A Note on New Exact Solutions for Some Unsteady Flows of Brinkman-Type Fluids over a Plane Wall

Farhad Ali, Ilyas Khan, Samiulhaq, and Sharidan Shafie

Department of Mathematical Sciences, Faculty of Science, University of Technology, 81310 UTM Skudai, Malaysia

Reprint requests to S. S.; E-mail: ridafie@yahoo.com

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Flows of a Brinkman fluid due to a plane boundary moving in its plane are studied using Laplace transform. The solutions that have been obtained for the velocity are presented in simple forms in terms of the complementary error function $\mathrm{erfc}(\cdot)$. They satisfy all imposed initial and boundary conditions and can easily be reduced to the similar solutions for Newtonian fluids.

Key words: Brinkman Type Fluids; Plane Wall; Exact Solutions.

1. Introduction

Among the motions of a fluid over a plane wall, the Stokes' problems as well as the motion induced by a constantly accelerating plate have been extensively studied in the literature. However, the first closed-form solutions for the motion of a viscous fluid over an oscillating plate have been late enough obtained by Erdogan [1]. These solutions, as well as those corresponding to the fluid motion due to a constantly or highly accelerating plate have been recently extended by Fetecau et al. [2] to fluids of Brinkman type [3 – 5].

The purpose of this note is to provide new exact solutions for the motion of a Brinkman fluid induced by a constantly or highly accelerating plate or due to an infinite oscillating plate. These dimensionless solutions, unlike those obtained in [2], are fully presented in analytical forms in terms of exponential and complementary error functions. They satisfy all imposed initial and boundary conditions and can easily be reduced to the similar solutions for Newtonian fluids.

2. Flow Induced by a Highly Accelerating Plate

Consider an incompressible fluid of Brinkman type [5] at rest over an infinite flat plate situated in the x,z-plane of a Cartesian coordinate system x,y, and z. At time $t = 0^+$, the plate starts motion in its plane with accelerating velocity. Owing to the shear, the fluid is gradually moved, and its velocity is of the form

$$\mathbf{v} = \mathbf{v}(y,t) = u(y,t)\mathbf{i}\,, \tag{1}$$

where $\mathbf{i}$ is the unit vector along the x-flow direction.

For such a flow, the governing equation is given by [2, Eq. (1.3)]

$$\nu\frac{\partial^2 u(y,t)}{\partial y^2} = \beta u(y,t) + \frac{\partial u(y,t)}{\partial t};\quad y,t>0\,, \tag{2}$$

where ν is the kinematic viscosity of the fluid, $\beta = \alpha/\rho$ (ρ being the constant density of the fluid), and α is the drag coefficient that is usually assumed to be positive.

The appropriate initial and boundary conditions are

$$u(y,0) = 0 \quad \text{for} \quad y>0;\quad u(0,t) = At^p \quad \text{for} \quad t\geq 0\,, \tag{3}$$

where A and $p>0$ are constants. Furthermore, the natural condition

$$u(y,t)\to 0 \quad \text{as} \quad y\to\infty \quad \text{and} \quad t\geq 0\,, \tag{4}$$

has also to be satisfied.

We introduce the following dimensionless quantities:

$$v_p = \frac{u}{(\nu^p A)^{\frac{1}{2p+1}}}\,,\quad \xi = y\left(\frac{A}{\nu^{p+1}}\right)^{\frac{1}{2p+1}}\,,\quad \tau = t\left(\frac{A^2}{\nu}\right)^{\frac{1}{2p+1}}\,. \tag{5}$$

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Equations (2) – (4) reduce to

$$\frac{\partial^2 v_p(\xi,\tau)}{\partial \xi^2} = \beta_p v_p(\xi,\tau) + \frac{\partial v_p(\xi,\tau)}{\partial \tau}; \quad \xi,\tau > 0, \tag{6}$$

$$v_p(\xi,0) = 0 \quad \text{for} \quad \xi > 0, \qquad v_p(0,\tau) = \tau^p \quad \text{for} \quad \tau \geq 0, \tag{7}$$

$$v_p(\xi,\tau) \to 0 \quad \text{as} \quad \xi \to \infty \quad \text{and} \quad \tau \geq 0, \tag{8}$$

where $\beta_p = \beta(\nu/A^2)^{\frac{1}{2p+1}}$. Applying the Laplace transform to (6) and bearing in mind (7) and (8), the solution in the transformed q-plane is

$$\bar{v}_p(\xi,q) = \frac{\Gamma(p+1)}{q^{p+1}} \mathrm{e}^{-\xi\sqrt{q+\beta_p}}, \tag{9}$$

where q is the transform parameter, and $\bar{v}_p(\xi,q)$ is the Laplace transform of $v_p(\xi,\tau)$.

