Infinite Series Symmetry Reduction Solutions to the Perturbed Coupled Nonlinear Schrödinger Equation

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By the approximate symmetry perturbation method, the perturbed coupled nonlinear Schrödinger equation (PCNLSE) is investigated. As a result, the approximate symmetries and infinite series symmetry reduction solutions are obtained. Specially, we take the symmetry reduction for soliton solutions as an example, where the effects of perturbations on soliton solutions are briefly discussed.

Key words: Perturbed Coupled Nonlinear Schrödinger Equation; Approximate Symmetry Perturbation Method; Infinite Series Symmetry Reduction Solution.
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1. Introduction

The highly idealized nonlinear evolution equations, such as the Korteweg–de Vries (KdV) equation, the nonlinear Schrödinger (NLS) equation, the Kadomtsev–Petviashvili (KP) equation, etc., may have been solved analytically many times by kinds of well-known methods, including the inverse scattering transformation (IST), the Hirota bilinear method, the Darboux transformation, and so on. However, to obtain the analytical solutions of their correction systems that differ from the standard ones by some small additional terms (so-called perturbations), it still does not seem to be an easy task. One of the most powerful techniques in dealing with these systems is based on the IST [1]. This method requires that the unperturbed equations must be exactly solvable via the IST, which seriously restricts the range of application. Besides, the direct perturbation method has also been frequently used to find approximate analytical solutions to perturbed partial differential equations. Supplements for this method can be found in [2 – 6] and the references therein. Another effective way to investigate perturbed nonlinear evolution equations is the approximate symmetry perturbation method, which is an integration of the perturbation method and the symmetry reduction method first proposed by Fushchich and Shtelen [7]. Recently, the approximate symmetry perturbation method has been further improved by Lou and his colleagues, and has been widely applied to many equations [8 – 11].

In the present paper, we shall introduce the approximate symmetry perturbation method to the perturbed coupled system of two NLS equations:

$$\mathrm{i}p_t+\beta_1 p_{xx}+(\kappa_1|p|^2+\gamma_1|q|^2)p = \varepsilon P(p,q,p_x,q_x,p_t,q_t,\ldots)\,, \tag{1a}$$

$$\mathrm{i}q_t+\beta_2 q_{xx}+(\kappa_2|q|^2+\gamma_2|p|^2)q = \varepsilon Q(q,p,q_x,p_x,p_t,q_t,\ldots)\,. \tag{1b}$$

Equation (1) has been applied in various physical fields, such as plasma physics [12], nonlinear optics [13, 14], condense matter physics [15, 16], geophysics fluid dynamics [17], etc. In the context of nonlinear optics, p and q in (1) describe the envelopes of the waves with mutually orthogonal linear polarization, x and t correspond to normalized distance and time, respectively. The parameters β_1, β_2 are the dispersion coefficients, κ_1, κ_2 the Landau constants which describe the self-modulation of the wave packets, and γ_1, γ_2 the wave–wave interaction coefficients which describe the cross-modulations of the wave packets. εP and εQ are perturbations of arbitrary form, and ε is a small parameter characterizing the magnitude of these perturbations. When $\varepsilon=0$, (1) reduces into the coupled nonlinear Schrödinger equation (CNLSE). Manakov [18] was the first to show that the CNLSE is

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integrable in terms of the method of IST for the case where $\beta_1 = \beta_2 = 1$, $\kappa_1 = \kappa_2 = \gamma_1 = \gamma_2$. Later Zakharov and Schulman [19] have thoroughly studied the integrability of system (1) with $\varepsilon = 0$, and they summarized that only when $\beta_1 = \beta_2$, $\kappa_1 = \kappa_2 = \gamma_1 = \gamma_2$ or $\beta_1 = -\beta_2$, $\kappa_1 = \kappa_2 = -\gamma_1 = -\gamma_2$ the CNLSE is integrable, i.e. it can be solved by IST. To put it in another way, with other more general $\beta_1, \beta_2, \kappa_1, \kappa_2, \gamma_1, \gamma_2$ values, system (1) is non-integrable, and the perturbed ones are included.

For the sake of simplicity and convenience, here we take account of the special case of (1) with coefficients $\beta_1 = \beta_2 = \kappa_1 = \kappa_2 = \gamma_1 = \gamma_2 = 1$ and $\varepsilon P(p,q,p_x,q_x,\ldots) = -\mathrm{i}\varepsilon p$, $\varepsilon Q(q,p,q_x,p_x,\ldots) = -\mathrm{i}\varepsilon q$, corresponding to the action of linear damping in fiber systems [20]. Thereby, system (1) becomes

$$\mathrm{i}p_t + p_{xx} + (|p|^2 + |q|^2)p + \mathrm{i}\varepsilon p = 0, \tag{2a}$$

$$\mathrm{i}q_t + q_{xx} + (|p|^2 + |q|^2)q + \mathrm{i}\varepsilon q = 0. \tag{2b}$$

As we know, several perturbation methods have been employed to obtain the zero-order solutions of (2) [21], but a perturbation method for deriving the first order or much higher-order modifications is still quite rare.

