Solution of the Coupled Burgers Equation Based on Operational Matrices of *d*-Dimensional Orthogonal Functions

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This paper aims to construct a general formulation for the d-dimensional orthogonal functions and their derivative and product matrices. These matrices together with the Tau method are utilized to reduce the solution of partial differential equations (PDEs) to the solution of a system of algebraic equations. The proposed method is applied to solve homogeneous and inhomogeneous two-dimensional parabolic equations. Also, the mentioned method is employed to find the solution of the coupled Burgers equation. Illustrative examples are included to demonstrate the validity and applicability of the presented technique.

Key words: Coupled Burgers Equation; Operational Matrix; Chebyshev and Legendre Polynomials; Tau Method.
Mathematics Subject Classification 2000: 34A08

1. Introduction

The Burgers equation has been found to describe various kinds of phenomena such as a mathematical model of turbulence [1] and the approximate theory of flow through a shock wave travelling in viscous fluid [2]. The coupled Burgers system was derived by Esipov [3]. It is a simple model of sedimentation or evolution of scaled volume concentrations of two kinds of particles in fluid suspensions or colloids under the effect of gravity [4]. Using the Hopf–Cole transformation, Fletcher [5] gave an analytical solution for the system of the two-dimensional Burgers equations. Several numerical methods for solving this equation have been given [6 – 13].

In this study, the system of coupled Burgers equations is investigated by applying the operational matrix. Accordingly, the orthogonal functions and their derivative and product matrices for d-dimensional time-depended partial differential equations are constructed. These matrices together with the Tau method are then utilized to reduce the solution of Burgers equations to the solution of a system of algebraic equations. The Tau approach is an approximation technique introduced by Lonczos [14] in 1938 to solve differential equations. The Tau method is based on expanding the required approximate solution as the elements of a complete set of orthogonal functions. This method may be viewed as a special case of the so-called Petrov–Galerkin method. But, unlike the Galerkin approximation, the expansion functions are not required to satisfy the boundary constraint individually [15 – 18].

2. Implementation of the Tau Method on the Heat Equation

In this section, we aim to convert the time-depended partial differential equations (PDEs) in the form of

$$\begin{aligned} &u_t(\mathbf{x},t) = \nabla u(\mathbf{x},t) + f(\mathbf{x},t) \ \text{ in } \Omega \times J\,, \\ &u(\mathbf{x},0) = g(\mathbf{x})\,,\ \ \mathbf{x} \in \Omega\,, \\ &Bu(\mathbf{x},t) = h(\mathbf{x},t) \ \text{ on } \partial\Omega \times J \end{aligned} \qquad (1)$$

to a system of algebraic equations by applying the operational matrix of orthogonal functions in $\Omega \times J$. Here, $\nabla = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + \cdots + \partial^2/\partial x_d^2$, $\Omega \subseteq \mathbb{R}^d$ is an open bounded domain with smooth boundary

$\partial\Omega$, $J=(0,T]$ with $T>0$, and $g(\mathbf{x})$ and $h(\mathbf{x},t)$ are known smooth functions.

In this way, we construct a family of orthogonal functions in $\Omega\times J$ by using orthogonal polynomials such as Legendre and Chebyshev. At first, we assume that $\Omega=\Omega_1\times\Omega_2\times\cdots\Omega_d$, $x_i\in\Omega_i\subseteq\mathbb{R}$ for $i=1,2,\cdots,d$, and the polynomials, denoted by $\phi_n^{[i]}(x_i)$, are orthogonal with weight functions $w_i(x_i)$ over Ω_i, and also $\psi_n(t)$ is orthogonal with weight functions $w(t)$ over J:

$$\int_{\Omega_i}\phi_n^{[i]}(x_i)\phi_m^{[i]}(x_i)w_i(x_i)\,\mathrm{d}x_i=b_n^{[i]}\delta_{nm},\ \ i=1,2,\cdots,d,$$
$$\int_0^T\psi_n(t)\psi_m(t)w(t)\,\mathrm{d}t=b_n\delta_{nm}, \tag{2}$$

where δ_{nm} is the Kronecker function, $b_n^{[i]}$ and b_n are known values for each polynomial $\phi_n^{[i]}(x_i)$ and $\psi_n(t)$, respectively, that are obtained as

$$\begin{aligned}b_n^{[i]}&=\int_{\Omega_i}\left(\phi_n^{[i]}(x_i)\right)^2w_i(x_i)\,\mathrm{d}x_i,\ \ i=1,2,\cdots,d,\\ b_n&=\int_0^T\left(\psi_n(t)\right)^2w(t)\,\mathrm{d}t.\end{aligned} \tag{3}$$

A function $f_i(x_i)$ defined over Ω_i can be expanded as

$$f_i(x_i)=\sum_{n=0}^{+\infty}a_n^{[i]}\ \phi_n^{[i]}(x_i),$$

and the coefficients $a_n^{[i]}$ are given by

$$a_n^{[i]}=\frac{1}{b_n^{[i]}}\int_{\Omega_i}f_i(x_i)\phi_n^{[i]}(x_i)w_i(x_i)\,\mathrm{d}x_i,\ \ n=0,1,2,\cdots.$$

