Trace Formula on Transmission Eigenvalues of the Sturm–Liouville Problem

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Z. Naturforsch. **67a,** 429 – 434 (2012) / DOI: 10.5560/ZNA.2012-0041
Received January 10, 2012

In this paper, we consider eigenvalues and traces of a special Sturm–Liouville problem, which originates from an acoustic scattering problem with a spherically symmetric speed of sound. Regularized trace formulae on transmission eigenvalues of the Sturm–Liouville problem are obtained.

Key words: Sturm–Liouville Problem; Transmission Eigenvalues; Trace Formula.
Mathematics Subject Classification 2000: 34A55, 34B24, 47E05

1. Introduction

As is known, the trace of a finite-dimensional matrix is the sum of all the eigenvalues. But in an infinite-dimensional space, in general, ordinary differential operators do not have a finite trace. Gelfand and Levitan [1] firstly obtained a trace formula for a self-adjoint Sturm–Liouville differential equation. After these studies, several mathematicians were interested in developing trace formulae for different differential operators. For the scalar Sturm–Liouville problems, there is an enormous literature on estimates of large eigenvalues and regularized trace formulae which may often be computed explicitly in terms of the coefficients of operators and boundary conditions. A detailed list of publications related to the present aspect can be found in [2].

The trace formulae can be used for an approximate calculation of the first eigenvalues of an operator [2] and in order to establish necessary and sufficient conditions for a set of complex numbers to be the spectrum of an operator [3].

Trace formulae for the matrix Sturm–Liouville problems were considered (see, e.g., [4 – 6], etc.). In this paper, we consider the acoustic scattering problem for a spherically symmetric inhomogeneity of compact support and study transmission eigenvalues corresponding to spherically symmetric eigenfunctions of the homogeneous interior transmission problem. This problem has been considered by a number of authors (see, e.g., [7 – 9]). However, the trace problem on transmission eigenvalues of the Sturm–Liouville problem has never been considered before.

2. Results

The main objective of this paper is to obtain the regularized trace formula for the matrix Sturm–Liouville problem

$$-Y''(x)+Q(x)Y(x)=\lambda^2 Y(x),\ \ x\in(0,1)\,, \qquad (1)$$

with the boundary conditions

$$\begin{aligned} &Y'(0)-HY(0)=0\,,\\ &Y(1)\cos(\lambda a)-Y'(1)\frac{\sin(\lambda a)}{\lambda}=0\,, \end{aligned} \qquad (2)$$

where λ is the spectral parameter and real number $0\le a\ne 1$, $Y(x)=(y_1(x),\cdots,y_d(x))^{\mathrm{t}}$, the entries of the $d\times d$ matrix-valued function $Q(x)$ belong to the space $C^1[0,\pi]$, and H is a $d\times d$ scalar matrix. In particular, $H=\infty$ means the Dirichlet boundary condition $Y(0)=0$. This special Sturm–Liouville problem is originated from the corresponding homogeneous eigenvalue problem with radially symmetric eigenfunctions in the acoustic scattering problem.

Denote

$$\mathbb{Z}=\{0,\pm1,\pm2,\cdots\}\,.$$

Theorem 1. *(i) For the case $H \neq \infty$, let $\lambda_n^{(j)}, j = \overline{1,d}, n \in \mathbb{Z}$, be the spectrum of problem (1) and (2), then for sufficiently large $|n|$*

$$\lambda_n^{(j)} = \frac{\left(n+\frac{1}{2}\right)\pi}{1-a} + \frac{\omega_{jj}^{(1)}}{\left(n+\frac{1}{2}\right)\pi} + \frac{(1-a)\sin\frac{(2n+1)\pi}{1-a}}{4\left(n+\frac{1}{2}\right)^2\pi^2} Q_j(1) + O\left(\frac{1}{n^3}\right), \tag{3}$$

where $\omega_{ij}^{(1)}$ denotes entry of matrix $\omega^{(1)}$ at the i-th row and j-th column, $i, j = 1, 2, \cdots, d$, and