2. Approximate Symmetry Perturbation Method to the Perturbed Coupled Nonlinear Schrödinger Equation

To search for the solutions of the PCNLSE (2), we would like to assume that the complex functions p and q are in exponential form:

$$p = u\,\mathrm{e}^{\mathrm{i}v}, \quad q = r\,\mathrm{e}^{\mathrm{i}s}, \tag{3}$$

where $u \equiv u(x,t)$, $v \equiv v(x,t)$, $r \equiv r(x,t)$, $s \equiv s(x,t)$. Substituting expression (3) into (2) and separating real and imaginary parts, (2) is then replaced by

$$\begin{aligned} &-uv_t + u_{xx} - uv_x^2 + u^3 + ur^2 = 0, \\ &u_t + 2u_x v_x + uv_{xx} + \varepsilon u = 0, \end{aligned} \tag{4a}$$

$$\begin{aligned} &-rs_t + r_{xx} - rs_x^2 + ru^2 + r^3 = 0, \\ &r_t + 2r_x s_x + rs_{xx} + \varepsilon r = 0. \end{aligned} \tag{4b}$$

According to the approximate symmetry perturbation theory, the solutions of (4) are expressed as series:

$$\begin{aligned} u &= \sum_{j=0}^{\infty} \varepsilon^j u_j, \quad v = \sum_{j=0}^{\infty} \varepsilon^j v_j, \\ r &= \sum_{j=0}^{\infty} \varepsilon^j r_j, \quad s = \sum_{j=0}^{\infty} \varepsilon^j s_j. \end{aligned} \tag{5}$$

u_j, v_j, r_j, and s_j are functions with respect to x and t. When one inserts expansion (5) into (4) and vanishes the coefficients of all different powers of ε, a system of partial differential equations is given:

$$-u_0 v_{0t} + u_{0xx} - u_0 v_{0x}^2 + u_0^3 + u_0 r_0^2 = 0, \tag{6a}$$

$$u_{0t} + 2u_{0x}v_{0x} + u_0 v_{0xx} = 0, \tag{6b}$$

$$-r_0 s_{0t} + r_{0xx} - r_0 s_{0x}^2 + r_0 u_0^2 + r_0^3 = 0, \tag{6c}$$

$$r_{0t} + 2r_{0x}s_{0x} + r_0 s_{0xx} = 0, \tag{6d}$$

$$\begin{aligned} &-u_0 v_{1t} - u_1 v_{0t} + u_{1xx} - u_1 v_{0x}^2 - 2u_0 v_{0x} v_{1x} \\ &+ 2u_0 r_0 r_1 + u_1 r_0^2 + 3u_0^2 u_1 = 0, \end{aligned} \tag{7a}$$

$$\begin{aligned} &u_{1t} + u_0 v_{1xx} + u_1 v_{0xx} + 2(u_{1x} v_{0x} + u_{0x} v_{1x}) \\ &+ u_0 = 0, \end{aligned} \tag{7b}$$

$$\begin{aligned} &-r_0 s_{1t} - r_1 s_{0t} + r_{1xx} - 2r_0 s_{0x} s_{1x} - r_1 s_{0x}^2 \\ &+ 2r_0 u_0 u_1 + r_1 u_0^2 + 3r_0^2 r_1 = 0, \end{aligned} \tag{7c}$$

$$\begin{aligned} &r_{1t} + r_0 s_{1xx} + r_1 s_{0xx} + 2(r_{0x} s_{1x} + r_{1x} s_{0x}) \\ &+ r_0 = 0, \end{aligned} \tag{7d}$$

$\ldots,$

$$\begin{aligned} &-\sum_{i=0}^{j} u_i v_{j-i,t} + u_{j,xx} - \sum_{i=0}^{j}\sum_{k=0}^{j-i} u_i v_{k,x} v_{j-i-k,x} \\ &+ \sum_{i=0}^{j}\sum_{k=0}^{j-i} u_i (r_k r_{j-i-k} + u_k u_{j-i-k}) = 0, \end{aligned} \tag{8a}$$

$$u_{j,t} + \sum_{i=0}^{j} u_i v_{j-i,xx} + 2\sum_{i=0}^{j} u_{i,x} v_{j-i,x} + u_{j-1} = 0, \tag{8b}$$

$$\begin{aligned} &-\sum_{i=0}^{j} r_i s_{j-i,t} + r_{j,xx} - \sum_{i=0}^{j}\sum_{k=0}^{j-i} r_i s_{k,x} s_{j-i-k,x} \\ &+ \sum_{i=0}^{j}\sum_{k=0}^{j-i} r_i (u_k u_{j-i-k} + r_k r_{j-i-k}) = 0, \end{aligned} \tag{8c}$$

$$r_{j,t} + \sum_{i=0}^{j} r_i s_{j-i,xx} + 2\sum_{i=0}^{j} r_{i,x} s_{j-i,x} + r_{j-1} = 0, \tag{8d}$$

with $u_{-1} = v_{-1} = r_{-1} = s_{-1} = 0$. To find the exact solutions of (8), we first construct its Lie point symmetries and then give the corresponding symmetry reductions. In what follows, we shall suppose that the Lie symmetry transformations satisfy

$$\sigma_{u_j} = X u_{j,x} + T u_{j,t} - U_j, \quad \sigma_{v_j} = X v_{j,x} + T v_{j,t} - V_j, \tag{9a}$$

$$\sigma_{r_j} = X r_{j,x} + T r_{j,t} - R_j, \quad \sigma_{s_j} = X s_{j,x} + T s_{j,t} - S_j, \tag{9b}$$

where X, T, U_j, V_j, R_j, and S_j are functions with respect to x, t, u_j, v_j, r_j, and s_j $(j = 0, 1, 2, \ldots)$, that is, (8) is invariant under the transformations

$$\{x,t,u_j,v_j,r_j,s_j\} \to \{x+\varepsilon X, t+\varepsilon T, u_j + \varepsilon U_j, v_j+\varepsilon V_j, r_j+\varepsilon R_j, s_j+\varepsilon S_j\} \tag{10}$$