By defining

$$\hat{\boldsymbol{\Phi}}^{[i]}(x_i)=\left[\phi_0^{[i]}(x_i),\phi_1^{[i]}(x_i),\cdots,\phi_n^{[i]}(x_i),\cdots\right]^{\mathrm{T}},$$
$$\hat{\boldsymbol{\Psi}}(t)=[\psi_0(t),\psi_1(t),\cdots,\psi_n(t),\cdots]^{\mathrm{T}},$$

the family of functions

$$\hat{\boldsymbol{\Phi}}(\mathbf{x},t)=\hat{\boldsymbol{\Phi}}^{[1]}(x_1)\otimes\hat{\boldsymbol{\Phi}}^{[2]}(x_2)\otimes\cdots\otimes\hat{\boldsymbol{\Phi}}^{[d]}(x_d)\otimes\hat{\boldsymbol{\Psi}}(t)$$

are orthogonal with weight functions $w(\mathbf{x},t)=w_1(x_1)w_2(x_2)\cdots w_d(x_d)w(t)$ over $\Omega\times J$; also $\otimes$ denotes the Kronecker product. It's clear that the orthogonality property appears as

$$\int_{\Omega\times J}\hat{\phi}_n(\mathbf{x},t)\hat{\phi}_m(\mathbf{x},t)w(\mathbf{x},t)\,\mathrm{d}\mathbf{x}\,dt=c_n\ \delta_{nm}. \tag{4}$$

A function $u(\mathbf{x},t)$ can be expanded as

$$u(\mathbf{x},t)=\sum_{n=0}^{+\infty}a_n\ \hat{\phi}_n(\mathbf{x},t),$$

and the coefficients a_n are given by

$$\begin{aligned}&a_n=\frac{1}{c_n}\int_{\Omega\times J}u(\mathbf{x},t)\hat{\phi}_n(\mathbf{x},t)w(\mathbf{x},t)\,\mathrm{d}\mathbf{x}\,\mathrm{d}t,\\ &n=0,1,2,\cdots.\end{aligned} \tag{5}$$

In practice, only the finite terms of $\hat{\boldsymbol{\Phi}}(\mathbf{x},t)$ are considered. Therefore, m_i-terms of $\hat{\boldsymbol{\Phi}}^{[i]}(x_i)$ and s-terms of $\hat{\psi}(t)$ are defined in the form

$$\boldsymbol{\Phi}^{[i]}(x_i)=\left[\phi_0^{[i]}(x_i),\phi_1^{[i]}(x_i),\cdots,\phi_{m_i-1}^{[i]}(x_i)\right]^{\mathrm{T}},$$
$$\boldsymbol{\Psi}(t)=[\psi_0(t),\psi_1(t),\cdots,\psi_{s-1}(t)]^{\mathrm{T}},$$

and consequently, N-terms of $\hat{\phi}_m(\mathbf{x})$ ($N=s\prod_{i=1}^{d}m_i$) can be obtained as

$$\boldsymbol{\Phi}(\mathbf{x},t)=\boldsymbol{\Phi}^{[1]}(x_1)\otimes\boldsymbol{\Phi}^{[2]}(x_2)\otimes\cdots\otimes\boldsymbol{\Phi}^{[d]}(x_d)\otimes\boldsymbol{\Psi}(t).$$

These functions approximate $u(\mathbf{x},t)$:

$$u_N(\mathbf{x},t)=\sum_{i=0}^{N-1}a_i\phi_i(\mathbf{x},t)=A^{\mathrm{T}}\boldsymbol{\Phi}(\mathbf{x},t)$$

with

$$\begin{aligned}&A=[a_0,a_1,\cdots,a_{N-1}]^{\mathrm{T}},\\ &\phi_i(\mathbf{x},t)=\phi_{i_1}^{[1]}(x_1)\phi_{i_2}^{[2]}(x_2)\cdots\phi_{i_d}^{[d]}(x_d)\psi_j(t),\\ &i=i_1m_2m_3\cdots m_ds+i_2m_3m_4\cdots m_ds+\cdots\\ &\quad+i_{d-1}m_ds+i_ds+j,\end{aligned} \tag{6}$$

$$\begin{aligned}i_k=&\left[\frac{i}{m_{k+1}m_{k+2}\cdots m_ds}\right]-i_1m_2m_3\cdots m_ks\\ &-i_2m_3\cdots m_ks-\cdots-i_{k-1}m_ks,\ \ k=1,2,\cdots,d,\end{aligned} \tag{7}$$

$$\begin{aligned}j=&\ i-i_1m_2m_3\cdots m_ds+i_2m_3m_4\cdots m_ds+\cdots\\ &+i_{d-1}m_ds+i_ds,\end{aligned} \tag{8}$$

where $[\cdot]$ denotes the integer part of the number.

