A Fractional Model of Gas Dynamics Equations and its Analytical Approximate Solution Using Laplace Transform

Sunil Kumar[a], Huseyin Kocak[b], and Ahmet Yıldırım[c]

[a] Department of Mathematics, National Institute of Technology, Jamshedpur, 831014, Jharkhand, India

[b] Department of Mathematical Sciences, University of Bath, Bath BA, United Kingdom

[c] Department of Applied Mathematics, Faculty of Science, Ege University, Bornova, Izmir, Turkey

Reprint requests to S. K.; E-mail: skiitbhu28@gmail.com

Z. Naturforsch. **67a,** 389 – 396 (2012) / DOI: 10.5560/ZNA.2012-0038
Received November 30, 2011 / revised April 4, 2012

In this study, the homotopy perturbation transform method (HPTM) is performed to give approximate and analytical solutions of nonlinear homogenous and non-homogenous time-fractional gas dynamics equations. Gas dynamics equations are based on the physical laws of conservation, namely, the laws of conservation of mass, conservation of momentum, conservation of energy etc. The HPTM is a combined form of the Laplace transform, the homotopy perturbation method, and He's polynomials. The nonlinear terms can be easily handled by the use of He's polynomials. The numerical solutions obtained by the proposed method indicate that the approach is easy to implement and accurate. Some numerical illustrations are given. These results reveal that the proposed method is very effective and simple to perform.

Key words: Homotopy Perturbation Method; Laplace Transform; Gas Dynamics Equation; Analytical Solution.

Mathematics Subject Classification 2000: 26A33, 34A08, 34A34

1. Introduction

Fractional-order ordinary differential equations, as generalizations of classical integer-order ordinary differential equations, are increasingly used to model problems in fluid flow, mechanics, viscoelasticity, biology, physics and engineering, and other applications. Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. Half-order derivatives and integrals proved to be more useful for the formulation of certain electrochemical problems than the classical models [1 – 6].

It is commonly known that the equation of gas dynamics is the mathematical expressions of conservation laws which exist in engineering practices such as conservation of mass, conservation of momentum, conservation of energy etc. The nonlinear equations of ideal gas dynamics are applicable for three types of nonlinear waves like shock fronts, rarefactions, and contact discontinuities. In 1981, Steger and Warming [7] addressed that the conservation-law form of the inviscid gas dynamic equation possesses a remarkable property by virtue of which the nonlinear flux vectors are homogeneous functions of degree one which permits the splitting of flux vectors into subvectors by similarity transformations, and as a result new explicit and implicit dissipative finite-difference schemes are developed for solving first-order hyperbolic systems of equations. The different types of gas dynamics equations in physics have been solved by Elizarova [8] and Evans and Bulut [9] by applying various kinds of analytical and numerical methods. In 1985, Aziz and Anderson [10] used a pocket computer to solve some problems arising in gas dynamics. Liu [11] has studied nonlinear hyperbolic-parabolic partial differential equations related to gas dynamics and mechanics. In 2003, Rasulov and Karaguler [12] applied the difference scheme for solving nonlinear system of equations of gas dynamic problems for a class of discontinuous functions. Recently, Biazar and Eslami [13] and Das and Kumar [14] have applied to obtain the solutions of the homogenous and non-homogenous time-fractional gas dynamics equation.

In this paper, the Laplace homotopy perturbation method (LHPM) basically illustrates how the Laplace transform can be used to approximate the solutions of the linear and nonlinear differential equations by manipulating the homotopy perturbation method. The perturbation methods which are generally used to solve nonlinear problems have some limitations, e.g., the approximate solution involves series of small parameters which poses a difficulty since the majority of nonlinear problems have no small parameters at all. Although appropriate choices of small parameters some time leads to an ideal solution, in most of the cases unsuitable choices lead to serious effects in the solutions. The homotopy perturbation method (HPM) was introduced and applied by He [15 – 19]. Recently, many researchers [20 – 24] have obtained the series solution of the integral equation and fractional differential equation by using HPM. The proposed method is a coupling of the Laplace transform, the homotopy perturbation method, and He's polynomials mainly due to Ghorbani [25, 26]. In recent years, many authors have paid attention to studying the solutions of linear and nonlinear partial differential equations by using various methods combined with the Laplace transform. Among these are the Laplace decomposition method [27, 28] and the Laplace homotopy perturbation method [29, 30].