$$\omega^{(1)} = H + \frac{1}{2}\int_0^1 Q(t)\,\mathrm{d}t\,.$$

(ii) For the case $H = \infty$, let $\lambda_n^{(j)}, j = \overline{1,d}, n \in \mathbb{Z}\setminus\{0\}$, be the spectrum of problem (1) and (2), then for sufficiently large $|n|$

$$\lambda_n^{(j)} = \frac{n\pi}{1-a} + \frac{\omega_{jj}^{(2)}}{n\pi} + \frac{(a-1)\sin\frac{2n\pi}{1-a}}{4n^2\pi^2} Q_j(1) + O\left(\frac{1}{n^3}\right), \tag{4}$$

where $\omega_{ij}^{(2)}$ denotes entry of matrix $\omega^{(2)}$ at the i-th row and j-th column, $i, j = 1, 2, \cdots, d$, and

$$\omega^{(2)} = \frac{1}{2}\int_0^1 Q(t)\,\mathrm{d}t\,.$$

It is seen from formulae (3) and (4) that the series

$$S_1 \stackrel{\text{def}}{=} \sum_{n\in\mathbb{Z}} \left\{ \sum_{j=1}^{d} \left[\left(\lambda_n^{(j)}\right)^2 - \frac{\left(n+\frac{1}{2}\right)^2\pi^2}{(1-a)^2} \right] - \frac{2}{1-a}\operatorname{tr}\omega^{(1)} \right\} \tag{5}$$

and

$$S_2 \stackrel{\text{def}}{=} \sum_{n\in\mathbb{Z}\setminus\{0\}} \left\{ \sum_{j=1}^{d} \left[\left(\lambda_n^{(j)}\right)^2 - \frac{n^2\pi^2}{(1-a)^2} \right] - \frac{2}{1-a}\operatorname{tr}\omega^{(2)} \right\} \tag{6}$$

are absolutely convergent, where $\operatorname{tr}A$ denotes the trace of a matrix A. In this work, we will find formulae for the sums of series (5) and (6), which are so-called regularized traces.

Theorem 2. *We have the trace formulae*

$$S_1 = \frac{1}{2}\operatorname{tr}Q(0) + \frac{1}{\pi}\left[2f\left(\frac{\pi}{1-a}\right) - f\left(\frac{2\pi}{1-a}\right) - \frac{\pi}{2}\right] \cdot \operatorname{tr}Q(1) - \operatorname{tr}H^2 \tag{7}$$

and

$$S_2 = -\frac{1}{2}\operatorname{tr}Q(0) + \frac{1}{\pi}\left[\frac{\pi}{2} - \frac{\pi}{1-a} - f\left(\frac{2\pi}{1-a}\right)\right] \cdot \operatorname{tr}Q(1) + \frac{1}{1-a}\operatorname{tr}\int_0^1 Q(t)\,\mathrm{d}t\,, \tag{8}$$

where $f(x)$ is a period function with 2π and

$$f(x) = \begin{cases} \frac{\pi-x}{2}, & 0 < x < \pi\,, \\ 0, & x = 0, \pi\,, \\ \frac{x-\pi}{2}, & -\pi < x < 0\,. \end{cases}$$

3. Proof

We only give the proof of (7). Analogously, we can also prove that (3), (4), and (8) hold.

Let $\Phi(x,\lambda)$ be the solution of (1) satisfying the initial conditions

$$\Phi(0,\lambda) = I_d\,,\ \ \Phi'(0,\lambda) = H\,,$$

where I_d is a $d \times d$ unit matrix. Let us show that $\Phi(x,\lambda)$ satisfies the integral equation

$$\Phi(x,\lambda) = \cos(\lambda x)I_d + H\frac{\sin(\lambda x)}{\lambda} + \int_0^x \frac{\sin[\lambda(x-t)]}{\lambda} Q(t)\Phi(t,\lambda)\,\mathrm{d}t\,. \tag{9}$$