with an infinitesimal parameter ε. Under the notation (9), σ_{u_j}, σ_{v_j}, σ_{r_j}, and σ_{s_j} are then the solutions of the symmetry equations, i.e., the linearized equations for (8) listed below:

$$-u_0\sigma_{v_0,t}-\sigma_{u_0}v_{0t}+\sigma_{u_0,xx}-\sigma_{u_0}v_{0x}^2-2u_0v_{0x}\sigma_{v_0,x} + 2u_0r_0\sigma_{r_0}+\sigma_{u_0}r_0^2+3u_0^2\sigma_{u_0}=0, \tag{11a}$$

$$\sigma_{u_0,t}+u_0\sigma_{v_0,xx}+\sigma_{v_0}v_{0xx}+2u_{0x}\sigma_{v_0,x} + 2\sigma_{u_0,x}v_{0x}=0, \tag{11b}$$

$$-r_0\sigma_{s_0,t}-\sigma_{r_0}s_{0t}+\sigma_{r_0,xx}-2r_0s_{0x}\sigma_{s_0,x}-\sigma_{r_0}s_{0x}^2 + 2r_0u_0\sigma_{u_0}+\sigma_{r_0}u_0^2+3r_0^2\sigma_{r_0}=0, \tag{11c}$$

$$\sigma_{r_0,t}+r_0\sigma_{s_0,xx}+\sigma_{r_0}s_{0xx}+2r_{0x}\sigma_{s_0,x} + 2\sigma_{r_0,x}s_{0x}=0, \tag{11d}$$

$$\begin{aligned}&-u_0\sigma_{v_1,t}-u_1\sigma_{v_0,t}-\sigma_{u_0}v_{1t}-\sigma_{u_1}v_{0t}+\sigma_{u_1,xx} \\ &-2\sigma_{u_0}v_{0x}v_{1x}-2u_0\sigma_{v_0,x}v_{1x}-2u_0v_{0x}\sigma_{v_1,x} \\ &-2u_1v_{0x}\sigma_{v_0,x}-\sigma_{u_1}v_{0x}^2+2r_0r_1\sigma_{u_0}+\sigma_{u_1}r_0^2 \\ &+6u_0u_1\sigma_{u_0}+2u_0r_1\sigma_{r_0}+2r_0u_0\sigma_{r_1}+2r_0u_1\sigma_{r_0} \\ &+3u_0^2\sigma_{u_1}=0,\end{aligned} \tag{12a}$$

$$\begin{aligned}&\sigma_{u_1,t}+u_0\sigma_{v_1,xx}+u_1\sigma_{v_0,xx}+\sigma_{u_0}v_{1xx}+\sigma_{u_1}v_{0xx} \\ &+2u_{0x}\sigma_{v_1,x}+2u_{1x}\sigma_{v_0,x}+2\sigma_{u_0,x}v_{1x}+2\sigma_{u_1,x}v_{0x} \\ &+\sigma_{u_0}=0,\end{aligned} \tag{12b}$$

$$\begin{aligned}&-r_0\sigma_{s_1,t}-r_1\sigma_{s_0,t}-\sigma_{r_0}s_{1t}-\sigma_{r_1}s_{0t}+\sigma_{r_1,xx} \\ &-2r_0s_{0x}\sigma_{s_1,x}-2r_0\sigma_{s_0,x}s_{1x}-2r_1s_{0x}\sigma_{s_0,x} \\ &-2\sigma_{r_0}s_{0x}s_{1x}-\sigma_{r_1}s_{0x}^2+2u_0r_0\sigma_{u_1}+2u_0u_1\sigma_{r_0} \\ &+2r_0u_1\sigma_{u_0}+2u_0r_1\sigma_{u_0}+6r_0r_1\sigma_{r_0}+\sigma_{r_1}u_0^2 \\ &+3r_0^2\sigma_{r_1}=0,\end{aligned} \tag{12c}$$

$$\begin{aligned}&\sigma_{r_1,t}+r_0\sigma_{s_1,xx}+r_1\sigma_{s_0,xx}+\sigma_{r_0}s_{1xx}+\sigma_{r_1}s_{0xx} \\ &+2r_{0x}\sigma_{s_1,x}+2r_{1x}\sigma_{s_0,x}+2\sigma_{r_0,x}s_{1x}+2\sigma_{r_1,x}s_{0x} \\ &+\sigma_{r_0}=0,\end{aligned} \tag{12d}$$

$\ldots,$

$$\begin{aligned}&-\sum_{i=0}^{j}(u_i\sigma_{v_{j-i},t}+\sigma_{u_i}v_{j-i,t})+\sigma_{u_j,xx} \\ &-\sum_{i=0}^{j}\sum_{k=0}^{j-i}\cdot(\sigma_{u_i}v_{k,x}v_{j-i-k,x}+u_i\sigma_{v_k,x}v_{j-i-k,x} \\ &+u_iv_{k,x}\sigma_{v_{j-i-k},x})+\sum_{i=0}^{j}\sum_{k=0}^{j-i}[\sigma_{u_i}(r_kr_{j-i-k}+u_ku_{j-i-k}) \\ &+u_i(\sigma_{r_k}r_{j-i-k}+r_k\sigma_{r_{j-i-k}}+\sigma_{u_k}u_{j-i-k}+u_k\sigma_{u_{j-i-k}})] \\ &=0,\end{aligned} \tag{13a}$$