2.1. Operational Matrix of Derivatives

At first, we describe a property of the Kronecker product as

$$\begin{aligned}&(A_1\otimes A_2\otimes\cdots\otimes A_n)(B_1\otimes B_2\otimes\cdots\otimes B_n)\\ &=A_1B_1\otimes A_2B_2\otimes\cdots\otimes A_nB_n.\end{aligned} \tag{9}$$

Lemma 1. *The derivative operator of the vector* $\boldsymbol{\Phi}(\boldsymbol{x},t)$ *can be expressed by*

$$\frac{\partial}{\partial x_i}\boldsymbol{\Phi}(\boldsymbol{x,t}) \simeq \boldsymbol{D}_i\boldsymbol{\Phi}(\boldsymbol{x,t}), \quad i=1,2,\cdots,d,$$

where

$$\boldsymbol{D}_i = I_{m_1}\otimes I_{m_2}\otimes\cdots\otimes I_{m_{i-1}}\otimes D_i\otimes I_{m_{i+1}}\otimes\cdots\otimes I_{m_d}\otimes I_s.$$

$\boldsymbol{D}_i$ *is the derivative matrix of vector* $\boldsymbol{\Phi}^{[i]}(x_i)$*, and* I_{m_j} *is the identity matrix of order* m_j*.*

Proof.

$$\frac{\partial}{\partial x_i}\boldsymbol{\Phi}(\mathbf{x},t) = \boldsymbol{\Phi}^{[1]}(x_1)\otimes\boldsymbol{\Phi}^{[2]}(x_2)\otimes\cdots \otimes D_{x_i}\boldsymbol{\Phi}^{[i]}(x_i)\otimes\cdots\otimes\boldsymbol{\Phi}^{[d]}(x_d)\otimes\boldsymbol{\Psi}(t),$$

where D_{x_i} is derivative operator of x_i. We assume $D_{x_i}\boldsymbol{\Phi}^{[i]}(x_i)\simeq D_i\boldsymbol{\Phi}^{[i]}(x_i)$ that $\mathbf{D}_i$ is the derivative matrix of vector $\boldsymbol{\Phi}^{[i]}(x_i)$.

Now by using (9), the Lemma can be proved. □

Also, the time-derivative matrix is obtained as

$$\frac{\partial}{\partial t}\boldsymbol{\Phi}(\mathbf{x,t}) \simeq \mathbf{D}_t\boldsymbol{\Phi}(\mathbf{x,t}),$$
$$\mathbf{D}_t = I_{m_1}\otimes I_{m_2}\otimes\cdots\otimes I_{m_d}\otimes D_t,$$

where $\mathbf{D}_t$ is the derivative matrix of vector $\boldsymbol{\Psi}(t)$.

Remark 1. The derivative matrix of vector $\boldsymbol{\Phi}^{[i]}(x_i)$ for shifted Chebyshev and Legendre polynomials is obtained in [19–21]. In general, we can obtain the derivative matrix of vector $\boldsymbol{\Phi}^{[i]}(x_i)$ by using the orthogonal property (2)

$$\mathbf{D}_i = < D_{x_i}\boldsymbol{\Phi}^{[i]}(x_i), \boldsymbol{\Phi}^{[i]}(x_i)^{\mathrm{T}} > \mathbf{d}_i^{-1},$$

where $< D_{x_i}\boldsymbol{\Phi}^{[i]}(x_i), \boldsymbol{\Phi}^{[i]}(x_i)^{\mathrm{T}} >$ and $\mathbf{d}_i$ are two $m_i\times m_i$ matrices defined as

$$< D_{x_i}\boldsymbol{\Phi}^{[i]}(x_i), \boldsymbol{\Phi}^{[i]}(x_i)^{\mathrm{T}} > = \left\{\int_{\Omega_i} D_{x_i}\phi_n^{[i]}(x_i)\phi_m^{[i]}(x_i)w_i(x_i)\,\mathrm{d}x_i\right\}_{n,m=0}^{m_i-1}, \quad (10)$$

$$\mathbf{d}_i = < \boldsymbol{\Phi}^{[i]}(x_i), \boldsymbol{\Phi}^{[i]}(x_i)^{\mathrm{T}} > = \mathrm{diag}\{b_n^{[i]}\}_{n=0}^{m_i-1}.$$

Denote that $\phi_n^{[i]}(x_i)$ are polynomials of degree n; one can write

$$\phi_n^{[i]}(x_i) = \sum_{j=0}^{n} h_{j,n}^{[i]}x_i^j,$$

where

$$h_{j,n}^{[i]} = \frac{1}{j!}\frac{\mathrm{d}^j}{\mathrm{d}x_i^j}\phi_n^{[i]}(x_i)\big|_{x_i=0}.$$

Subsequently, $D_{x_i}\phi_n^{[i]}(x_i)$ is obtained as

$$D_{x_i}\phi_n^{[i]}(x_i) = \sum_{j=1}^{n} jh_{j,n}^{[i]}x_i^{j-1}.$$

Thus, by using above equation, the appeared integral in (10) can be obtained as

$$\int_{\Omega_i} D_{x_i}^{\alpha}\phi_n^{[i]}(x_i)\phi_m^{[i]}(x_i)w_i(x_i)\,\mathrm{d}x_i = \sum_{j=1}^{n} jh_{j,n}^{[i]}\int_{\Omega_i} x_i^{j-1}\phi_m^{[i]}(x_i)w_i(x_i)\,\mathrm{d}x_i.$$