Indeed, the Volterra integral equation

$$Y(x,\lambda) = \cos(\lambda x)I_d + H\frac{\sin(\lambda x)}{\lambda} + \int_0^x \frac{\sin[\lambda(x-t)]}{\lambda} Q(t)Y(t,\lambda)\,\mathrm{d}t$$

has a unique solution by the theory of Volterra integral equations. On the other hand, if a certain function $Y(x,\lambda)$ satisfies this equation, we get by differentiation

$$-Y''(x) + Q(x)Y(x) = \lambda^2 Y(x),\ \ x \in (0,1)\,,$$

and

$$Y(0,\lambda) = I_d,\ \ Y'(0,\lambda) = H\,.$$

Thus, $Y(x,\lambda)=\Phi(x,\lambda)$ and (9) is valid. Differentiating (9), we calculate

$$\Phi'(x,\lambda)=-\lambda\sin(\lambda x)+H\cos(\lambda x)+\int_0^x\cos[\lambda(x-t)]Q(t)\Phi(t,\lambda)\,\mathrm{d}t. \tag{10}$$

We see that $\Phi(x,\lambda)$ satisfies the boundary condition at the point zero in (2), thus the general solutions of systems (1) have the form

$$\phi(x,\lambda)=\Phi(x,\lambda)C,$$

where $C=(c_1,c_2,\cdots,c_d)^{\mathrm{t}}, c_k\in\mathbf{C}, k=1,2,\cdots,d$, and A^{t} denotes the transpose of matrix A. If $\phi(x,\lambda)=\Phi(x,\lambda)C$ is a nontrivial solution of problem (1) and (2), there exists a non-vanishing vector C satisfying the matrix equation (i.e., boundary condition at the point 1 in (2))

$$\left(\Phi(1,\lambda)\cos(\lambda a)-\Phi'(1,\lambda)\frac{\sin(\lambda a)}{\lambda}\right)C=0.$$

Therefore, λ is an eigenvalue of problem (1) and (2) if and only if the matrix

$$W(\lambda)=\Phi(1,\lambda)\cos(\lambda a)-\Phi'(1,\lambda)\frac{\sin(\lambda a)}{\lambda}$$

is singular.

As $|\lambda|$ tends to infinity through any part of the complex plane from (9) and (10), one can obtain the following representations:

$$\Phi(1,\lambda)=\cos\lambda I_d+\omega^{(1)}\frac{\sin\lambda}{\lambda}+\omega^{(1,1)}\frac{\cos\lambda}{\lambda^2}+o\left(\frac{\mathrm{e}^{\tau}}{\lambda^2}\right) \tag{11}$$

and

$$\Phi'(1,\lambda)=-\lambda\sin\lambda I_d+\omega^{(1)}\cos\lambda+\omega^{(1,2)}\frac{\sin\lambda}{\lambda}+o\left(\frac{\mathrm{e}^{\tau}}{\lambda}\right), \tag{12}$$

where $\tau=|\mathrm{Im}\lambda|$,

$$\begin{aligned}\omega^{(1)}&=\frac{1}{2}\int_0^1 Q(t)dt+H,\\ \omega^{(1,1)}&=\frac{Q(1)-Q(0)}{4}-\frac{1}{8}\left(\int_0^1 Q(t)\,\mathrm{d}t\right)^2-\frac{1}{2}H\int_0^1 Q(t)\,\mathrm{d}t,\\ \omega^{(1,2)}&=\frac{Q(1)+Q(0)}{4}+\frac{1}{8}\left(\int_0^1 Q(t)\,\mathrm{d}t\right)^2+\frac{1}{2}H\int_0^1 Q(t)\,\mathrm{d}t.\end{aligned} \tag{13}$$

From (11) and (12), we have

$$W(\lambda)=\cos[\lambda(1-a)]I_d+\omega^{(1)}\frac{\sin[\lambda(1-a)]}{\lambda}+\omega^{(1,1)}\frac{\cos\lambda\cos(\lambda a)}{\lambda^2}-\omega^{(1,2)}\frac{\sin\lambda\sin(\lambda a)}{\lambda^2}+o\left(\frac{\mathrm{e}^{\tau(1-a)}}{\lambda^2}\right). \tag{14}$$