$$\begin{aligned}&\sigma_{u_j,t}+\sum_{i=0}^{j}(u_i\sigma_{v_{j-i},xx}+\sigma_{u_i}v_{j-i,xx}) \\ &+2\sum_{i=0}^{j}(u_{i,x}\sigma_{v_{j-i},x}+\sigma_{u_i,x}v_{j-i,x})+\sigma_{u_{j-1}}=0,\end{aligned} \tag{13b}$$

$$\begin{aligned}&-\sum_{i=0}^{j}(r_i\sigma_{s_{j-i},t}+\sigma_{r_i}s_{j-i,t})+\sigma_{r_j,xx} \\ &-\sum_{i=0}^{j}\sum_{k=0}^{j-i}(\sigma_{r_i}s_{k,x}s_{j-i-k,x}+r_i\sigma_{s_k,x}s_{j-i-k,x} \\ &+r_is_{k,x}\sigma_{s_{j-i-k},x})+\sum_{i=0}^{j}\sum_{k=0}^{j-i}[\sigma_{r_i}(u_ku_{j-i-k}+r_kr_{j-i-k}) \\ &+r_i(\sigma_{u_k}u_{j-i-k}+u_k\sigma_{u_{j-i-k}}+\sigma_{r_k}r_{j-i-k}+r_k\sigma_{r_{j-i-k}})]=0,\end{aligned} \tag{13c}$$

$$\begin{aligned}&\sigma_{r_j,t}+\sum_{i=0}^{j}(r_i\sigma_{s_{j-i},xx}+\sigma_{r_i}s_{j-i,xx}) \\ &+2\sum_{i=0}^{j}(r_{i,x}\sigma_{s_{j-i},x}+\sigma_{r_i,x}s_{j-i,x})+\sigma_{r_{j-1}}=0,\end{aligned} \tag{13d}$$

and $\sigma_{u_{-1}}=\sigma_{v_{-1}}=\sigma_{r_{-1}}=\sigma_{s_{-1}}=0$. Substituting (9) into the linearized equation system (13) and eliminating u_{jt}, v_{jt}, r_{jt}, and s_{jt} according to (8) yield thousands of determining equations by vanishing all coefficients of the different partial derivatives of u_j, v_j, r_j, and s_j for the functions X, T, U_j, V_j, R_j, and S_j. Solving these determining equations, we conclude

$$X=\frac{c_0}{2}x+c_1t+x_0,\quad T=c_0t+t_0, \tag{14a}$$

$$U_0=-\frac{c_0u_0}{2},\quad R_0=-\frac{c_0r_0}{2},\quad V_0=S_0=\frac{c_1x}{2}, \tag{14b}$$

$$U_1=\frac{c_0u_1}{2},\quad R_1=\frac{c_0r_1}{2},\quad V_1=c_0v_1,\quad S_1=c_0s_1, \tag{14c}$$

$\ldots,$

$$U_j=\frac{(2j-1)c_0u_j}{2},\quad V_j=jc_0v_j+\frac{c_1x}{2}\delta_{j,0}, \tag{14d}$$

$$R_j=\frac{(2j-1)c_0r_j}{2},\quad S_j=jc_0s_j+\frac{c_1x}{2}\delta_{j,0}. \tag{14e}$$

c_0, c_1, x_0, and t_0 are all arbitrary constants. Thus the Lie symmetries take the forms

$$\sigma_{u_0}=\left(\frac{c_0}{2}x+c_1t+x_0\right)u_{0x}+(c_0t+t_0)u_{0t}+\frac{c_0u_0}{2}, \tag{15a}$$

$$\sigma_{v_0}=\left(\frac{c_0}{2}x+c_1t+x_0\right)v_{0x}+(c_0t+t_0)v_{0t}-\frac{c_1x}{2}, \tag{15b}$$

$$\sigma_{r_0} = \left(\frac{c_0}{2}x + c_1t + x_0\right) r_{0x} + (c_0t + t_0) r_{0t} + \frac{c_0 r_0}{2}, \tag{15c}$$

$$\sigma_{s_0} = \left(\frac{c_0}{2}x + c_1t + x_0\right) s_{0x} + (c_0t + t_0) s_{0t} - \frac{c_1 x}{2}, \tag{15d}$$

$$\sigma_{u_1} = \left(\frac{c_0}{2}x + c_1t + x_0\right) u_{1x} + (c_0t + t_0) u_{1t} - \frac{c_0 u_1}{2}, \tag{16a}$$

$$\sigma_{v_1} = \left(\frac{c_0}{2}x + c_1t + x_0\right) v_{1x} + (c_0t + t_0) v_{1t} - c_0 v_1, \tag{16b}$$

$$\sigma_{r_1} = \left(\frac{c_0}{2}x + c_1t + x_0\right) r_{1x} + (c_0t + t_0) r_{1t} - \frac{c_0 r_1}{2}, \tag{16c}$$

$$\sigma_{s_1} = \left(\frac{c_0}{2}x + c_1t + x_0\right) s_{1x} + (c_0t + t_0) s_{1t} - c_0 s_1, \ldots, \tag{16d}$$

$$\sigma_{u_j} = \left(\frac{c_0}{2}x + c_1t + x_0\right) u_{j,x} + (c_0t + t_0) u_{j,t} - \frac{(2j-1)c_0 u_j}{2}, \tag{17a}$$

$$\sigma_{v_j} = \left(\frac{c_0}{2}x + c_1t + x_0\right) v_{j,x} + (c_0t + t_0) v_{j,t} - jc_0 v_j - \frac{c_1 x}{2}\delta_{j,0}, \tag{17b}$$

$$\sigma_{r_j} = \left(\frac{c_0}{2}x + c_1t + x_0\right) r_{j,x} + (c_0t + t_0) r_{j,t} - \frac{(2j-1)c_0 r_j}{2}, \tag{17c}$$

$$\sigma_{s_j} = \left(\frac{c_0}{2}x + c_1t + x_0\right) s_{j,x} + (c_0t + t_0) s_{j,t} - jc_0 s_j - \frac{c_1 x}{2}\delta_{j,0}. \tag{17d}$$