In orthogonal polynomials one can say that the elements of the derivative matrix $\mathbf{D}_i$ are achieved as

$$\{\mathbf{D}_i\}_{n,m=0}^{m_i-1} = \begin{cases} 0, & n-1<m, \\ \sum_{j=1}^{n}\sum_{k=0}^{m} j\frac{h_{j,n}^{[i]}h_{k,m}^{[i]}}{b_n^{[i]}}\int_{\Omega_i} x_i^{k+j-1}w_i(x_i)\,\mathrm{d}x_i, & n-1\geq m. \end{cases}$$

For shifted Legendre polynomials defined in $[0,L_i]$, we have

$$h_{j,n}^{[i]} = \frac{(-1)^{j+n}(j+n)!}{(n-j)!(j!)^2L_i^j}, \quad b_n^{[i]} = \frac{L_i}{2n+1}, \quad w_i(x_i)=1,$$

$$\{\mathbf{D}_i\}_{n,m=0}^{m_i-1} = \begin{cases} 0, & n-1<m \text{ or } n+m \text{ is even}, \\ \frac{4m+2}{L_i}, & \text{otherwise}. \end{cases}$$

Also, for shifted Chebyshev polynomials defined in $[0,L_i]$, one has

$$h_{j,n}^{[i]} = n\frac{(-1)^{n-j}(n+j-1)!2^{2j}}{(n-j)!(2j)!L_i^j},$$

$$b_0^{[i]} = \pi, \qquad b_n^{[i]} = \frac{\pi}{2}, \qquad n\geq 1,$$

$$w_i(x_i) = \frac{1}{\sqrt{L_ix-x^2}},$$

$$\{\mathbf{D}_i\}_{n,m=0}^{m_i-1} = \begin{cases} 0, & n-1<m \text{ or } n+m \text{ is even}, \\ \frac{2n}{L_i}, & m=0, \\ \frac{4n}{L_i}, & m\neq 0. \end{cases}$$

Remark 2. The ∇-operational matrix can be constructed as

$$\nabla = \frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\cdots+\frac{\partial^2}{\partial x_d^2},$$
$$\overline{\nabla} = \mathbf{D}_1^2+\mathbf{D}_2^2\cdots+\mathbf{D}_d^2.$$

2.2. The Product Operational Matrix

The following property of the product of two orthogonal vectors will also be applied:

$$\phi^{[i]}(x_i)\phi^{[i]}(x_i)^{\mathrm{T}}V \simeq \tilde{V}\phi^{[i]}(x_i),$$

where $\tilde{V}$ is an $m_i \times m_i$ product operational matrix for the vector V. Using above equation and by the orthogonal property mentioned in (2), the elements $\{\tilde{V}_{nm}\}_{n,m=0}^{m_i-1}$ can be calculated from

$$\tilde{V}_{nm} = \frac{1}{b_n^{[i]}}\sum_{k=0}^{m_i-1} v_k g_{nmk}^{[i]},$$

where $g_{nmk}^{[i]}$ is given by

$$g_{nmk}^{[i]} = \int_{\Omega_i} \phi_n^{[i]}(x_i)\phi_m^{[i]}(x_i)\phi_k^{[i]}(x_i)w_i(x_i)\,\mathrm{d}x_i.$$

Remark 3. $g_{nmk}^{[i]}$ for the shifted Chebyshev and Legendre polynomials can be obtained as

i) shifted Chebyshev polynomials defined in $[0,L_i]$:

$$g_{nmk}^{[i]} = \frac{L_i c_k}{2}\left(\delta_{n+m,k} + \delta_{|n-m|,k}\right),$$
$$c_0 = \pi, \qquad c_k = \frac{\pi}{2}, \qquad k = 1,2,\cdots,m_i-1.$$

ii) shifted Legendre polynomials defined in $[0,L_i]$:

$$g_{nmk}^{[i]} = \frac{L_i}{2}\begin{cases} \frac{d_{m-l}d_l d_{n-l}}{(2n+2m-2l+1)d_{n+m-l}}, & k = n+m-2l, \\ & l = 0,1,\cdots,m, \\ 0, & k \neq n+m-2l, \\ & l = 0,1,\cdots,m, \end{cases}$$

where $m \le n$ and $d_l = (2l)!/2^l(l!)^2$.