The eigenvalues of problem (1) and (2) coincide with the zeros of the function $\det W(\lambda)$. Using the Laplace expansion of determinants, we obtain from (14)

$$\begin{aligned}\omega(\lambda)\overset{\text{def}}{=}\det W(\lambda)=\prod_{i=1}^{d}\Big[&\cos[\lambda(1-a)]+\omega_{ii}^{(1)}\frac{\sin[\lambda(1-a)]}{\lambda}+\omega_{ii}^{(1,1)}\frac{\cos\lambda\cos(\lambda a)}{\lambda^2}\\ &-\omega_{ii}^{(1,2)}\frac{\sin\lambda\sin(\lambda a)}{\lambda^2}+o\left(\frac{\mathrm{e}^{\tau(1-a)}}{\lambda^2}\right)\Big]\\ &+a\cos^{d-2}[\lambda(1-a)]\frac{\sin^2[\lambda(1-a)]}{\lambda^2}+O\left(\lambda^{-3}\mathrm{e}^{d\tau}\right),\end{aligned} \tag{15}$$

where

$$a=\begin{cases}-\sum_{i=1}^{d-1}\sum_{i<j}\omega_{ij}^{(1)}\omega_{ji}^{(1)} & (d\geq 2),\\ 0 & (d=1).\end{cases} \tag{16}$$

Define

$$\omega_0(\lambda)=\cos^d[\lambda(1-a)], \tag{17}$$

and denote by $\lambda_n^{(0,j)}\overset{\text{def}}{=}\lambda_n^0$, $n\in\mathbb{Z}$, $j=\overline{1,d}$, zeros of the function $\omega_0(\lambda)$, then

$$\lambda_n^0=\frac{(n+\frac{1}{2})\pi}{1-a}, \tag{18}$$

and zeros of the function $\omega_0(\lambda)$ are multiplicities d.

For an integer N_0, let Γ_{N_0} be the counterclockwise square contours, with

$$A=\frac{(N_0+\frac{1}{2}+\varepsilon)\pi}{|1-a|}(1-\mathrm{i}),$$

$$B=\frac{\left(N_0+\frac{1}{2}+\varepsilon\right)\pi}{|1-a|}(1+\mathrm{i}),$$
$$C=\frac{\left(N_0+\frac{1}{2}+\varepsilon\right)\pi}{|1-a|}(-1+\mathrm{i}),$$
$$D=\frac{\left(N_0+\frac{1}{2}+\varepsilon\right)\pi}{|1-a|}(-1-\mathrm{i}).$$

Obviously, $\lambda_n^{(0,j)}$, which are the zeros of function $\omega_0(\lambda)$, don't lie on the contour Γ_{N_0}.

Combining (15) and (17) and arranging the terms on the right-hand side in decreasing order of powers of λ gives

$$\begin{aligned}\frac{\omega(\lambda)}{\omega_0(\lambda)}&=\det\Bigg[I_d+\omega^{(1)}\frac{\tan[\lambda(1-a)]}{\lambda}\\&+\omega^{(1,1)}\frac{\cos\lambda\cos(\lambda a)}{\lambda^2\cos[\lambda(1-a)]}-\omega^{(1,2)}\frac{\sin\lambda\sin(\lambda a)}{\lambda^2\cos[\lambda(1-a)]}\\&+o\left(\frac{1}{\lambda^2}\right)\Bigg]\\&=\prod_{i=1}^{d}\Bigg[1+\omega_{ii}^{(1)}\frac{\tan[\lambda(1-a)]}{\lambda}+\omega_{ii}^{(1,1)}\frac{\cos\lambda\cos(\lambda a)}{\lambda^2\cos[\lambda(1-a)]}\\&-\omega_{ii}^{(1,2)}\frac{\sin\lambda\sin(\lambda a)}{\lambda^2\cos[\lambda(1-a)]}+o\left(\frac{1}{\lambda^2}\right)\Bigg]\\&+a\frac{\tan^2[\lambda(1-a)]}{\lambda^2}+o\left(\frac{1}{\lambda^2}\right)\\&=1+\sum_{i=1}^{d}\omega_{ii}^{(1)}\frac{\tan[\lambda(1-a)]}{\lambda}\\&+\sum_{i=1}^{d}\omega_{ii}^{(1,1)}\frac{\cos\lambda\cos(\lambda a)}{\lambda^2\cos[\lambda(1-a)]}\\&-\sum_{i=1}^{d}\omega_{ii}^{(1,2)}\frac{\sin\lambda\sin(\lambda a)}{\lambda^2\cos[\lambda(1-a)]}\\&+\sum_{i<j}^{d}\omega_{ii}^{(1)}\omega_{jj}^{(1)}\frac{\tan^2[\lambda(1-a)]}{\lambda^2}+a\frac{\tan^2[\lambda(1-a)]}{\lambda^2}\\&+o\left(\frac{1}{\lambda^2}\right)\text{ on }\Gamma_{N_0}.\end{aligned}$$

