Boundary Layer Theory and Symmetry Analysis of a Williamson Fluid

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Boundary layer equations are derived for the first time for a Williamson fluid. Using Lie group theory, a symmetry analysis of the equations is performed. The partial differential system is transferred to an ordinary differential system via symmetries, and the resulting equations are numerically solved. Finally, the effects of the non-Newtonian parameters on the solutions are discussed.

Key words: Non-Newtonian Fluid; Williamson Fluid; Boundary Layer Theory; Lie Group Theory.

1. Introduction

A wide range of fluids exhibits a complex behaviour which can not be examined within the context of Newtonian fluid theory which predicts a linear relationship between the shear stress and the velocity gradient. Usually, the stress constitutive relations inherit complexities which lead to highly nonlinear equations of motion with many terms. To simplify the extremely complex equations with excess terms, one alternative is to use the boundary layer assumptions which are known to effectively reduce the complexity of the Navier-Stokes equations and reduce drastically the computational time. Since there are many non-Newtonian models and new models are being proposed continuously, boundary layer theories for each proposed model also appear in the literature. It is beyond the scope of this work to review vast literature on the boundary layers of non-Newtonian fluids. A limited work on the topic can be referred as examples [1 – 21].

In this work, a boundary layer theory is developed for the Williamson fluid, a non-Newtonian fluid model well known in the literature. For fluids with pseudoplastic behaviour, Williamson's model fits well with experimental data of polymer solutions and particle suspensions [22]. To the best of authors' knowledge, application of boundary layer assumptions to the Williamson fluid did not appear in the literature previously. A complete symmetry analysis of the boundary layer equations is presented for the first time for such fluids. Using a special symmetry (scaling symmetry), the partial differential system is transformed into an ordinary differential system. Since the resulting equations are highly nonlinear, they are solved numerically. The effect of non-Newtonian parameters on the velocity profiles is shown in graphs.

Some of the recent work on Williamson fluids is as follows: Lyubimov and Perminov [23] investigated the motion of a thin layer of the Williamson fluid over an inclined surface performing translational vibrations in its plane. Dapra and Scarpi [24] presented a perturbation solution for the unsteady flow of a Williamson fluid between parallel plates. The pressure gradient is assumed to have a mean and harmonic component. Nadeem and Akram [25] considered the peristaltic flow of a Williamson fluid in an asymmetric channel and found a regular perturbation solution valid for small Weissenberg number. Same authors [26] have also included the influence of an inclined magnetic field on the peristaltic flow of a Williamson fluid. As mentioned before, boundary layer equations and their analytical treatment using symmetry analysis is non-existent in the literature.

2. Boundary Layer Equations

The Cauchy stress for a Williamson fluid is

$$\tau = \left(\mu_\infty + \frac{\mu_0 - \mu_\infty}{1+\lambda|\dot{\gamma}|}\right)\dot{\gamma}, \quad (1)$$

where

$$\dot{\gamma} = \begin{bmatrix} 2\frac{\partial u^*}{\partial x^*} & \frac{\partial u^*}{\partial y^*} + \frac{\partial v^*}{\partial x^*} \\ \frac{\partial u^*}{\partial y^*} + \frac{\partial v^*}{\partial x^*} & 2\frac{\partial v^*}{\partial y^*} \end{bmatrix}, \quad (2)$$

$$|\dot{\gamma}| = \left[2\left(\frac{\partial u^*}{\partial x^*}\right)^2 + 2\left(\frac{\partial v^*}{\partial y^*}\right)^2 + \left(\frac{\partial u^*}{\partial y^*} + \frac{\partial v^*}{\partial x^*}\right)^2\right]^{1/2}. \quad (3)$$

μ_0 and μ_∞ are the limiting viscosities at zero and at infinite shear rate, respectively, and λ is a rheological parameter [24]. The steady-state two dimensional, incompressible equations of motion including mass conservation can be written as