In this article, the homotopy perturbation method is used to solve the time-fractional gas dynamics equation with coupling of Laplace transform and He's polynomials. Using the initial condition, the approximate analytical expressions of $u(x,t)$ for different fractional Brownian motions and also for standard motion are obtained. The approximate solution is obtained numerically and depicted graphically. The elegance of this article can be attributed to the simplistic approach in seeking the approximate analytical solution of the problem.

2. Basic Definitions of Fractional Calculus

In this section, we give some basic definitions and properties of the fractional calculus theory which shall be used in this paper:

Definition 1. A real function $f(t), t > 0$, is said to be in the space $C_\mu, \mu \in \mathbb{R}$, if there exists a real number $p > \mu$, such that $f(t) = t^p f_1(t)$ where $f_1(t) \in C(0,\infty)$, and it is said to be in the space C_n if and only if $f^{(n)} \in C_\mu, n \in \mathbb{N}$.

Definition 2. The left sided Riemann–Liouville fractional integral operator of order $\mu \geq 0$ of a function $f \in C_\alpha$, $\alpha \geq -1$, is defined as follows [31, 32]:

$$I^\mu f(t) = \begin{cases} \dfrac{1}{\Gamma(\mu)} \displaystyle\int_0^t (t-\tau)^{\mu-1} f(\tau) d\tau, & \mu > 0, t > 0, \\ f(t), & \mu = 0, \end{cases} \tag{1}$$

where $\Gamma(.)$ is the well-known Gamma function.

Definition 3. The left sided Caputo fractional derivative of $f, f \in C_{-1}^m, m \in \mathbb{N} \cup \{0\}$ is defind as follows [4, 33]:

$$D_*^\mu f(t) = \frac{\partial^\mu f(t)}{\partial t^\mu} = \begin{cases} I^{m-\mu}\left[\dfrac{\partial^m f(t)}{\partial t^m}\right], & m-1 < \mu < m, \ m \in N, \\ \dfrac{\partial^m f(t)}{\partial t^m}, & \mu = m. \end{cases} \tag{2}$$

Note that [4, 35]

(i) $I_t^\mu f(x,t) = \dfrac{1}{\Gamma(\mu)} \displaystyle\int_0^t \frac{f(x,t)}{(t-s)^{1-\mu}} dt, \ \mu > 0, t > 0.$

(ii) $D_*^\mu f(x,t) = I_t^{m-\mu} \dfrac{\partial^m f(x,t)}{\partial t^m}, \ m-1 < \mu \leq m.$

Definition 4. The Mittag–Leffler function $E_\alpha(z)$ with $\alpha > 0$ is defined by the following series representation, valid in the whole complex plane [34]:

$$E_\alpha(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(\alpha n + 1)}. \tag{3}$$

Definition 5. The Laplace transform of $f(t)$ is

$$F(s) = L[f(t)] = \int_0^\infty \mathrm{e}^{-st} f(t)\, \mathrm{d}t. \tag{4}$$

Definition 6. The Laplace transform $L[f(t)]$ of the Riemann–Liouville fractional integral is defined as [2]

$$L[I^\alpha f(t)] = s^{-\alpha} F(s). \tag{5}$$

Definition 7. The Laplace transform $L[f(t)]$ of the Caputo fractional derivative is defined as [2]

$$L[D^\alpha f(t)] = s^\alpha F(s) - \sum_{k=0}^{n-1} s^{(\alpha-k-1)} f^{(k)}(0), \quad n-1 < \alpha \leq n. \tag{6}$$