Applying the inverse Laplace transform to (9) and using the convolution theorem, we find that

$$v_p(\xi,\tau) = \frac{\xi}{2\sqrt{\pi}} \int_0^\tau \frac{(\tau-s)^p}{s\sqrt{s}} \exp\left(-\frac{\xi^2}{4s} - \beta_p s\right) \mathrm{d}s. \tag{10}$$

By making $\beta_p = 0$ into above relation, the corresponding Newtonian solution

$$v_{p_N}(\xi,\tau) = \frac{\xi}{2\sqrt{\pi}} \int_0^\tau \frac{(\tau-s)^p}{s\sqrt{s}} \mathrm{e}^{-\frac{\xi^2}{4s}} \mathrm{d}s, \tag{11}$$

is obtained. Direct computations show that the velocity $u(y,t)$, resulting from (5) and (11), has the same form as the shear stress $\tau(y,t)$ obtained from [6, Eq. (24)] for $f(t) = ft^p$. This is important (see [6, Eq. (35)] and the remark of that section), because it is a positive proof of our solution's correctness.

For convenience, let us write (9) in the equivalent form

$$\bar{v}_p(\xi,q) = \frac{\Gamma(p+1)}{\Gamma(p)} \cdot \frac{\Gamma(p)}{q^p} \cdot \frac{1}{q} \mathrm{e}^{-\xi\sqrt{q+\beta_p}}. \tag{12}$$

Applying the inverse Laplace transform to (12) and using the convolution theorem, we find an equivalent form for $v_p(\xi,\tau)$, namely

$$v_p(\xi,\tau) = \frac{\Gamma(p+1)}{2\Gamma(p)} \int_0^\tau (\tau-s)^{p-1} \cdot \left\{ \mathrm{e}^{-\xi\sqrt{\beta_p}} \mathrm{erfc}\left(\frac{\xi}{2\sqrt{\tau}} - \sqrt{\beta_p \tau}\right) + \mathrm{e}^{\xi\sqrt{\beta_p}} \mathrm{erfc}\left(\frac{\xi}{2\sqrt{\tau}} + \sqrt{\beta_p \tau}\right) \right\} \mathrm{d}\tau. \tag{13}$$

The corresponding form of solution for Newtonian fluids is

$$v_{pN}(\xi,\tau) = \frac{\Gamma(p+1)}{\Gamma(p)} \int_0^\tau (\tau-s)^{p-1} \cdot \mathrm{erfc}\left(\frac{\xi}{2\sqrt{\tau}}\right) \mathrm{d}\tau. \tag{14}$$

Of course, (13) and (14) are equivalent to the equalities (2.13) and (2.14) from [2] for each natural number p.

Finally, in view of Eqs. (1), (5), and of the recurrence relations (7) – (10) from [6], it is worth pointing out that the general solutions $v_p(\xi,\tau)$ can be written in terms of the complementary error function erf$c(\cdot)$ for each natural number p. For $p = 1$, for instance, the corresponding solution is (see (A1) or [7, Eq. (7)])

$$v_1(\xi,\tau) = \frac{1}{2}\left\{ \left(\tau - \frac{\xi}{2\sqrt{\beta_1}}\right) \mathrm{e}^{-\xi\sqrt{\beta_1}} \mathrm{erfc}\left(\frac{\xi}{2\sqrt{\tau}} - \sqrt{\beta_1 \tau}\right) + \left(\tau + \frac{\xi}{2\sqrt{\beta_1}}\right) \mathrm{e}^{\xi\sqrt{\beta_1}} \mathrm{erfc}\left(\frac{\xi}{2\sqrt{\tau}} + \sqrt{\beta_1 \tau}\right) \right\}. \tag{15}$$