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0, \quad (4)$$

$$\rho\left(u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*}\right) = -\frac{\partial p^*}{\partial x^*} + \frac{\partial \tau_{xx}}{\partial x^*} + \frac{\partial \tau_{xy}}{\partial y^*}, \quad (5)$$

$$\rho\left(u^*\frac{\partial v^*}{\partial x^*} + v^*\frac{\partial v^*}{\partial y^*}\right) = -\frac{\partial p^*}{\partial y^*} + \frac{\partial \tau_{xy}}{\partial x^*} + \frac{\partial \tau_{yy}}{\partial y^*}, \quad (6)$$

where x^* is the spatial coordinate along the surface, y^* is vertical to it, u* and v* are the velocity components in the x^* and y^* coordinates. The shear stress components are inserted into the equations of motion and the usual boundary layer assumptions are made, i.e. $x^* \sim \mathrm{O}(1)$, $y^* \sim \mathrm{O}(\delta)$, $u^* \sim \mathrm{O}(1)$, $v^* \sim \mathrm{O}(\delta)$. The highest order terms are retained and the variables are made non-dimensional. The x-momentum equation then reads

$$\begin{aligned} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = &-\frac{\partial p}{\partial x} + \frac{\varepsilon_2}{\delta^2}\frac{\partial^2 u}{\partial y^2} \\ &+ \frac{\varepsilon_1 - \varepsilon_2}{\delta^2}\left[1 + \frac{\varepsilon_3}{\delta}\left|\frac{\partial u}{\partial y}\right|\right]^{-1}\frac{\partial^2 u}{\partial y^2} \\ &- \frac{(\varepsilon_1-\varepsilon_2)\varepsilon_3}{\delta^3}\left[1 + \frac{\varepsilon_3}{\delta}\left|\frac{\partial u}{\partial y}\right|\right]^{-2}\frac{\partial^2 u}{\partial y^2}\frac{\partial u}{\partial y}, \end{aligned} \quad (7)$$

with the dimensionless variables and parameters defined as follows:

$$\begin{aligned} &x = \frac{x^*}{L}, \; y = \frac{y^*}{L}, \; u = \frac{u^*}{V}, \; v = \frac{v^*}{V}, \\ &p = \frac{p^*}{\rho V^2}\varepsilon_1 = \frac{\mu_0}{\rho V L}, \; \varepsilon_2 = \frac{\mu_\infty}{\rho V L}, \; \varepsilon_3 = \frac{\lambda V}{L}, \end{aligned} \quad (8)$$

where L is a characteristic length and V a reference velocity. Requiring ε_1, ε_2 to be $\mathrm{O}(\delta^2)$ and ε_3 to be $\mathrm{O}(\delta)$, the final boundary layer equations become

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (9)$$

$$\begin{aligned} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = &\, U\frac{\mathrm{d}U}{\mathrm{d}x} + k_2\frac{\partial^2 u}{\partial y^2} \\ &+ (k_1 - k_2)\left[1 + k_3\left|\frac{\partial u}{\partial y}\right|\right]^{-1}\frac{\partial^2 u}{\partial y^2} \\ &- (k_1 - k_2)k_3\left[1 + k_3\left|\frac{\partial u}{\partial y}\right|\right]^{-2}\frac{\partial^2 u}{\partial y^2}\frac{\partial u}{\partial y}, \end{aligned} \quad (10)$$

where

$$\varepsilon_1 = k_1\delta^2, \; \varepsilon_2 = k_2\delta^2, \; \varepsilon_3 = k_3\delta, \quad (11)$$

and $U(x)$ is the usual outer velocity.

The classical boundary conditions for the problem are

$$u(x,0) = 0, \; v(x,0) = 0, \; u(x,\infty) = U(x). \quad (12)$$

For $k_2 = k_3 = 0$, the equations reduce to those of a Newtonian fluid.