3. Fractional Homotopy Perturbation Transform Method

In order to elucidate the solution procedure of the fractional Laplace homotopy perturbation method, we consider the following nonlinear fractional differential equation:

$$\begin{aligned} &D_t^{\alpha}u(x,t)+R[x]u(x,t)+N[x]u(x,t)\\ &=q(x,t)\,,\ \ t>0\,,x\in\mathbb{R}\,,0<\alpha\leq 1\,,\\ &u(x,0)=h(x)\,, \end{aligned} \tag{7}$$

where $D^{\alpha}=\frac{\partial^{\alpha}}{\partial t^{\alpha}}$, $R[x]$ is the linear operator in x, $N[x]$ is the general nonlinear operator in x, and $q(x,t)$ are continuous functions. Now, the methodology consists of applying the Laplace transform first on both sides of (7):

$$\begin{aligned} &L[D^{\alpha}u(x,t)]+L[R[x]u(x,t)+N[x]u(x,t)]\\ &=L[q(x,t)]\,. \end{aligned} \tag{8}$$

Using the differentiation property of the Laplace transform, we have

$$\begin{aligned} L[u(x,t)]&=s^{-1}h(x)-s^{-\alpha}L[q(x,t)]\\ &\quad+s^{-\alpha}L[R[x]u(x,t)+N[x]u(x,t)]\,. \end{aligned} \tag{9}$$

Operating the inverse Laplace transform on both sides in (9), we get

$$\begin{aligned} u(x,t)&=G(x,t)\\ &\quad-L^{-1}(s^{-\alpha}L[R[x]u(x,t)+N[x]u(x,t)])\,, \end{aligned} \tag{10}$$

where $G(x,t)$ represents the term arising from the source term and the prescribed initial conditions. Now, applying the classical perturbation technique, we can assume that the solution can be expressed as a power series in p as given below:

$$u(x,t)=\sum_{n=0}^{\infty}p^n u_n(x,t)\,, \tag{11}$$

where the homotopy parameter p is considered as a small parameter ($p\in[0,1]$). The nonlinear term can be decomposed as

$$Nu(x,t)=\sum_{n=0}^{\infty}p^n H_n(u)\,, \tag{12}$$

where H_n are He's polynomials (see [26, 27]) of $u_0,u_1,u_2,\ldots,u_n$, and it can be calculated by the following formula:

$$H_n(u_0,u_1,u_2,\ldots,u_n)=\frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{i=0}^{\infty}p^i u_i\right)\right]_{p=0},$$
$$n=0,1,2,3,\ldots.$$

Substituting (11) and (12) in (10) and using HPM [15–19], we get

$$\begin{aligned} &\sum_{n=0}^{\infty}p^n u_n(x,t)=G(x,t)\\ &-p\left(L^{-1}\left[s^{-\alpha}L\left[R\sum_{n=0}^{\infty}p^n u_n(x,t)+\sum_{n=0}^{\infty}p^n H_n(u)\right]\right]\right). \end{aligned} \tag{13}$$

This is coupling of the Laplace transform and homotopy perturbation method using He's polynomials. Now, equating the coefficients of the corresponding power of p on both sides, the following approximations are obtained:

$$\begin{aligned} &p^0: u_0(x,t)=G(x,t)\,,\\ &p^1: u_1(x,t)=L^{-1}\left(s^{-\alpha}L[R[x]u_0(x,t)+H_0(u)]\right),\\ &p^2: u_2(x,t)=L^{-1}\left(s^{-\alpha}L[R[x]u_1(x,t)+H_1(u)]\right),\\ &p^3: u_3(x,t)=L^{-1}\left(s^{-\alpha}L[R[x]u_2(x,t)+H_2(u)]\right),\\ &\vdots \end{aligned}$$

Proceeding in this same manner, the rest of the components $u_n(x,t), n\geq 4$, can be completely obtained, and the series solutions are thus entirely determined.

Finally, we approximate the analytical solution $u(x,t)$ by the truncated series

$$u(x,t)=\lim_{N\to\infty}\sum_{n=1}^{N}u_n(x,t)\,. \tag{14}$$

The above series solutions generally converge very rapidly. A classical approach of convergence of this type of series is already presented by Abbaoui and Cherruault [35].

4. Numerical Examples

In this section, three examples on nonlinear fractional-order homogeneous and non-homogeneous time-fractional gas dynamic equations are solved to demonstrate the performance and efficiency of the HPM with coupling of Laplace transform and He's polynomials.