Note that the notation $\delta_{j,i}$ satisfies $\delta_{j,j} = 1$ and $\delta_{j,i} = 0 (j \neq i)$. Subsequently, the similarity solutions to (8) can be obtained by solving the characteristic equations

$$\frac{dx}{X} = \frac{dt}{T} = \frac{du_0}{U_0} = \frac{dv_0}{V_0} = \frac{dr_0}{R_0} = \frac{ds_0}{S_0} = \ldots = \frac{du_j}{U_j} = \frac{dv_j}{V_j} = \frac{dr_j}{R_j} = \frac{ds_j}{S_j} = \ldots. \tag{18}$$

In the following paragraphs, two subcases are distinguished concerning the solutions to (18).

Case 1. When $c_0 \neq 0$, we choose the group invariant as

$$\xi = \frac{c_0^2 x - 2c_0c_1t + 2c_0x_0 - 4c_1t_0}{c_0^2\sqrt{c_0t + t_0}}.$$

Then the similarity solutions for fields u_j, v_j, r_j, and s_j are

$$u_0 = U_0(\xi)(c_0t + t_0)^{-\frac{1}{2}}, \ u_1 = U_1(\xi)(c_0t + t_0)^{\frac{1}{2}}, \ldots, \quad u_j = U_j(\xi)(c_0t + t_0)^{\frac{2j-1}{2}}, \tag{19a}$$

$$v_0 = V_0(\xi) + f, \ v_1 = V_1(\xi)(c_0t + t_0), \ldots, \quad v_j = V_j(\xi)(c_0t + t_0)^j + \delta_{j,0} f, \tag{19b}$$

$$r_0 = R_0(\xi)(c_0t + t_0)^{-\frac{1}{2}}, \ r_1 = R_1(\xi)(c_0t + t_0)^{\frac{1}{2}}, \ldots, \quad r_j = R_j(\xi)(c_0t + t_0)^{\frac{2j-1}{2}}, \tag{19c}$$

$$s_0 = S_0(\xi) + f, \ s_1 = S_1(\xi)(c_0t + t_0), \ldots, \quad s_j = S_j(\xi)(c_0t + t_0)^j + \delta_{j,0} f. \tag{19d}$$

As follows from (19), the perturbation series solutions to (4) are described by the equations

$$u = \sum_{j=0}^{\infty} \varepsilon^j U_j(\xi)(c_0t + t_0)^{\frac{2j-1}{2}}, \quad v = f + \sum_{j=0}^{\infty} \varepsilon^j V_j(\xi)(c_0t + t_0)^j, \tag{20a}$$

$$r = \sum_{j=0}^{\infty} \varepsilon^j R_j(\xi)(c_0t + t_0)^{\frac{2j-1}{2}}, \quad s = f + \sum_{j=0}^{\infty} \varepsilon^j S_j(\xi)(c_0t + t_0)^j, \tag{20b}$$

where

$$f = \frac{c_1}{c_0^3}\left[(c_1t_0 - c_0x_0)\ln(c_0t + t_0) + c_0^2 x - c_0c_1t + 2c_0x_0 - 4c_1t_0\right]$$

and $U_{-1} = V_{-1} = R_{-1} = S_{-1} = 0$. Inserting similarity solutions (19) into (8), we summarize the similarity reduction equations

$$U_{0\xi\xi} - U_0V_{0\xi}^2 + \frac{1}{2}c_0\xi U_0V_{0\xi} + U_0\left(U_0^2 + R_0^2\right) + \lambda U_0 = 0, \tag{21a}$$

$$2U_0V_{0\xi\xi} + 4U_{0\xi}V_{0\xi} - c_0\xi U_{0\xi} - c_0U_0 = 0, \tag{21b}$$

$$R_{0\xi\xi} - R_0S_{0\xi}^2 + \frac{1}{2}c_0\xi R_0S_{0\xi} + R_0\left(U_0^2 + R_0^2\right) + \lambda R_0 = 0, \tag{21c}$$

$$2R_0S_{0\xi\xi} + 4R_{0\xi}S_{0\xi} - c_0\xi R_{0\xi} - c_0R_0 = 0, \tag{21d}$$

$$\begin{aligned}&U_{1\xi\xi}-2U_0V_{1\xi}V_{0\xi}-U_1V_{0\xi}^2+\frac{1}{2}c_0\xi(U_0V_{1\xi}+U_1V_{0\xi})\\&+2U_0R_0R_1+U_1R_0^2+3U_0^2U_1-c_0U_0V_1\\&+\lambda U_1=0,\end{aligned}\tag{22a}$$

$$\begin{aligned}&2(U_0V_{1\xi\xi}+U_1V_{0\xi\xi})+4(U_{0\xi}V_{1\xi}+U_{1\xi}V_{0\xi})\\&-c_0\xi U_{1\xi}+c_0U_1+2U_0=0,\end{aligned}\tag{22b}$$

$$\begin{aligned}&R_{1\xi\xi}-2R_0S_{1\xi}S_{0\xi}-R_1S_{0\xi}^2+\frac{1}{2}c_0\xi(R_0S_{1\xi}+R_1S_{0\xi})\\&+2R_0U_0U_1+R_1U_0^2+3R_0^2R_1-c_0R_0S_1\\&+\lambda R_1=0,\end{aligned}\tag{22c}$$

$$\begin{aligned}&2(R_0S_{1\xi\xi}+R_1S_{0\xi\xi})+4(R_{0\xi}S_{1\xi}+R_{1\xi}S_{0\xi})\\&-c_0\xi R_{1\xi}+c_0R_1+2R_0=0,\end{aligned}\tag{22d}$$