3. Lie Group Theory and Symmetry Reductions

Lie group theory is employed in search of symmetries of the equations. Details of the theory can be found in Bluman and Kumei [27] and Stephani [28]. The infinitesimal generator for the problem is

$$\begin{aligned} X = &\,\xi_1(x,y,u,v)\frac{\partial}{\partial x} + \xi_2(x,y,u,v)\frac{\partial}{\partial y} \\ &+ \eta_1(x,y,u,v)\frac{\partial}{\partial u} + \eta_2(x,y,u,v)\frac{\partial}{\partial v}. \end{aligned} \quad (13)$$

A straightforward and tedious calculation (see [27, 28] for details) yields

$$\begin{aligned} &\xi_1 = 3ax + b, \; \xi_2 = ay + c(x), \; \eta_1 = au, \\ &\eta_2 = c'(x)u - av. \end{aligned} \quad (14)$$

The classifying relation for the outer velocity is

$$(3ax+b)\frac{\mathrm{d}}{\mathrm{d}x}(UU')+a(UU')=0\,. \tag{15}$$

Symbolic packages developed to calculate symmetries fail to produce the above results due to the arbitrary outer velocity function and hand calculation becomes inevitable for precise results. There are two finite parameter Lie group symmetries represented by parameters 'a' and 'b', the former corresponding to scaling symmetry and the latter to translational symmetry in the x coordinate. There is an additional infinite parameter Lie group symmetry represented by the function $c(x)$. For a general stress tensor which is an arbitrary function of the velocity gradient, the symmetries were calculated previously [20]. Our results for the special case of a Williamson fluid confirm the previous calculations (see principal Lie Algebra presented in [20]).

Usually, the boundary conditions put much restriction on the symmetries which may lead to a removal of all the symmetries. In our case however, some of the symmetries remain stable after imposing the boundary conditions. For nonlinear equations, the generators should be applied to the boundaries and boundary conditions also [27]. Applying the generator to the boundary $y=0$ yields $c(x)=0$. Applying the generator to the boundary conditions do not impose further restrictions and, hence, the symmetries valid for the equations and boundary conditions reduce to

$$\xi_1=3ax+b\,,\ \ \xi_2=ay\,,\ \ \eta_1=au\,,\ \ \eta_2=-av\,, \tag{16}$$

with the classifying relation (15) remaining unchanged.

Selecting parameter 'a' in the symmetries, the associated equations which define similarity variables are

$$\frac{\mathrm{d}x}{3x}=\frac{\mathrm{d}y}{y}=\frac{\mathrm{d}u}{u}=\frac{\mathrm{d}v}{-v}\,. \tag{17}$$

Solving the system yields the similarity variables

$$\xi=\frac{y}{x^{1/3}}\,,\ \ u=x^{1/3}f(\xi)\,,\ \ v=\frac{g(\xi)}{x^{1/3}}\,. \tag{18}$$

From (15) with $b=0$, $U(x)=x^{1/3}$ corresponds to the external flow velocity over a wedge with an included angle of $\pi/2$ [29]. Substituting all into the boundary layer equations yields the ordinary differential system

$$f-\xi f'+3g'=0\,, \tag{19}$$

$$\begin{aligned} &f^2-\xi ff'+3gf'=1+3k_2f'' \\ &+3(k_1-k_2)(1+k_3|f'|)^{-1}f'' \\ &-3(k_1-k_2)k_3\,(1+k_3|f'|)^{-2}f''f'. \end{aligned} \tag{20}$$

The boundary conditions also transform:

$$f(0)=0\,,\ \ g(0)=0\,,\ \ f(\infty)=1\,. \tag{21}$$

The non-dimensional form of the shear stress is given as

$$\tau_{xy}=k_2\frac{\partial u}{\partial y}+\frac{(k_1-k_2)\dfrac{\partial u}{\partial y}}{\left(1+k_3\dfrac{\partial u}{\partial y}\right)}\,. \tag{22}$$