Similarly for $p = 2$ and 3, the corresponding solutions are (see (A2) and (A3))

$$v_2(\xi,\tau) = \frac{1}{2}\left\{ \left(\tau^2 - \frac{\tau\xi}{\sqrt{\beta_2}} + \frac{\xi^2}{4\beta_2} + \frac{\xi}{4\beta_2\sqrt{\beta_2}}\right) \cdot \mathrm{e}^{-\xi\sqrt{\beta_2}} \mathrm{erfc}\left(\frac{\xi}{2\sqrt{\tau}} - \sqrt{\beta_2 \tau}\right) + \left(\tau^2 + \frac{\tau\xi}{\sqrt{\beta_2}} + \frac{\xi^2}{4\beta_2} - \frac{\xi}{4\beta_2\sqrt{\beta_2}}\right) \cdot \mathrm{e}^{\xi\sqrt{\beta_2}} \mathrm{erfc}\left(\frac{\xi}{2\sqrt{\tau}} + \sqrt{\beta_2 \tau}\right) - \frac{\xi}{2\beta_2}\sqrt{\frac{\tau}{\pi}} \exp\left(-\frac{\xi^2}{4\tau} - \beta_2 \tau\right) \right\}, \tag{16}$$

$$v_3(\xi,\tau) = \left[\frac{\tau^3}{12} + \frac{\xi^2\tau}{16\beta_3} - \frac{\xi^2}{32\beta_3}\right] f(\xi,\tau) + \frac{1}{6}\left\{\left(\frac{3\xi}{4\beta_3} - \frac{\xi\tau}{\beta_3}\right)\sqrt{\frac{\tau}{\pi}}\exp\left(\frac{\xi^2}{4\tau} - \beta_3\tau\right)\right\} \left[-\frac{\xi\tau^2}{8\sqrt{\beta_3}} + \frac{\xi\tau}{16\beta_3\sqrt{\beta_3}} - \frac{\xi^3}{96\beta_3\sqrt{\beta_3}} - \frac{\xi}{32\beta_3\sqrt{\beta_3}}\right] \cdot g(\xi,\tau), \tag{17}$$

where

$$g(\xi,\tau) = \left\{ e^{-\xi\sqrt{\beta_3}}\operatorname{erfc}\left(\frac{\xi}{2\sqrt{\tau}} - \sqrt{\beta_3\tau}\right) - e^{\xi\sqrt{\beta_3}}\operatorname{erfc}\left(\frac{\xi}{2\sqrt{\tau}} + \sqrt{\beta_3\tau}\right)\right\},$$
$$f(\xi,\tau) = \left\{ e^{-\xi\sqrt{\beta_3}}\operatorname{erfc}\left(\frac{\xi}{2\sqrt{\tau}} - \sqrt{\beta_3\tau}\right) + e^{\xi\sqrt{\beta_3}}\operatorname{erfc}\left(\frac{\xi}{2\sqrt{\tau}} + \sqrt{\beta_3\tau}\right)\right\}. \tag{18}$$

3. Flow over an Oscillating Plate (Stokes' Second Problem)

The motion of a fluid due to an oscillating wall has many engineering applications. At time $t = 0^+$, the plate starts to oscillate in its plane according to

$$\mathbf{v} = U\sin(\omega t)\mathbf{i}, \tag{19}$$

where U is the amplitude of the motion, and ω is the frequency of vibrations. Due to the shear, the fluid is gradually moved. Its velocity is of the form (1), the governing equation is given by (2) while the initial and boundary conditions are the same as given by (3) and (4), excepting

$$u(0,t) = U\sin(\omega t) \quad \text{for} \quad t \geq 0. \tag{20}$$

Introducing the dimensionless quantities

$$v = \frac{u}{U}, \quad \xi = \frac{U}{\nu}y, \quad \tau = \frac{U^2}{\nu}t, \quad \omega^* = \frac{\nu}{U^2}\omega, \tag{21}$$

and dropping out the star notation, the governing equation takes the same form (6) with $\gamma = \frac{\nu\beta}{U^2}$ instead of β_p and the boundary condition (20) becomes

$$v(0,\tau) = \sin(\omega\tau) \quad \text{for} \quad \tau > 0. \tag{22}$$

In this case, the solution in the transformed q-plane is given by

$$\bar{v}(\xi,q) = \frac{\omega}{q^2+\omega^2}e^{-\xi\sqrt{q+\gamma}}. \tag{23}$$