Expanding $\ln\frac{\omega(\lambda)}{\omega_0(\lambda)}$ by the Maclaurin formula, we find that on Γ_{N_0}

$$\begin{aligned}\ln\frac{\omega(\lambda)}{\omega_0(\lambda)}&=\sum_{i=1}^{d}\omega_{ii}^{(1)}\frac{\tan[\lambda(1-a)]}{\lambda}\\&+\sum_{i=1}^{d}\omega_{ii}^{(1,1)}\frac{\cos\lambda\cos(\lambda a)}{\lambda^2\cos[\lambda(1-a)]}\\&-\sum_{i=1}^{d}\omega_{ii}^{(1,2)}\frac{\sin\lambda\sin(\lambda a)}{\lambda^2\cos[\lambda(1-a)]}\\&+\left[a-\frac{1}{2}\sum_{i=1}^{d}(\omega_{ii}^{(1)})^2\right]\frac{\tan^2[\lambda(1-a)]}{\lambda^2}+o\left(\frac{1}{\lambda^2}\right).\end{aligned}\tag{19}$$

Denote

$$N_0^*=\{0,1,\cdots,N_0\}\cup\{-1,-2,\cdots,-(N_0+1)\}.$$

Asymptotic formula (3) implies that, for all sufficiently large N_0, the numbers $\lambda_n^{(j)}$, which are the zeros of the function $\omega(\lambda)$, with $n\in N_0^*$, are inside Γ_{N_0} and the number $\lambda_n^{(j)}$, with $n\notin N_0^*$, are outside Γ_{N_0}. By the residue theorem, it follows that

$$\begin{aligned}&\sum_{\Gamma_{N_0}}\left[(\lambda_n^{(j)})^2-(\lambda_n^{(0,j)})^2\right]\\&=\frac{1}{2\pi\mathrm{i}}\oint_{\Gamma_{N_0}}\lambda^2\left[\frac{\omega'(\lambda)}{\omega(\lambda)}-\frac{\omega_0'(\lambda)}{\omega_0(\lambda)}\right]\mathrm{d}\lambda\\&=\frac{1}{2\pi\mathrm{i}}\oint_{\Gamma_{N_0}}\lambda^2\,\mathrm{d}\ln\frac{\omega(\lambda)}{\omega_0(\lambda)}\\&=-\frac{1}{2\pi\mathrm{i}}\oint_{\Gamma_{N_0}}2\lambda\ln\frac{\omega(\lambda)}{\omega_0(\lambda)}\,\mathrm{d}\lambda,\end{aligned}\tag{20}$$

where $\lambda_n^{(j)}$ are zeros of entire functions $\omega(\lambda)$ inside the contour Γ_{N_0} listed with multiplicity, respectively.