The shear stress acting on the surface can also be expressed in terms of similarity variables as follows:

$$\tau_{xy}(0)=k_2f'(0)+\frac{(k_1-k_2)f'(0)}{(1+k_3f'(0))}\,. \tag{23}$$

4. Numerical Results

Equations (19) and (20) are numerically integrated using a finite difference scheme subject to the boundary conditions (21). All numerical procedurse are carried out in Matlab environment with the bvp4c function for boundary value problems of differential equations. bvp4c is based on three stage Lobatto IIIa finite difference code with collacation [30]. Mesh selection and error control algorithm in bvp4c depend on the residual of the continuous solution. In Figure 1, function f and in Figure 2, function g, related to the x and y components of the velocities, are drawn for different parameters k_1 which are related to zero shear rate viscosities. The boundary layer becomes thicker for an increase in k_1. There is an increase in the y component of velocity (not in the absolute sense) for an increase in k_1 as can be seen from Figure 2. A similar trend is observed for the parameter k_2 which is related to the infinite shear rate viscosity as is evident from Figures 3 and 4. On the contrary, a reverse effect is observed for parameter k_3 which is related to the rheological parameter, i.e. as k_3 increases, the boundary layer becomes narrower (see Fig. 5). Higher values of k_3 suppress the non-Newtonian term by making the denominator very

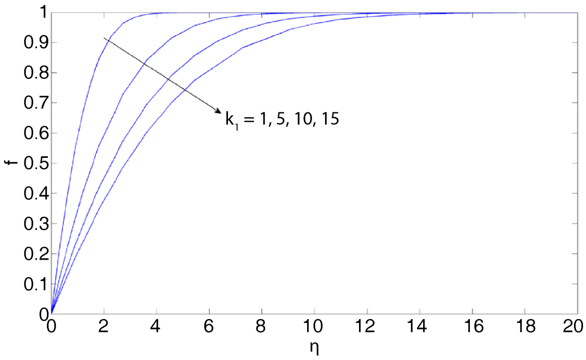

Fig. 1 (colour online). Effect of the parameter k_1 on the similarity function f related to the x-component of velocity ($k_2 = 1, k_3 = 1$).

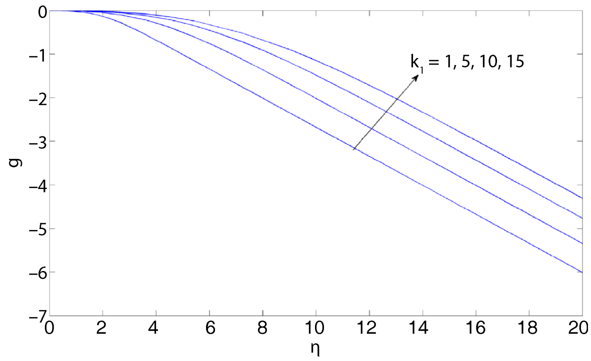

Fig. 2 (colour online). Effect of the parameter k_1 on the similarity function g related to the y-component of velocity ($k_2 = 1, k_3 = 1$).

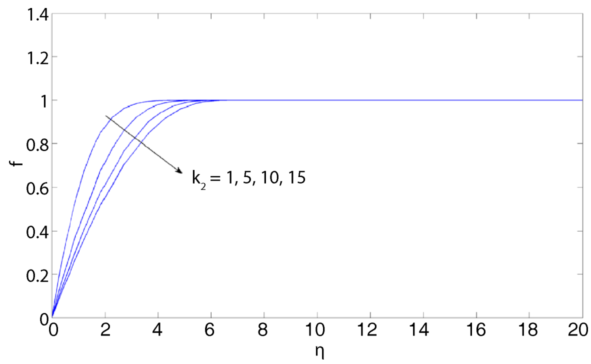

Fig. 3 (colour online). Effect of the parameter k_2 on the similarity function f related to the x-component of velocity ($k_1 = 1, k_3 = 1$).

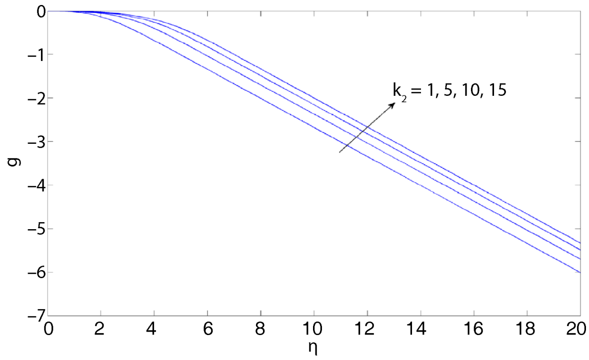

Fig. 4 (colour online). Effect of the parameter k_2 on the similarity function g related to the y-component of velocity ($k_1 = 1, k_3 = 1$).