The product of two orthogonal vectors will also be applied as

$$\boldsymbol{\Phi}(\mathbf{x},t)\boldsymbol{\Phi}(\mathbf{x},t)^{\mathrm{T}}A \simeq \tilde{A}\boldsymbol{\Phi}(\mathbf{x},t),$$

where $\tilde{A}$ is an $N \times N$ product operational matrix for the vector A. Using above equation and by the orthogonal property (4), the elements $\{\tilde{A}_{ij}\}_{i,j=0}^{N-1}$ can be calculated from

$$\tilde{A}_{ij} = \frac{1}{c_i}\sum_{k=0}^{N-1} a_k g_{ijk},$$

where g_{ijk} and c_i are given by

$$g_{ijk} = \int_{\Omega\times J} \phi_i(\mathbf{x},t)\phi_j(\mathbf{x},t)\phi_k(\mathbf{x},t)w(\mathbf{x},t)\,\mathrm{d}\mathbf{x}\,\mathrm{d}t,$$
$$g_{ijk} = g_{i_1 j_1 k_1}^{[1]} g_{i_2 j_2 k_2}^{[2]} \cdots g_{i_d j_d k_d}^{[d]} g_{rpq},$$
$$c_i = b_{i_1}^{[1]} b_{i_2}^{[2]} \cdots b_{i_d}^{[d]} b_r,$$

where i_s, j_s, k_s, r, p, and q are obtained by applying (6)–(8).

3. Numerical Examples

In this section, two examples of homogeneous and inhomogeneous two-dimensional parabolic equations are given to illustrate our results. Also, in the last example, we apply our combined method to the coupled Burgers equations. In all experiments, we consider the shifted Legendre polynomials defined in $(0,\pi)$ as basis functions. It is noticeable that the results in shifted Chebyshev or other polynomials have no more difference. Comparisons between present results and corresponding analytical solutions are given. For these comparisons, the root mean square (RMS) error of the following form is applied:

$$\mathrm{RMS} = \sqrt{\frac{\sum_{k=1}^{M}\left(u(\mathbf{x}_k,t_k) - u_N(\mathbf{x}_k,t_k)\right)^2}{M}},$$

where $u(\mathbf{x}_k,t_k)$ and $u_N(\mathbf{x}_k,t_k)$ are achieved by the exact and the approximate solution on $(\mathbf{x}_k,t_k)$, and M is the number of test points.

3.1. The Inhomogeneous Two-Dimensional Heat Equation

Consider the following equation [12, 13]:

$$f(x_1,x_2,t) = \sin(x_1)\cdot\sin(x_2)\cdot \mathrm{e}^{-t} - 4,$$
$$g(x_1,x_2) = \sin(x_1)\cdot\sin(x_2) + x_1^2 + x_2^2,$$
$$0 < x_1,x_2 < \pi, \; t > 0,$$

with boundary conditions

$$u(0,x_2,t) = x_2^2, \qquad u(x_1,0,t) = x_1^2,$$
$$u(\pi,x_2,t) = x_2^2 + \pi^2, \qquad u(x_1,\pi,t) = x_1^2 + \pi^2.$$

The exact solution of this problem is

$$(x_1,x_2,t) = \sin(x_1)\cdot\sin(x_2)\cdot \mathrm{e}^{-t} + x_1^2 + x_2^2.$$

By applying the mentioned method, (1) is written as

$$A^{\mathrm{T}}\left[\mathbf{D}_t - \overline{\nabla}\right]\boldsymbol{\Phi}(x_1,x_2,t) = F^{\mathrm{T}}\boldsymbol{\Phi}(x_1,x_2,t),$$

where by using (5), we have $f(\mathbf{x},t) \simeq F^{\mathrm{T}}\boldsymbol{\Phi}(x_1,x_2,t)$. Now similar to the Tau method [18, 20], one has

$$\mathbf{E}A = F, \quad \text{where} \quad \mathbf{E} = \left[\mathbf{D}_t - \overline{\nabla}\right]^{\mathrm{T}}. \tag{11}$$

Also the initial and boundary conditions are given as

$$\begin{aligned}
&\left[I_{m_1} \otimes I_{m_2} \otimes \boldsymbol{\Psi}(0)\right]^{\mathrm{T}} A = G,\\
&g(x_1,x_2) \simeq G^{\mathrm{T}}\boldsymbol{\Phi}^{[1]}(x_1) \otimes \boldsymbol{\Phi}^{[2]}(x_2),\\
&\left[\boldsymbol{\Phi}^{[1]}(0) \otimes I_{m_2} \otimes I_s\right]^{\mathrm{T}} A = H_1,\\
&x_2^2 \simeq H_1^{\mathrm{T}}\boldsymbol{\Phi}^{[2]}(x_2) \otimes \boldsymbol{\Psi}(t),\\
&\left[I_{m_1} \otimes \boldsymbol{\Phi}^{[2]}(0) \otimes I_s\right]^{\mathrm{T}} A = H_2,\\
&x_1^2 \simeq H_2^{\mathrm{T}}\boldsymbol{\Phi}^{[1]}(x_1) \otimes \boldsymbol{\Psi}(t),\\
&\left[\boldsymbol{\Phi}^{[1]}(\pi) \otimes I_{m_2} \otimes I_s\right]^{\mathrm{T}} A = H_3,\\
&x_2^2 + \pi^2 \simeq H_3^{\mathrm{T}}\boldsymbol{\Phi}^{[2]}(x_2) \otimes \boldsymbol{\Psi}(t),\\
&\left[I_{m_1} \otimes \boldsymbol{\Phi}^{[2]}(\pi) \otimes I_s\right]^{\mathrm{T}} A = H_4,\\
&x_1^2 + \pi^2 \simeq H_4^{\mathrm{T}}\boldsymbol{\Phi}^{[1]}(x_1) \otimes \boldsymbol{\Psi}(t).
\end{aligned}$$