Note that

$$\begin{aligned}&\sum_{\Gamma_{N_0}}\left[(\lambda_n^{(j)})^2-(\lambda_n^{(0,j)})^2\right]\\&=\sum_{n\in N_0^*}\sum_{j=1}^{d}\left[(\lambda_n^{(j)})^2-\frac{\left(n+\frac{1}{2}\right)^2\pi^2}{(1-a)^2}\right].\end{aligned}$$

Thus, from (19) and (20), we have

$$\begin{aligned}&\sum_{n\in N_0^*}\sum_{j=1}^{d}\left[(\lambda_n^{(j)})^2-\frac{\left(n+\frac{1}{2}\right)^2\pi^2}{(1-a)^2}\right]\\&=-\frac{1}{2\pi\mathrm{i}}\oint_{\Gamma_{N_0}}2\lambda\ln\frac{\omega(\lambda)}{\omega_0(\lambda)}\,\mathrm{d}\lambda\\&=-\frac{1}{2\pi\mathrm{i}}\oint_{\Gamma_{N_0}}\Bigg\{2\sum_{i=1}^{d}\omega_{ii}^{(1)}\tan[\lambda(1-a)]\\&+2\sum_{i=1}^{d}\omega_{ii}^{(1,1)}\frac{\cos\lambda\cos(\lambda a)}{\lambda\cos[\lambda(1-a)]}\\&-2\sum_{i=1}^{d}\omega_{ii}^{(1,2)}\frac{\sin\lambda\sin(\lambda a)}{\lambda\cos[\lambda(1-a)]}+2\left[a-\frac{1}{2}\sum_{i=1}^{d}(\omega_{ii}^{(1)})^2\right]\end{aligned}$$

$$\cdot \frac{\tan^2[\lambda(1-a)]}{\lambda}+o\left(\frac{1}{\lambda}\right)\Big\}\,\mathrm{d}\lambda$$
$$=\frac{4N_0+4}{1-a}\sum_{i=1}^{d}\omega_{ii}^{(1)}-2A_1(N_0)\sum_{i=1}^{d}\omega_{ii}^{(1,1)}$$
$$+2A_2(N_0)\sum_{i=1}^{d}\omega_{ii}^{(1,2)}+2a-\sum_{i=1}^{d}(\omega_{ii}^{(1)})^2+O\left(\frac{1}{N_0}\right)$$
$$=\frac{4N_0+4}{1-a}\sum_{i=1}^{d}\omega_{ii}^{(1)}-2[A_1(N_0)-1]\sum_{i=1}^{d}\omega_{ii}^{(1,1)}$$
$$+2A_2(N_0)\sum_{i=1}^{d}\omega_{ii}^{(1,2)}+\frac{1}{2}\sum_{i=1}^{d}[Q_{ii}(0)-Q_{ii}(1)]$$
$$-\sum_{i=1}^{d}H_{ii}^2+o(1)\,,$$

where

$$A_1(N_0)=1+\sum_{n\in N_0^*}\frac{(-1)^{n-1}\cos\left(\lambda_n^0\right)\cos\left(\lambda_n^0 a\right)}{\lambda_n^0(1-a)}$$

and

$$A_2(N_0)=\sum_{n\in N_0^*}\frac{(-1)^{n-1}\sin\left(\lambda_n^0\right)\sin\left(\lambda_n^0 a\right)}{\lambda_n^0(1-a)}\,.$$

The above equation implies that

$$\sum_{n\in N_0^*}\left\{\sum_{j=1}^{d}\left[(\lambda_n^{(j)})^2-\frac{\left(n+\frac{1}{2}\right)^2\pi^2}{(1-a)^2}\right]-\frac{2}{1-a}\mathrm{tr}\omega^{(1)}\right\}$$
$$=-2[A_1(N_0)-1]\sum_{i=1}^{d}\omega_{ii}^{(1,1)}+2A_2(N_0)\sum_{i=1}^{d}\omega_{ii}^{(1,2)} \quad (21)$$
$$+\frac{1}{2}\sum_{i=1}^{d}[Q_{ii}(0)-Q_{ii}(1)]-\sum_{i=1}^{d}H_{ii}^2+o(1)\,.$$

Passing to the limit as $N_0\to\infty$ in (21), we find that

$$S_1=-2[A_1(+\infty)-1]\sum_{i=1}^{d}\omega_{ii}^{(1,1)}+2A_2(+\infty)\sum_{i=1}^{d}\omega_{ii}^{(1,2)}$$
$$+\frac{1}{2}\sum_{i=1}^{d}[Q_{ii}(0)-Q_{ii}(1)]-\sum_{i=1}^{d}H_{ii}^2\,. \quad (22)$$