$\ldots,$

$$\begin{aligned}&U_{j,\xi\xi}-\sum_{i=0}^{j}\sum_{k=0}^{j-i}U_iV_{k,\xi}V_{j-i-k,\xi}\\&+\sum_{i=0}^{j}\sum_{k=0}^{j-i}U_i(R_kR_{j-i-k}+U_kU_{j-i-k})\\&+\frac{1}{2}c_0\xi\sum_{i=0}^{j}U_iV_{j-i,\xi}-c_0\sum_{i=0}^{j}(j-i)U_iV_{j-i}+\lambda U_j=0,\end{aligned}\tag{23a}$$

$$\begin{aligned}&2\sum_{i=0}^{j}U_iV_{j-i,\xi\xi}+4\sum_{i=0}^{j}U_{i,\xi}V_{j-i,\xi}+c_0\xi U_{j,\xi}\\&+(2j-1)c_0U_j+2U_{j-1}=0,\end{aligned}\tag{23b}$$

$$\begin{aligned}&R_{j,\xi\xi}-\sum_{i=0}^{j}\sum_{k=0}^{j-i}R_iS_{k,\xi}S_{j-i-k,\xi}\\&+\sum_{i=0}^{j}\sum_{k=0}^{j-i}R_i(R_kR_{j-i-k}+U_kU_{j-i-k})\\&+\frac{1}{2}c_0\xi\sum_{i=0}^{j}R_iS_{j-i,\xi}-c_0\sum_{i=0}^{j}(j-i)R_iV_{j-i}+\lambda R_j=0,\end{aligned}\tag{23c}$$

$$\begin{aligned}&2\sum_{i=0}^{j}R_iS_{j-i,\xi\xi}+4\sum_{i=0}^{j}R_{i,\xi}S_{j-i,\xi}+c_0\xi R_{j,\xi}\\&+(2j-1)c_0R_j+2R_{j-1}=0.\end{aligned}\tag{23d}$$

The constant λ is $\lambda=\frac{1}{c_0^2}(c_0c_1x_0-c_1^2t_0)$.

Case 2. In the case $c_0=0$, the similarity solutions become

$$\begin{aligned}&u_j=U_j(\xi),\;\; v_j=V_j(\xi)+g\delta_{j,0},\\&r_j=R_j(\xi),\;\; s_j=S_j(\xi)+g\delta_{j,0}\end{aligned}\tag{24}$$

with the similarity variable ξ and function g being taken as

$$\xi=\frac{c_1t^2+2x_0t-2t_0x}{2t_0},$$
$$g=-\frac{(c_1t+x_0)}{6c_1t_0^2}(c_1^2t^2+2c_1x_0t-3c_1t_0x+x_0^2).$$

Hence, the perturbation series solutions to (4) are

$$\begin{aligned}&u=\sum_{j=0}^{\infty}\varepsilon^jU_j(\xi),\;\; v=g+\sum_{j=0}^{\infty}\varepsilon^jV_j(\xi),\\&r=\sum_{j=0}^{\infty}\varepsilon^jR_j(\xi),\;\; s=g+\sum_{j=0}^{\infty}\varepsilon^jS_j(\xi).\end{aligned}\tag{25}$$

And the similarity reduction equations related to similarity solutions (24) can be expressed by

$$U_{0\xi\xi}-U_0V_{0\xi}^2+U_0(R_0^2+U_0^2)+\kappa U_0=0,\tag{26a}$$
$$U_0V_{0\xi\xi}+2V_{0\xi}U_{0\xi}=0,\tag{26b}$$
$$R_{0\xi\xi}-R_0S_{0\xi}^2+R_0(R_0^2+U_0^2)+\kappa R_0=0,\tag{26c}$$
$$R_0S_{0\xi\xi}+2S_{0\xi}R_{0\xi}=0,\tag{26d}$$

$$\begin{aligned}&U_{1\xi\xi}-2U_0V_{0\xi}V_{1\xi}-U_1V_{0\xi}^2+2U_0R_0R_1\\&+3U_0^2U_1+U_1R_0^2+\kappa U_1=0,\end{aligned}\tag{27a}$$

$$\begin{aligned}&U_0V_{1\xi\xi}+U_1V_{0\xi\xi}+2(V_{0\xi}U_{1\xi}+V_{1\xi}U_{0\xi})\\&+U_0=0,\end{aligned}\tag{27b}$$

$$\begin{aligned}&R_{1\xi\xi}-2R_0S_{0\xi}S_{1\xi}-R_1S_{0\xi}^2+2R_0U_0U_1\\&+3R_0^2R_1+R_1U_0^2+\kappa R_1=0,\end{aligned}\tag{27c}$$

$$\begin{aligned}&R_0S_{1\xi\xi}+R_1S_{0\xi\xi}+2(S_{0\xi}R_{1\xi}+S_{1\xi}R_{0\xi})\\&+R_0=0,\end{aligned}\tag{27d}$$