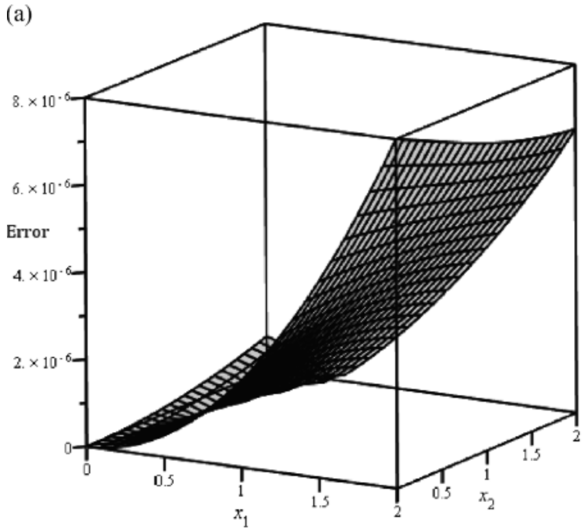
By substituting above equations in (11), one can obtain a system of algebraic equations and give a unique solution for the unknown coefficients $\{a_i\}_{i=0}^{N-1}$. We solved this problem for different N and compared our result with the exact solution in Table 1. The graphs of the absolute error functions for $m_1 = 7$, $m_2 = 7$, and $s = 4$ are shown in Figure 1. These errors illustrate that the approximate solution is in good agreement with the exact solution.

3.2. *The Homogeneous Two-Dimensional Heat Equation*

In this example, we consider the two-dimensional linear homogeneous Burgers equation given by [12, 13]

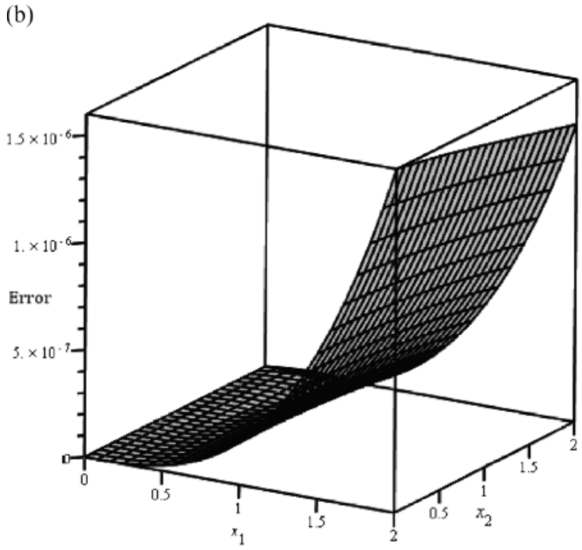
$$g(x_1,x_2) = \sin(x_1)\cdot\sin(x_2), \quad 0 < x_1,x_2 < \pi, \quad t > 0,$$

(a)



Graph of the absolute error with $t = 0$ for Example 1.

(b)



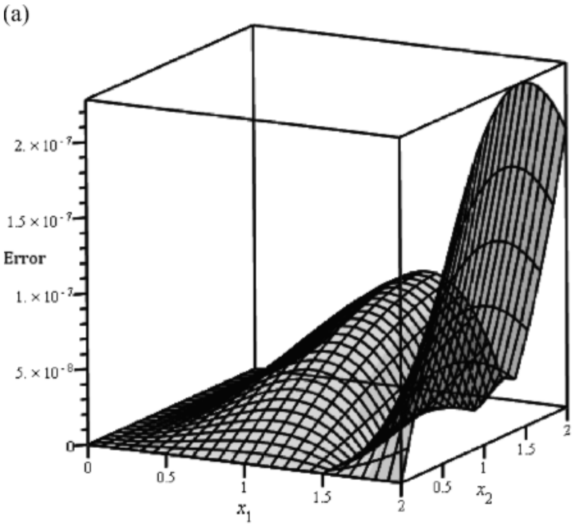
Graph of the absolute error with $t = 1$ for Example 1.

Fig. 1. Graph of the absolute error with $N = 196$ for Example 1.

Table 1. RMS error with some values of $N = m_1 m_2 s$ in Examples 1, 2, and 3.

