.

$$c_1 = c_2 = \frac{3}{4}, \; d_1 = d_2 = \frac{3}{8}, \tag{15}$$

then from the Darboux transformation (6)–(8), the three fields f, u and v become

$$f = \frac{-4\sqrt{2}\mathrm{i}[\sinh(-\sqrt{2}x+\sqrt{2}t)-\sin(\sqrt{2}x+\sqrt{2}t)]}{5\mathrm{i}\sin(\sqrt{2}x+\sqrt{2}t)\sinh(\sqrt{2}t-\sqrt{2}x)-3\cos(\sqrt{2}x+\sqrt{2}t)\cosh(\sqrt{2}t-\sqrt{2}x)-3}, \tag{16}$$

$$u = \frac{B}{A}, \tag{17}$$

$$v = \frac{C}{A}, \tag{18}$$

where

$$\begin{aligned} A = {} & 36\cosh(-\sqrt{2}x+\sqrt{2}t)\cos(\sqrt{2}x+\sqrt{2}t)+35 \\ & -8\cos(2\sqrt{2}x+2\sqrt{2}t)-60\mathrm{i}\sinh(-\sqrt{2}x+\sqrt{2}t) \\ & \cdot\sin(\sqrt{2}x+\sqrt{2}t)+17\cosh(-2\sqrt{2}x+2\sqrt{2}t) \\ & \cdot\cos(2\sqrt{2}x+2\sqrt{2}t)-15\mathrm{i}\sin(2\sqrt{2}x+2\sqrt{2}t) \\ & \cdot\sinh(-2\sqrt{2}x+2\sqrt{2}t)-8\cosh(-2\sqrt{2}x+2\sqrt{2}t), \end{aligned} \tag{19}$$

$$\begin{aligned} B = {} & 8[15\mathrm{i}\cos(\sqrt{2}x+\sqrt{2}t)\cosh(-\sqrt{2}x+\sqrt{2}t) \\ & +17\sinh(-\sqrt{2}x+\sqrt{2}t)\sin(\sqrt{2}x+\sqrt{2}t)+15\mathrm{i} \\ & +2\cosh(-2\sqrt{2}x+2\sqrt{2}t)-2\cos(2\sqrt{2}x+2\sqrt{2}t)], \end{aligned} \tag{20}$$

and

$$\begin{aligned} C = {} & -4[3\mathrm{i}\sin(2\sqrt{2}x+2\sqrt{2}t)\sinh(-\sqrt{2}x+\sqrt{2}t) \\ & +5\sinh(-2\sqrt{2}x+2\sqrt{2}t)\sin(\sqrt{2}x+\sqrt{2}t) \\ & +3\mathrm{i}\cosh(-2\sqrt{2}x+2\sqrt{2}t)\cos(\sqrt{2}x+\sqrt{2}t) \\ & -3\mathrm{i}\sinh(-2\sqrt{2}x+2\sqrt{2}t)\sin(\sqrt{2}x+\sqrt{2}t) \\ & +21\mathrm{i}\cosh(-\sqrt{2}x+\sqrt{2}t)-5\cos(\sqrt{2}x+\sqrt{2}t) \\ & +3\mathrm{i}\cos(2\sqrt{2}x+2\sqrt{2}t)\cosh(-\sqrt{2}x+\sqrt{2}t) \\ & -5\cos(2\sqrt{2}x+2\sqrt{2}t)\cosh(-\sqrt{2}x+\sqrt{2}t) \\ & +5\sin(2\sqrt{2}x+2\sqrt{2}t)\sinh(-\sqrt{2}x+\sqrt{2}t) \\ & +5\cosh(-2\sqrt{2}x+2\sqrt{2}t)\cos(\sqrt{2}x+\sqrt{2}t) \\ & +21\mathrm{i}\cos(\sqrt{2}x+\sqrt{2}t)+5\cosh(-\sqrt{2}x+\sqrt{2}t)]. \end{aligned} \tag{21}$$

It is seen that (16)–(18) with (19)–(21) are complex solutions for the coupled KdV–mKdV system. Separating the real and imaginary parts of the complexiton solutions for the fields f, u, v, we can construct the nonsingular complexiton solutions for the coupled KdV–mKdV system. The detailed structures are given out directly in Figures 9–11.

In this section, with the help of Darboux transformation for the coupled KdV–mKdV system, we have obtained the positon, negaton, and complexiton solutions by selecting the real or complex spectral parameter in its Lax pair. These new negaton solutions for the coupled KdV–mKdV system are singular or nonsingular due to the different constants selections. But the real and imaginary parts of the complexiton solutions are just analytical and may be regarded as the usual periodic waves or quasi-periodic waves.