$\ldots,$

$$\begin{aligned}&U_{j,\xi\xi}-\sum_{i=0}^{j}\sum_{k=0}^{j-i}U_iV_{k,\xi}V_{j-i-k,\xi}\\&+\sum_{i=0}^{j}U_i(R_kR_{j-i-k}+U_kU_{j-i-k})+\kappa U_j=0,\end{aligned}\tag{28a}$$

$$\sum_{i=0}^{j}U_iV_{j-i,\xi\xi}+2\sum_{i=0}^{j}U_{i,\xi}V_{j-i,\xi}+U_{j-1}=0,\tag{28b}$$

$$\begin{aligned}&R_{j,\xi\xi}-\sum_{i=0}^{j}\sum_{k=0}^{j-i}R_iS_{k,\xi}V_{j-i-k,\xi}\\&+\sum_{i=0}^{j}R_i(R_kR_{j-i-k}+U_kU_{j-i-k})+\kappa R_j=0,\end{aligned}\tag{28c}$$

$$\sum_{i=0}^{j} R_i S_{j-i,\xi\xi} + 2\sum_{i=0}^{j} R_{i,\xi} S_{j-i,\xi} + R_{j-1} = 0\,, \tag{28d}$$

where $U_{-1} = V_{-1} = R_{-1} = S_{-1} = 0$, and the parameter κ is determined by $\kappa = \frac{2c_1 t_0 \xi + x_0^2}{4t_0^2}$.

On observation of (23) and (28), it is not difficult to find that the jth similarity reduction equations contain $U_0, U_1, \ldots, U_j$, $V_0, V_1, \ldots, V_j$, $R_0, R_1, \ldots, R_j$, and $S_0, S_1, \ldots, S_j$. When the previous $U_0, U_1, \ldots, U_{j-1}$, $V_0, V_1, \ldots, V_{j-1}$, $R_0, R_1, \ldots, R_{j-1}$, and $S_0, S_1, \ldots, S_{j-1}$ are known, the jth similarity reduction equations reduce to the second-order linear ordinary differential equations of U_j, V_j, R_j, and S_j. One can then solve them one after another.

3. Symmetry Perturbation Reduction for One-Soliton Solutions

For more detailed and further studying, let us take the simple case $c_1 = c_0 = 0$ for example and consider the effects of perturbations on soliton solutions of the CNLSE with the help of Maple. Now the group variable ξ becomes

$$\xi = t - \frac{t_0}{x_0} x\,.$$

Calculating as above steps, it is easy to get the similarity reduction equations (for convenience, we do not write out them in detail). Solving of the reduction equations leads to the solutions of different orders. Here for simplicity, we give the perturbation series solutions up to order 1, that is,

$$u_0 = U_0(\xi) = k_0 \tanh\left(\frac{\sqrt{2}x_0^2\xi}{4t_0^2} + b_1\right), \tag{29a}$$

$$r_0 = R_0(\xi) = \frac{\sqrt{-\left(4k_0^2t_0^2 + x_0^2\right)}\tanh\left(\frac{\sqrt{2}x_0^2\xi}{4t_0^2} + b_1\right)}{2t_0}, \tag{29b}$$

$$s_0 = v_0 = S_0(\xi) = V_0(\xi) = -\frac{x_0^2\xi}{2t_0^2}, \tag{29c}$$

$$u_1 = U_1(\xi) = \frac{k_1}{2\cosh^2\left(\frac{\sqrt{2}x_0^2\xi}{4t_0^2} + b_1\right)}, \tag{30a}$$

$$r_1 = R_1(\xi) = \frac{k_1\sqrt{-\left(4k_0^2t_0^2 + x_0^2\right)}}{4k_0t_0\cosh^2\left(\frac{\sqrt{2}x_0^2\xi}{4t_0^2} + b_1\right)}, \tag{30b}$$

$$s_1 = v_1 = S_1(\xi) = V_1(\xi) = -\frac{x_0^2\xi^2}{2t_0^2} + \frac{4\sqrt{2}x_0^2\xi\,\mathrm{e}^{\frac{\sqrt{2}x_0^2\xi}{2t_0^2}+2b_1} - 16t_0^2}{x_0^2\left(\mathrm{e}^{\frac{\sqrt{2}x_0^2\xi}{2t_0^2}+2b_1} - 1\right)}, \tag{30c}$$

$\ldots,$

where k_0, k_1, b_1 are arbitrary constants.

In Figures 1 and 2, the evolution of intensity profiles $|p|^2$ and $|q|^2$ of the one-soliton solutions of (2) without and with perturbations, respectively, are plotted. The parameters in (29) and (30) are fixed at

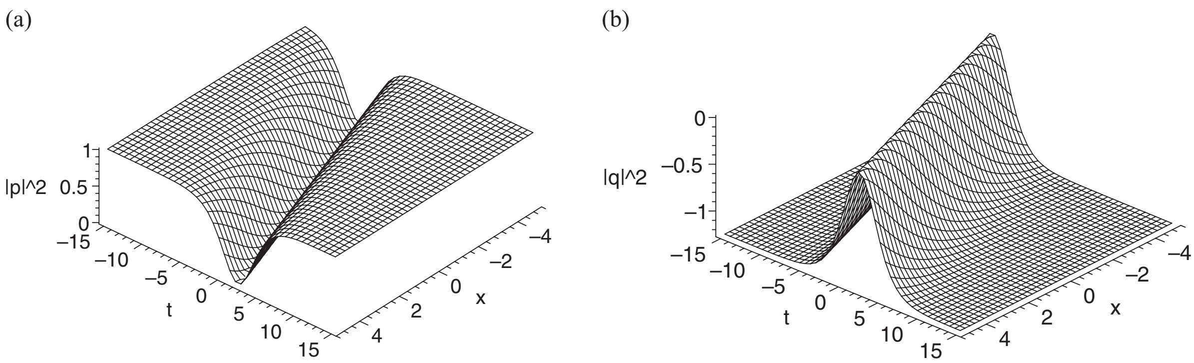

Fig. 1. Evolution of intensity profile $|p|^2$ (a) and $|q|^2$ (b) of the one-soliton solutions of (2) with no perturbation, where the functions u, v, r, s in (3) are respectively replaced by u_0, v_0, r_0, s_0 shown in (29). The parameters are set at $x_0 = t_0 = b_1 = k_0 = 1$.