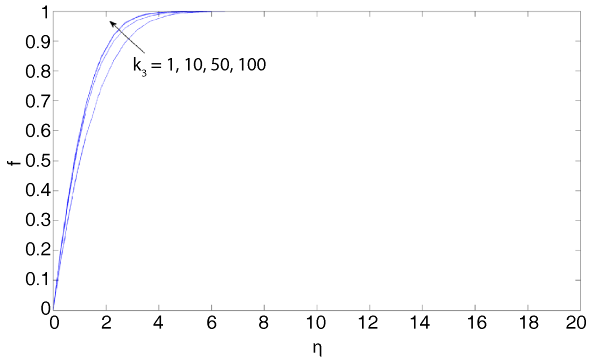

Fig. 5 (colour online). Effect of the parameter k_3 on the similarity function f related to the x-component of velocity ($k_1 = 2, k_2 = 1$).

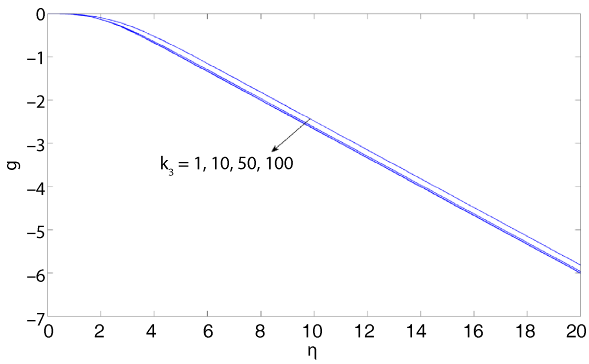

Fig. 6 (colour online). Effect of the parameter k_3 on the similarity function g related to the y-component of velocity ($k_1 = 2, k_2 = 1$).

large which in turn makes the solutions look qualitatively similar to those of Newtonian case. For high values of the parameter, the marginal increase does not affect much the solutions (see Figs. 5 and 6).

According to the fluid material constants, the shear stress at the surface is given numerically in Table 1. Some conclusions that can be drawn from Table 1 are as follows:

k_1	k_2	k_3	τ_{xy}
1	1	1	0.7574
		5	0.7574
		10	0.7574
	5	1	1.0753
		5	1.3668
		10	1.4780
	10	1	1.3019
		5	1.7742
		10	1.9683
5	1	1	1.5470
		5	1.1973
		10	1.0307
	5	1	1.6937
		5	1.6937
		10	1.6937
	10	1	1.8343
		5	2.0542
		10	2.1581
10	1	1	2.2230
		5	1.7025
		10	1.3690
	5	1	2.3052
		5	2.0865
		10	1.9628
	10	1	2.3953
		5	2.3953
		10	2.3953

Table 1. Numerical values of the shear stress at the surface.

- The shear stress increases as k_1 and k_2 increase.
- When the fluid property k_2 is greater than k_1, an increase in k_3 results in an increase in the shear stress.
- When the fluid property k_1 is greater than k_2, an increase in k_3 results in a decrease in the shear stress.
- When the fluid properties k_1 and k_2 are equal to each other, there is no variation in the shear stress values with k_3.

5. Concluding Remarks

Boundary layer equations for a Williamson fluid are derived for the first time. The Lie group theory is applied to the equations. The equations admit two finite parameter Lie group transformations and an infinite parameter Lie group transformation. The infinite parameter Lie group transformation is not stable with respect to usual boundary layer conditions. Using the scaling symmetry which is one of the finite parameter transformations, the partial differential system is transferred into an ordinary differential system which is highly nonlinear. The resulting equations are solved numerically using a finite difference scheme. Effects of zero shear rate viscosities, infinite shear rate viscosities, and rheological parameters on the boundary layers are discussed in detail.

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