Example 1. We consider the following homogeneous nonlinear time-fractional gas dynamic equation [14]:

$$\frac{\partial^\alpha u}{\partial t^\alpha} + u\frac{\partial u}{\partial x} - u(1-u) = 0\,, \;\; 0 < \alpha \le 1\,, \quad (15)$$

with initial condition $u(x,0) = \mathrm{e}^{-x}$, and $u(x,t) = \mathrm{e}^{t-x}$ is the exact solution for $\alpha = 1$.

Applying Laplace transform on both sides in (15), we get

$$L[u(x,t)] = s^{-1}\mathrm{e}^{-x} + s^{-\alpha}L[u - u^2 - uu_x]\,. \quad (16)$$

The inverse Laplace transform on both sides implies that

$$u(x,t) = \mathrm{e}^{-x} + L^{-1}\left(s^{-\alpha}L[u - u^2 - uu_x]\right). \quad (17)$$

Now, we apply the homotopy perturbation method [15–19]:

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = \mathrm{e}^{-x} + p\left(L^{-1}\left[s^{-\alpha}L[\sum_{n=0}^{\infty} p^n H_n(u)]\right]\right), \quad (18)$$

where $H_n(u)$ are He's polynomials [26, 27] that represent the nonlinear terms. The first few components of He's polynomials, for example, are given by

$$H_0(u) = u_0 - u_0^2 - u_0u_{0x}\,,$$
$$H_1(u) = u_1 - 2u_0u_1 - (u_0u_{1x} + u_1u_{0x})\,,$$
$$H_2(u) = u_2 - (u_1^2 + 2u_0u_2) - (u_0u_{2x} + u_1u_{1x} + u_2u_{0x})\,,$$
$$H_3(u) = u_3 - (2u_1u_2 + 2u_0u_3) - (u_0u_{3x} + u_1u_{2x} + u_2u_{1x} + u_3u_{0x})\,.$$

Equating the coefficients of corresponding power of p on both sides in (18), we get

$$p^0 : u_0(x,t) = \mathrm{e}^{-x}\,,$$
$$p^1 : u_1(x,t) = L^{-1}\left(s^{-\alpha}L[H_0(u)]\right) = \mathrm{e}^{-x}\frac{t^\alpha}{\Gamma(\alpha+1)}\,,$$
$$p^2 : u_2(x,t) = L^{-1}\left(s^{-\alpha}L[H_1(u)]\right) = \mathrm{e}^{-x}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\,,$$
$$p^3 : u_3(x,t) = L^{-1}\left(s^{-\alpha}L[H_2(u)]\right) = \mathrm{e}^{-x}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\,,$$
$$\vdots$$
$$p^n : u_n(x,t) = L^{-1}\left(s^{-\alpha}L[H_{n-1}(u)]\right) = \mathrm{e}^{-x}\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}\,.$$

Using the above terms, the solution $u(x,t)$ is given as

$$u(x,t) = \mathrm{e}^{-x}\left(1 + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \ldots + \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} + \ldots\right) = \mathrm{e}^{-x}\sum_{k=0}^{\infty}\frac{t^{k\alpha}}{\Gamma(k\alpha+1)} = \mathrm{e}^{-x}E_\alpha(t^\alpha)\,.$$

As $\alpha = 1$, this series has the closed form e^{t-x}, which is an exact solution of the classical gas dynamics. The above result is in complete agreement with Biazar and Eslami [13].

Example 2. Consider the following homogeneous nonlinear time-fractional gas dynamic equation:

$$\frac{\partial^\alpha u}{\partial t^\alpha} + u\frac{\partial u}{\partial x} - u(1-u)\log a = 0\,, \quad a > 0,\;\; 0 < \alpha \le 1\,, \quad (19)$$

with initial condition $u(x,0) = a^{-x}$. The exact solution for $\alpha = 1$ is $u(x,t) = a^{t-x}$.