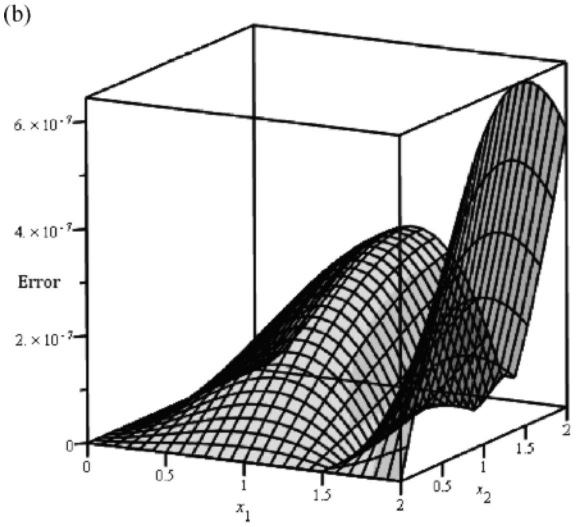
Example	$m_1 = m_2 = s = 4$	$m_1 = m_2 = s = 5$	$m_1 = m_2 = s = 6$	$m_1 = m_2 = s = 7$
1	$2.85 \cdot 10^{-3}$	$1.54 \cdot 10^{-4}$	$6.73 \cdot 10^{-5}$	$4.94 \cdot 10^{-7}$
2	$4.57 \cdot 10^{-3}$	$3.15 \cdot 10^{-4}$	$8.61 \cdot 10^{-5}$	$5.45 \cdot 10^{-7}$
3	$2.67 \cdot 10^{-3}$	$1.45 \cdot 10^{-4}$	$5.36 \cdot 10^{-5}$	$1.37 \cdot 10^{-7}$

(a)



Graph of the absolute error with $t = 0$ for Example 2.

(b)



Graph of the absolute error with $t = 1$ for Example 2.

Fig. 2. Graph of the absolute error with $N = 196$ for Example 2.

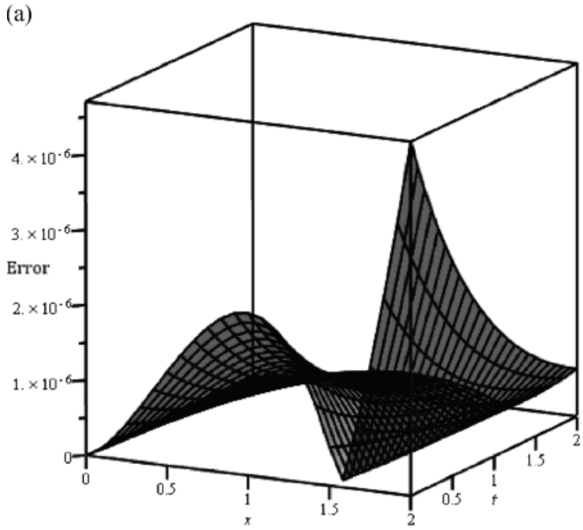
with boundary conditions

$$u(0,x_2,t) = 0\,,\;\; u(x_1,0,t) = 0\,,$$
$$u(\pi,x_2,t) = 0\,,\;\; u(x_1,\pi,t) = 0\,.$$

The exact solution of this problem is

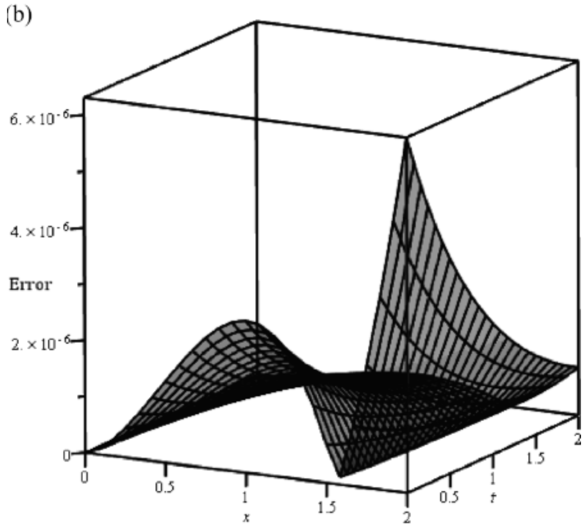
$$u(x_1,x_2,t) = \sin(x_1)\cdot\sin(x_2)\cdot \mathrm{e}^{-2t}.$$

(a)



Graph of the absolute error $u(x,t)$ for coupled Burgers equation.

(b)



Graph of the absolute error $v(x,t)$ for coupled Burgers equation.

Fig. 3. Graph of the absolute error for the coupled Burgers equation with $N = 64$.

By applying the mentioned method, (1) is written as

$$A^{\mathrm{T}}\left[\mathbf{D}_t - \overline{\nabla}\right]\Phi(x_1,x_2,t) = 0\,.$$

Now, similar to Tau method [18, 20], one has

$$\mathbf{E}A = 0, \quad \text{where} \quad \mathbf{E} = \left[\mathbf{D}_t - \overline{\nabla}\right]^{\mathrm{T}}. \tag{12}$$