Following the same way as in the previous section and avoiding repetition, we obtain the result (see (A4))

$$v_s(\xi,\tau) = \frac{e^{i\omega\tau}}{4i}\left[e^{-\xi\sqrt{\gamma+i\omega}}\operatorname{erfc}\left(\frac{\xi}{2\sqrt{\tau}} - \sqrt{(\gamma+i\omega)\tau}\right) + e^{\xi\sqrt{\gamma+i\omega}}\operatorname{erfc}\left(\frac{\xi}{2\sqrt{\tau}} + \sqrt{(\gamma+i\omega)\tau}\right)\right] - \frac{e^{-i\omega\tau}}{4i}\left[e^{-\xi\sqrt{\gamma-i\omega}}\operatorname{erfc}\left(\frac{\xi}{2\sqrt{\tau}} - \sqrt{(\gamma-i\omega)\tau}\right) + e^{\xi\sqrt{\gamma-i\omega}}\operatorname{erfc}\left(\frac{\xi}{2\sqrt{\tau}} + \sqrt{(\gamma-i\omega)\tau}\right)\right], \tag{24}$$

where $\operatorname{erfc}(x+iy)$ is the complementary error function of complex argument $x+iy$ which can be calculated in terms of tabulated functions [8]. Of course, by making $\gamma \to 0$ into previous relation, the solution

$$v_{sN}(\xi,\tau) = -\frac{e^{-i\omega\tau}}{4i}\left[e^{-\xi\sqrt{-i\omega}}\operatorname{erfc}\left(\frac{\xi}{2\sqrt{\tau}} - \sqrt{-i\omega\tau}\right) + e^{\xi\sqrt{-i\omega}}\operatorname{erfc}\left(\frac{\xi}{2\sqrt{\tau}} + \sqrt{-i\omega\tau}\right)\right] + \frac{e^{i\omega\tau}}{4i}\left[e^{-\xi\sqrt{i\omega}}\operatorname{erfc}\left(\frac{\xi}{2\sqrt{\tau}} - \sqrt{i\omega\tau}\right) + e^{\xi\sqrt{i\omega}}\operatorname{erfc}\left(\frac{\xi}{2\sqrt{\tau}} + \sqrt{i\omega\tau}\right)\right], \tag{25}$$

corresponding to Newtonian fluids is recovered. Indeed, bearing in mind the initial variables, it is easy to see that (25) is identical to Eq. (15) from [1]. Furthermore, as it clearly results from Figure 1, the solution given by (24) is equivalent to that obtained in [2, Eq. (3.6)] by a different technique.

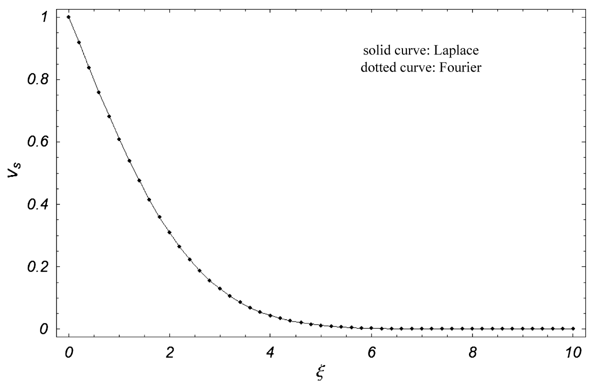

Fig. 1. Comparative diagram of (24) and [2, Eq. (3.6)] when $\gamma = 1$, $\tau = 2$, $\omega = 0.2$, and $\omega\tau = \frac{\pi}{2}$.

4. Conclusions

New exact solutions for some unsteady motions of a Brinkman fluid due to a plane boundary moving in its plane are established using Laplace transforms. These solutions for velocity are presented in simple forms in terms of the elementary function $\exp(\cdot)$ and the complementary error function $\mathrm{erf}c(\cdot)$. They satisfy all imposed initial and boundary conditions and can immediately be reduced to the similar solutions for Newtonian fluids.

The solutions corresponding to the second problem of Stokes are presented in terms of the complementary error function of complex argument $x+\mathrm{i}y$. This function can be calculated in terms of tabulated functions [8]. Finally, for a check of results, the equivalence of our solutions (24) to the known results from the literature [2, Eq. (3.6)] is graphically showed.