Taking the Laplace transform on both sides of (19), we get

$$L[u(x,t)] = s^{-1}a^{-x} + s^{-\alpha}L[(u-u^2)\log a - uu_x]\,. \quad (20)$$

We apply the inverse Laplace transform on both sides and get

$$u(x,t) = a^{-x} + L^{-1}\left(s^{-\alpha}L[(u-u^2)\log a - uu_x]\right). \quad (21)$$

Using the aforesaid homotopy perturbation method, we have

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = a^{-x} + p\left(L^{-1}\left(s^{-\alpha}L[\sum_{n=0}^{\infty} p^n H_n(u)]\right)\right). \quad (22)$$

Equating the corresponding power of p on both sides, we get

$$p^0 : u_0(x,t) = a^{-x}\,,$$
$$p^1 : u_1(x,t) = L^{-1}\left(s^{-\alpha}L[H_0(u)]\right) = a^{-x}\frac{t^\alpha \log a}{\Gamma(\alpha+1)}\,,$$
$$p^2 : u_2(x,t) = L^{-1}\left(s^{-\alpha}L[H_1(u)]\right) = a^{-x}\frac{(t^\alpha \log a)^2}{\Gamma(2\alpha+1)}\,,$$

$$p^3 : u_3(x,t) = L^{-1}\left(s^{-\alpha}L[H_2(u)]\right) = a^{-x}\frac{(t^\alpha \log a)^3}{\Gamma(3\alpha+1)},$$

$$\vdots$$

so that the solution $u(x,t)$ is given as

$$u(x,t) = a^{-x}\left(1 + \frac{t^\alpha \log a}{\Gamma(\alpha+1)} + \frac{(t^\alpha \log a)^2}{\Gamma(2\alpha+1)}\right.$$
$$= \left. + \frac{(t^\alpha \log a)^3}{\Gamma(3\alpha+1)} + \ldots\right)$$
$$= a^{-x}\sum_{k=0}^{\infty}\frac{(t^\alpha \log a)^k}{\Gamma(k\alpha+1)} = a^{-x}E_\alpha(t^\alpha \log a).$$

Now for the standard case, i.e. for $\alpha = 1$, the series has the form a^{t-x}, which is the closed form of the solution.

Example 3. In this example, we consider the following inhomogeneous fractional gas dynamics equation [13]:

$$\frac{\partial^\alpha u}{\partial t^\alpha} + u\frac{\partial u}{\partial x} + (1+t)^2u^2 = x^2, \ 0 < \alpha \le 1, \tag{23}$$

with initial condition $u(x,0) = x$ and exact solution $u(x,t) = \frac{x}{(1+t)}$ for $\alpha = 1$.

Taking the known Laplace transform on both sides in (23), we get

$$L[u(x,t)] = s^{-1}x - s^{-\alpha}L[uu_x + (1+t)^2u^2 - x^2]. \tag{24}$$

Then the inverse Laplace transform on both sides implies that

$$u(x,t) = x - L^{-1}\left(s^{-\alpha}L[uu_x + (1+t)^2u^2 - x^2]\right). \tag{25}$$

We apply the homotopy perturbation method [15–19] and get

$$\sum_{n=0}^{\infty}p^n u_n(x,t)$$
$$= x - p\left(L^{-1}\left[s^{-\alpha}L[\sum_{n=0}^{\infty}p^nH_n(u) - x^2]\right]\right). \tag{26}$$

The first component of He's polynomials is given as

$$H_0(u) = u_0u_{0x} + (1+t)^2u_0^2, H_1(u)$$
$$= u_0u_{1x} + u_1u_{0x} + 2(1+t)^2u_0u_1.$$

Equating the coefficients of corresponding power of p on both sides in (26), we get

$$p^0 : u_0(x,t) = x,$$

$$p^1 : u_1(x,t) = -L^{-1}\left(s^{-\alpha}L[H_0(u) - x^2]\right)$$
$$= -t^\alpha\left(\frac{x}{\Gamma(\alpha+1)} + \frac{2x^2t}{\Gamma(\alpha+2)} + \frac{2x^2t^2}{\Gamma(\alpha+3)}\right),$$

$$p^2 : u_2(x,t) = -L^{-1}\left(s^{-\alpha}L[H_1(u)]\right) = \frac{2x(x+1)t^{2\alpha}}{\Gamma(2\alpha+1)}$$
$$+ \frac{2x^2(2x+2\alpha+5)t^{2\alpha+1}}{\Gamma(2\alpha+2)}$$
$$+ \frac{2x^2(4\alpha x+10x+\alpha^2+3\alpha+5)t^{2\alpha+2}}{\Gamma(2\alpha+3)}$$
$$+ \frac{4x^3(\alpha+3)(\alpha+4)t^{2\alpha+3}}{\Gamma(2\alpha+4)} + \frac{4x^3(\alpha+3)(\alpha+4)t^{2\alpha+4}}{\Gamma(2\alpha+5)},$$

$$\vdots$$

Proceeding in this manner, the rest of the components $u_n(x,t), n \ge 2$, can be completely obtained, and the series solutions are thus entirely determined.