Note that

$$A_1(+\infty)-1=2\sum_{n=0}^{\infty}\frac{(-1)^{n-1}\cos(\lambda_n^0)\cos(\lambda_n^0 a)}{\lambda_n^0(1-a)}\,,$$
$$A_2(+\infty)=2\sum_{n=0}^{\infty}\frac{(-1)^{n-1}\sin(\lambda_n^0)\sin(\lambda_n^0 a)}{\lambda_n^0(1-a)}\,.$$

From (18), by trigonometrical calculation, we have

$$\sin\left(\lambda_n^0\right)\sin\left(\lambda_n^0 a\right)=\frac{1}{2}(-1)^{n-1}\sin\frac{(2n+1)\pi}{1-a}$$
$$=-\cos\left(\lambda_n^0\right)\cos\left(\lambda_n^0 a\right),$$

which implies that

$$A_1(+\infty)-1=-\frac{2}{\pi}\sum_{n=0}^{\infty}\frac{\sin\frac{(2n+1)\pi}{1-a}}{2n+1}\,,$$
$$A_2(+\infty)=\frac{2}{\pi}\sum_{n=0}^{\infty}\frac{\sin\frac{(2n+1)\pi}{1-a}}{2n+1}\,. \quad (23)$$

From (13), (22), and (23), it yields that

$$S_1=\frac{2}{\pi}\sum_{n=0}^{\infty}\frac{\sin\frac{(2n+1)\pi}{1-a}}{2n+1}\sum_{i=1}^{d}Q_{ii}(1)$$
$$+\frac{1}{2}\sum_{i=1}^{d}[Q_{ii}(0)-Q_{ii}(1)]-\sum_{i=1}^{d}H_{ii}^2\,. \quad (24)$$

Expanding the function $f(x)=\frac{\pi-x}{2}$, $0<x<\pi$, to a sine series yields

$$f(x)=\sum_{n=1}^{\infty}\frac{\sin(nx)}{n}=\begin{cases}\frac{\pi-x}{2}, & 0<x<\pi,\\ 0, & x=0,\pi,\\ \frac{x-\pi}{2}, & -\pi<x<0.\end{cases}$$

Denote

$$\sum_{n=1}^{\infty}\frac{\sin\frac{2n\pi}{1-a}}{n}=f\left(\frac{2\pi}{1-a}\right).$$

Note that

$$\sum_{n=0}^{\infty}\frac{\sin(2n+1)x}{2n+1}$$
$$=\sum_{n=1}^{\infty}\frac{\sin(nx)}{n}-\sum_{n=1}^{\infty}\frac{\sin(2nx)}{2n}$$
$$=\sum_{n=1}^{\infty}\frac{\sin(nx)}{n}-\frac{1}{2}\sum_{n=1}^{\infty}\frac{\sin(2nx)}{n}$$
$$=f(x)-\frac{1}{2}f(2x)\,,$$

thus, we have

$$\sum_{n=0}^{\infty}\frac{\sin\frac{(2n+1)\pi}{1-a}}{2n+1}=f\left(\frac{\pi}{1-a}\right)-\frac{1}{2}f\left(\frac{2\pi}{1-a}\right). \quad (25)$$

Together with

$$\sum_{i=1}^{d} Q_{ii}(1) = \operatorname{tr} Q(1)\,,$$

$$\sum_{i=1}^{d} [Q_{ii}(0) - Q_{ii}(1)] = \operatorname{tr}\left(Q(0) - Q(1)\right),$$

and

$$\sum_{i=1}^{d} H_{ii}^2 = \operatorname{tr} H^2,$$

substituting (25) into (24), we find that formula (7) holds. The proof of the theorem is finished.

Acknowledgements

This work was supported by the National Natural Science Foundation of China (11171152/A010602), Natural Science Foundation of Jiangsu Province of China (BK 2010489), and the Outstanding Plan-Zijin Star Foundation of NUST (AB 41366).

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