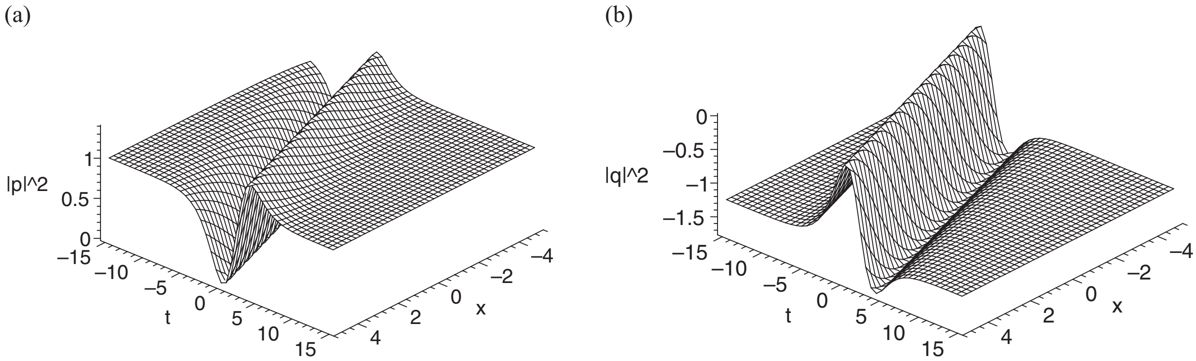

Fig. 2. Evolution of intensity profile $|p|^2$ (a) and $|q|^2$ (b) of PCNLSE (2) with $\varepsilon = 0.9$. The zeroth-order and the first-order solutions are chosen as (29) and (30), where parameter $k_1 = 2$ and others are the same as those in Figure 1.

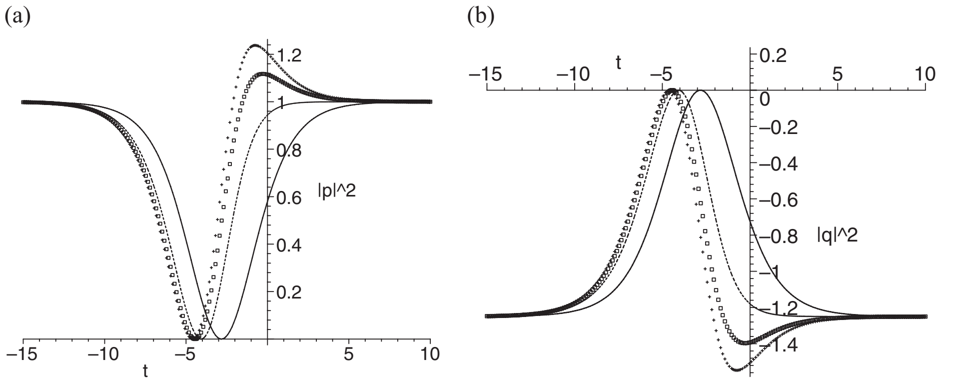

Fig. 3. Time evolution of amplitudes of the one-soliton solutions of PCNLSE (2) with different ε values for $x = 0$. $\varepsilon = 0, 0.5, 0.7, 0.8$ from right to left.

$x_0 = t_0 = k_0 = b_1 = 1$, $k_1 = 2$, $\varepsilon = 0.9$. Some general behaviours of solitons in this case can also be found in [22]. Under the action of perturbations $-\mathrm{i}\varepsilon p$ and $-\mathrm{i}\varepsilon q$, the solitons become deformed. For larger ε values, the distortion is more serious and the width of solitons is narrower and narrower. Another effect of the perturbations on solitons is that the perturbations have also changed the initial positions of solitons. It can be found obviously in Figure 3, which illustrates the time evolution of amplitudes of solutions of (2) with different ε values, where the parameter $\varepsilon = 0, 0.5, 0.7, 0.8$, respectively, from right to left.

4. Summary and Discussion

In summary, approximate similarity reductions of the PCNLSE were studied in the frame of the approximate symmetry reduction method, and the approximate symmetries and similarity symmetry solutions of different orders are given. From the results (23) and (28), it is not difficult to discover that the jth-order similarity reduction equations are linear variable coefficient ordinary differential equations of U_j, V_j, R_j, and S_j, which depend on particular solutions of the previous similarity reduction equations from zero order to $(j-1)$ order and U_j, V_j, R_j, and S_j can be solved step by step. Compared with other perturbation methods, the approximate symmetry perturbation method is much easier for treating the perturbed partial differential equations. Besides, with the help of Maple, we have also analyzed the effects of perturbations on the one-soliton solutions, where we found that the perturbations $-\mathrm{i}\varepsilon p$ and $-\mathrm{i}\varepsilon q$ have not only changed the shapes of the solitons, but also the initial positions of the solitons. Also, if choosing other solutions as initial approximate, we can analyze the effects of perturbations on them in the similar way. Moreover, extending the approximate symmetry perturbation method to more other perturbed nonlinear evolution equations is worthy of further study.

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