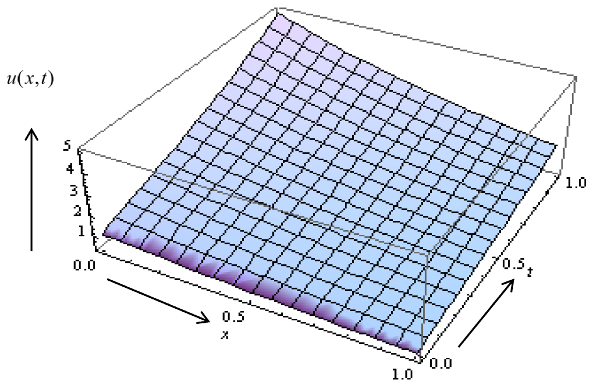

Fig. 1 (colour online). Plot of the exact solution $u(x,t)$ for Example 1.

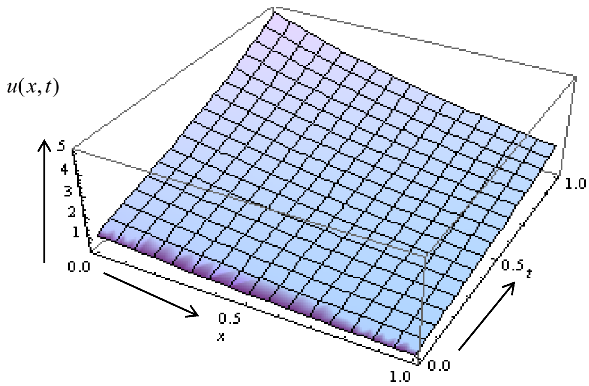

Fig. 2 (colour online). Plot of the approximate solution $u(x,t)$ at $\alpha = 1$ for Example 1. It is very close to the exact one in Figure 1.

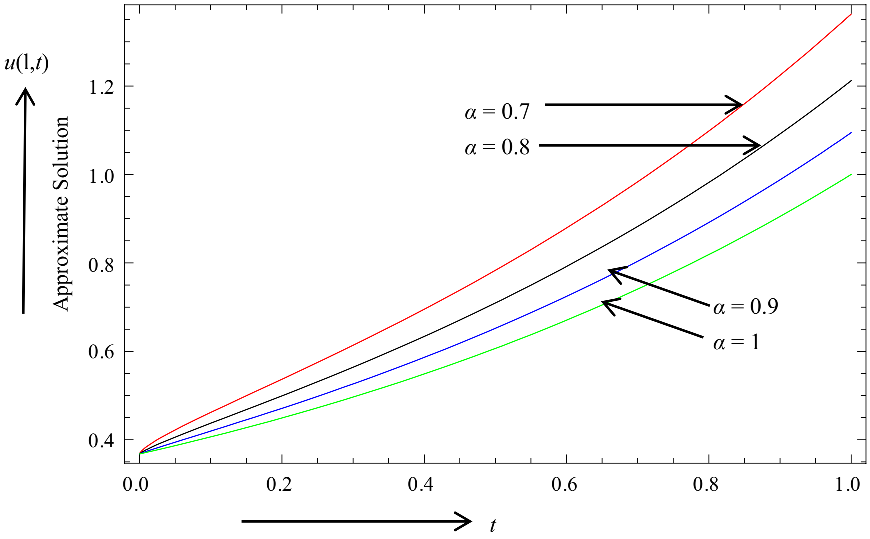

Fig. 3 (colour online). Plot of $u(x,t)$ vs. time t at $x = 1$ for different values of α for Example 1.