5. Appendix

$$\begin{aligned}
&\mathcal{L}^{-1}\left\{\frac{1}{q^2}\mathrm{e}^{-a\sqrt{q+b}}\right\} \\
&= \frac{1}{2}\left[\left(t-\frac{a}{2\sqrt{b}}\right)\mathrm{e}^{-a\sqrt{b}}\mathrm{erf}c\left(\frac{a}{2\sqrt{t}}-\sqrt{bt}\right)\right. \\
&\left.+\left(t+\frac{a}{2\sqrt{b}}\right)\mathrm{e}^{a\sqrt{b}}\mathrm{erf}c\left(\frac{a}{2\sqrt{t}}+\sqrt{bt}\right)\right],
\end{aligned} \tag{A1}$$

$$\begin{aligned}
&\mathcal{L}^{-1}\left\{\frac{1}{q^3}\mathrm{e}^{-a\sqrt{q+b}}\right\} = \frac{1}{4}\left\{\left(t^2-\frac{ta}{\sqrt{b}}+\frac{a^2}{4b}+\frac{a}{4b\sqrt{b}}\right)\right. \\
&\cdot \mathrm{e}^{-a\sqrt{b}}\mathrm{erf}c\left(\frac{a}{2\sqrt{t}}-\sqrt{bt}\right) \\
&+\left(t^2+\frac{ta}{\sqrt{b}}+\frac{a^2}{4b}-\frac{a}{4b\sqrt{b}}\right)\mathrm{e}^{a\sqrt{b}}\mathrm{erf}c\left(\frac{a}{2\sqrt{t}}+\sqrt{bt}\right) \\
&\left.-\frac{a}{b}\sqrt{\frac{t}{\pi}}\exp(-\frac{a^2}{4t}-bt)\right\},
\end{aligned} \tag{A2}$$

$$\begin{aligned}
&\mathcal{L}^{-1}\left\{\frac{1}{q^4}\mathrm{e}^{-a\sqrt{q+b}}\right\} = \frac{1}{6}\left\{\left(\frac{3a}{4b^2}-\frac{at}{b}\right)\right. \\
&\left.\cdot\sqrt{\frac{t}{\pi}}\exp\left(\frac{a^2}{4t}-bt\right)\right\}+\left[-\frac{a^2}{32b^2}+\frac{a^2t}{16b}+\frac{t^3}{12}\right] \\
&\cdot\left\{\mathrm{e}^{-a\sqrt{b}}\mathrm{erf}c\left(\frac{a}{2\sqrt{t}}-\sqrt{bt}\right)\right. \\
&\left.+\mathrm{e}^{a\sqrt{b}}\mathrm{erf}c\left(\frac{a}{2\sqrt{t}}+\sqrt{bt}\right)\right\} \\
&+\left\{\mathrm{e}^{-a\sqrt{b}}\mathrm{erf}c\left(\frac{a}{2\sqrt{t}}-\sqrt{bt}\right)\right. \\
&\left.-\mathrm{e}^{a\sqrt{b}}\mathrm{erf}c\left(\frac{a}{2\sqrt{t}}+\sqrt{bt}\right)\right\} \\
&\cdot\left[-\frac{a}{32b^2\sqrt{b}}-\frac{a^3}{96b\sqrt{b}}+\frac{at}{16b\sqrt{b}}-\frac{at^2}{8\sqrt{b}}\right],
\end{aligned} \tag{A3}$$

$$\begin{aligned}
&\mathcal{L}^{-1}\left\{\frac{1}{q+\mathrm{i}\omega}\mathrm{e}^{-a\sqrt{q+b}}\right\} \\
&= \frac{\mathrm{e}^{-\mathrm{i}\omega t}}{2}\left\{\mathrm{e}^{-a\sqrt{b-\mathrm{i}\omega}}\mathrm{erf}c\left(\frac{a}{2\sqrt{t}}-\sqrt{(b-\mathrm{i}\omega)t}\right)\right. \\
&\left.+\mathrm{e}^{a\sqrt{b-\mathrm{i}\omega}}\mathrm{erf}c\left(\frac{a}{2\sqrt{t}}+\sqrt{(b-\mathrm{i}\omega)t}\right)\right\}.
\end{aligned} \tag{A4}$$

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