4. Summary and Discussions

In this paper, three kinds of new exact and explicit solutions called positons, negatons, and complexitons

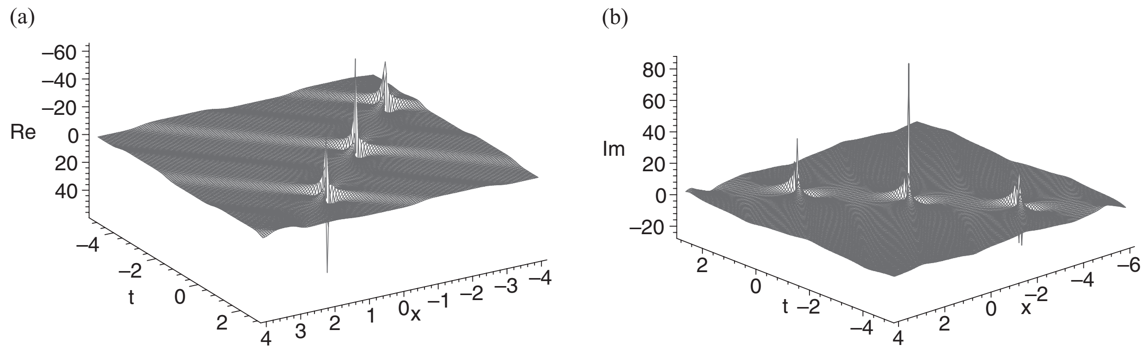

Fig. 9. Nonsingular complexiton solution for potential f with constants (16). (a) Real part; (b) Imaginary part.

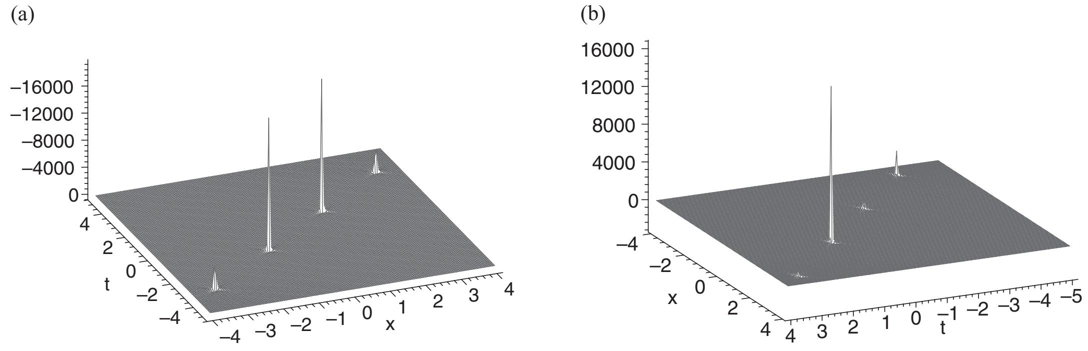

Fig. 10. Nonsingular complexiton solution for potential u with constants (17), (19), and (20); (a) Real part; (b) Imaginary part.

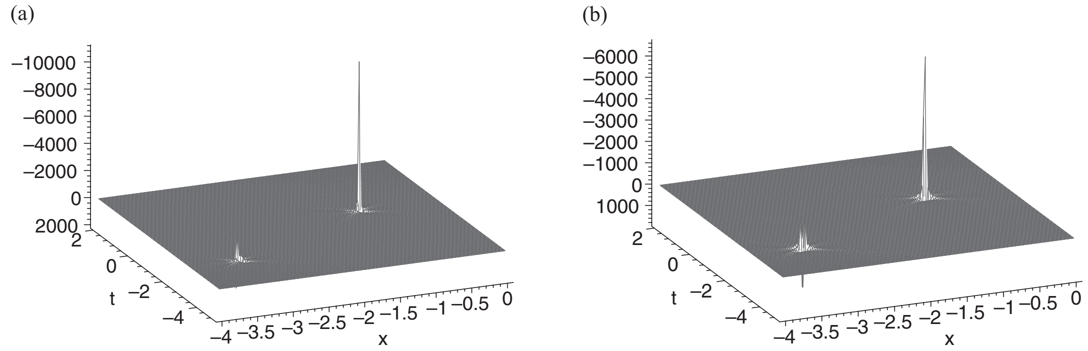

Fig. 11. Analytical complexiton solution for potential v with constants (18), (19), and (21); (a) Real part; (b) Imaginary part.

for the coupled KdV–mKdV system are obtained by means of Darboux transformation from constant seed solution. New analytical or singular positon, negaton, and complexiton solutions for the coupled KdV–mKdV system are constructed and given out analytically and graphically with the help of its Darboux transformation by different constant selections.

The positon for the potential f of the coupled KdV–mKdV system is nonsingular, but the positon solutions for u and v are analytical when the constants in Lax pair (9) and (10) are fixed as $c_1 = c_2 = c$, $d_1 = d_2 = ck$ and $k \leq 1$, but singularities arise when $k > 1$. And the negaton solutions are all nonsingular. That is to say, we can construct different positon and negaton solutions for the coupled KdV–mKdV system by different selections of four arbitrary constants in the wave functions of the Lax pair. The real and imaginary parts of complexiton solutions for the coupled KdV–mKdV system are analytical and can be regarded as the usual periodic or quasi-periodic waves.

On the other hand, the positon, negaton, and complexiton solutions for the integrable systems have been studied extensively by many authors under different approaches, such as the Hirota bilinear form, Casoratian technique, Wronskian determinant, and Bäcklund transformations, etc. But how to construct the Bäcklund transformation for the coupled KdV–mKdV system in order to give out the corresponding positon, negaton, and complexiton solutions is also an interesting and significant topic, and other integrable properties, such as Painlevé property and symmetry reduction, for the coupled KdV–mKdV system should also be studied. And the more about this coupled system and the possible applications of new positon, negaton, and complexiton solutions proposed in this article is worthy of studying further.

Acknowledgements

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