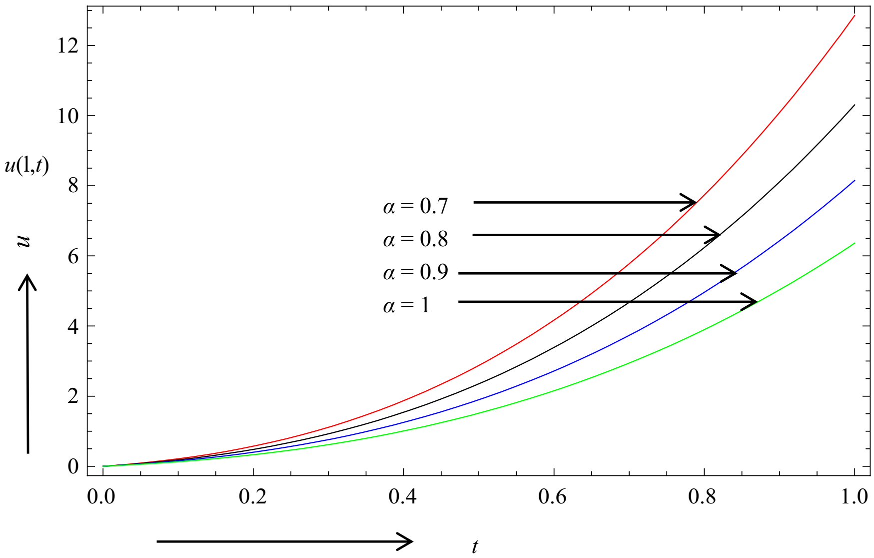

Fig. 4 (colour online). Plot of $u(x,t)$ vs. time t at different values of α for Example 3.

Finally, we approximate the analytical solution $u(x,t)$ by truncated series:

$$u(x,t) = \lim_{n\to\infty} \Psi_n(x,t)\,, \tag{27}$$

where $\Psi_N(x,t) = \sum_{n=0}^{N-1} u_n(x,t)$. The above series solutions generally converge very rapidly in real physical problems. The rapid convergence means that only few terms are required to get approximate solutions.

5. Numerical Results and Discussion

In this section, Figures 1–4 show the evaluation results of the approximate solution for Examples 1 and 3, respectively. Figures 3 and 4 show the behaviour of the approximate solution for different fractional Brownian motions and standard motions, i.e. for the standard motion $\alpha = 1$ at the value of $x = 1$.

Figures 1–3 show the evaluation results of the approximate solution for Example 1. Figures 1–2 show the comparison between the exact solution and the approximate solution (which is obtained by aforesaid method). It can be seen from these figures that the analytical solution obtained by the present method is nearly identical to the exact solution of the standard gas dynamics, i.e. for the standard motion $\alpha = 1$.

Figure 3 shows the behaviour of the approximate solution $u(x,t)$ for different fraction Brownian motion $\alpha = 0.7, 0.8, 0.9$, and for standard motion, i.e. at $\alpha = 1$ for Example 1. It is seen from Figure 3 that the solution obtained by HPTM increases very rapidly with the increases in t at the value of $x = 1$.

Figure 4 shows the behaviour of the approximate solution $u(x,t)$ for different values of $\alpha = 0.7, 0.8, 0.9$, and for the standard gas dynamics equation, i.e. at $\alpha = 1$ for the inhomogenous equation (23). It is seen from Figure 4 that the solution obtained by the present method increases very rapidly with the increase in t at the value of $x = 1$. The accuracy of the result can be improved by introducing more terms of the approximate solutions.

6. Conclusion

This paper develops an effective modification of the homotopy perturbation method, which is a coupling with the Laplace transform and He's polynomials, and studied its validity in a wide range with three examples of linear and nonlinear time-fractional gas dynamics equations. The method is applied in a direct way without using linearization, transformation, discritization or restrictive assumptions. The method gives more realistic series solutions that converge very rapidly in physical problems. It is worth mentioning that the method is capable of reducing the volume of the computational work as compare to the classical methods with high accuracy of the numerical result and will considerably benefit mathematicians and scientists working in the field of partial differential equations. It may be concluded that the LHPM methodology is very powerful and efficient in finding approximate solutions as well as analytical solutions.

Acknowledgement

The authors are very grateful to the referees for carefully reading the paper and for their comments and suggestions which have improved the paper. The first author is highly grateful to the Department of Mathematics, National Institute of Technology, Jamshedpur, India for the provision of some excellent facilities and research environment.

[1] A. Carpinteri and F. Mainardi, Fractals and Fractional Calculus in Continuum Mechanics, Springer Verlag, Wien, New York 1997.
[2] K. S. Miller and B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, Wiley, New York 1993.
[3] K. B. Oldham and J. Spanier, The Fractional Calculus, Academic Press, New York 1974.
[4] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego 1999.
[5] R. Gorenflo and F. Mainardi, Fractals and Fractional Calculus in Continuum Mechanics (Ed. F. Mainardi) Springer, New York 1997, p. 223.
[6] K. Diethelm, Electron. Trans. Numer. Anal. **5**, 1 (1997).
[7] J. L. Steger and R. F. Warming, J. Comput. Phys. **40**, 263 (1981).
[8] T. G. Elizarova, Quasi-Gas Dynamics Equations, ISBN: 978-3-642-00291-5, Springer Verlag 2009.
[9] D. J. Evans and H. Bulut, Int. J. Comput. Math. **79**, 817 (2002).
[10] A. Aziz and D. Anderson, Comput. Educat. **9**, 41 (1985).
[11] T. P. Liu, Non-Linear Waves in Mechanics and Gas Dynamics, Defence Technical Information Centre, Accession Number: ADA 238340, 1990.
[12] M. Rasulov and T. Karaguler, Appl. Math. Comput. **143**, 45 (2003).
[13] J. Biazar and M. Eslami, Int. J. Phys. Sci. **6**, 1203 (2011).
[14] S. Das and R. Kumar, Appl. Math. Comput. **217**, 9905 (2011).
[15] J. H. He, Comput. Meth. Appl. Mech. Eng. **178**, 257 (1999).
[16] J. H. He, Chaos Soliton. Fract. **26**, 695 (2005).
[17] J. H. He, Int. J. Nonlinear Mech. **35**, 37 (2000).
[18] J. H. He, Int. J. Modern Phys. B **20**, 1141 (2006).
[19] J. H. He, J. Sound Vibr. **229**, 1257 (2006).
[20] S. Kumar and O. P. Singh, Z. Naturforsch. **65a**, 677 (2010).
[21] A. Yıldırım and H. Kocak, Adv. Water Resour. **32**, 1711 (2009).
[22] A. Yıldırım and Y. Gulkanat, Commun. Theor. Phys. **53**, 1005 (2010).
[23] A. Yıldırım, J. King Saud Univ. **22**, 257 (2010).

[24] A. Yıldırım and H. Kocak, J. King Saud Univ. **23**, 183 (2011).
[25] A. Ghorbani and J. Saberi-Nadjafi, Int. J. Nonlinear Sci. Numer. Simul. **8**, 229 (2007).
[26] A. Ghorbani, Chaos Soliton. Fract. **39**, 1486 (2009).
[27] M. Khan and M. Hussain, Numer. Algor. **56**, 211 (2011).
[28] H. Jafari, C. M. Khalique, and M. Nazari, Appl. Math. Lett. **24**, 1799 (2011).
[29] Y. Khan and Q. Wu, Comput. Math. Appl. **61**, 1963 (2011).
[30] M. Madani, M. Fathizadeh, Y. Khan, and A. Yıldırım, Math. Comput. Model. **53**, 1937 (2011).
[31] Y. Luchko and R. Gorenflo, Acta Math. Vietnamica **24**, 207 (1999).
[32] O. L. Moustafa, Chaos Soliton. Fract. **18**, 135 (2003).
[33] G. Samko, A. A. Kilbas, and O. I. Marichev, Fractional Integral and Derivative: Theory and Applications, Gordan and Breach, Yverdon 1993.
[34] F. Mainardi, On the initial value problem for the fractional diffusion equation, in: Wave and Stability in Continuous Media, S. Rionero, T. Ruggeeri (Eds.), Word Scientific, Singapore, 246, 1994.
[35] K. Abbaoui and Y. Cherruault, Comput. Math. Appl. **29**, 103 (1995).