Also the initial and boundary conditions are obtained as

$$\begin{aligned}
&\left[I_{m_1} \otimes I_{m_2} \otimes \Psi(0)\right]^{\mathrm{T}} A = G, \\
&g(x_1,x_2) \simeq G^{\mathrm{T}} \Phi^{[1]}(x_1) \otimes \Phi^{[2]}(x_2), \\
&\left[\Phi^{[1]}(0) \otimes I_{m_2} \otimes I_s\right]^{\mathrm{T}} A = 0, \\
&\left[I_{m_1} \otimes \Phi^{[2]}(0) \otimes I_s\right]^{\mathrm{T}} A = 0, \\
&\left[\Phi^{[1]}(\pi) \otimes I_{m_2} \otimes I_s\right]^{\mathrm{T}} A = 0, \\
&\left[I_{m_1} \otimes \Phi^{[2]}(\pi) \otimes I_s\right]^{\mathrm{T}} A = 0.
\end{aligned}$$

By substituting above equations in (12), one can obtain a system of algebraic equations and give a unique solution for the unknown coefficients $\{a_i\}_{i=0}^{N-1}$. We solved this problem for different N and compared it with the exact solution in Table 1. The graphs of the absolute error functions for $m_1 = 7, m_2 = 7$, and $s = 4$ are shown in Figure 2. These errors demonstrate that the approximate solution obtained by using this scheme is in good agreement with the exact solution.

3.3. The Coupled Burgers Equations

Finally, we consider the following nonlinear system of equations given by [12, 13]:

$$u_t - \nabla u - 2uu_x + (uv)_x = 0, \tag{13}$$
$$v_t - \nabla v - 2vv_x + (uv)_x = 0, \tag{14}$$

subject to the initial and boundary conditions

$$\begin{aligned}
&u(x,0) = \sin(x), \;\; v(x,0) = \sin(x), \\
&u(0,t) = 0, \;\; v(0,t) = 0, \\
&u(\pi,t) = 0, \;\; v(\pi,t) = 0.
\end{aligned}$$

The exact solutions of this system are

$$u(x,t) = \mathrm{e}^{-t}\sin(x), \;\; v(x,t) = \mathrm{e}^{-t}\sin(x).$$

By assuming $u(x,t) = A^{\mathrm{T}}\Phi(x,t)$ and $v(x,t) = B^{\mathrm{T}}\Phi(x,t)$ and applying the proposed method, (13) and (14) are obtained as

$$\begin{aligned}
&A^{\mathrm{T}}\left[\mathbf{D}_t - \overline{\nabla} - 2\mathbf{D}_x\tilde{A} + \mathbf{D}_x\tilde{B} + \tilde{V}\right]\Phi(x,t) = 0, \\
&B^{\mathrm{T}}\left[\mathbf{D}_t - \overline{\nabla} - 2\mathbf{D}_x\tilde{B} + \mathbf{D}_x\tilde{A} + \tilde{W}\right]\Phi(x,t) = 0,
\end{aligned}$$

where $V = \mathbf{D}_x^{\mathrm{T}}B$ and $W = \mathbf{D}_x^{\mathrm{T}}A$. Now, similar to the Tau method, we have

$$\left[\mathbf{D}_t - \overline{\nabla} - 2\mathbf{D}_x\tilde{A} + \mathbf{D}_x\tilde{B} + \tilde{V}\right]^{\mathrm{T}} A = 0, \tag{15}$$
$$\left[\mathbf{D}_t - \overline{\nabla} - 2\mathbf{D}_x\tilde{B} + \mathbf{D}_x\tilde{A} + \tilde{W}\right]^{\mathrm{T}} B = 0. \tag{16}$$

The conditions of the problem are obtained as

$$\begin{aligned}
&\left[I_{m_1} \otimes \Psi(0)\right]^{\mathrm{T}} A = S, \;\; \left[I_{m_1} \otimes \Psi(0)\right]^{\mathrm{T}} B = S, \\
&\left[\Phi^{[1]}(0) \otimes I_s\right]^{\mathrm{T}} A = 0, \;\; \left[\Phi^{[1]}(0) \otimes I_s\right]^{\mathrm{T}} B = 0, \\
&\left[\Phi^{[1]}(\pi) \otimes I_s\right]^{\mathrm{T}} A = 0, \;\; big[\Phi^{[1]}(\pi) \otimes I_s\right]^{\mathrm{T}} B = 0,
\end{aligned}$$

where $\sin(x) = S^{\mathrm{T}}\Phi^{[1]}(x)$. By substituting above equations in (15) and (16), we can obtain a system of nonlinear algebraic equations and have a solution for the unknown coefficients $\{a_i, b_i\}_{i=0}^{N-1}$. This problem is solved for different values of N and the accomplished comparison with the exact solution is shown in Table 1. The graphs of the absolute error functions for $m_1 = 8$ and $s = 8$ are shown in Figure 3. These errors reveal that the approximate solution is in good agreement with the exact solution.

4. Conclusions

In this paper, the Tau method for solving the coupled Burgers equation is applied. The orthogonal functions based on orthogonal polynomials for solving this time-depended equation by the means of the Kronecker product have been constructed. Also, a general formulation for the operational matrices of derivative and product has been derived. The achieved operational matrices along with the Tau method are used to reduce the problem to a system of algebraic equations. In order to demonstrate the efficiency and reliability of the proposed technique, the root mean square (RMS) error are applied.

[1] J. M. Burgers, Adv. Appl. Mech. **1**, 171 (1948).
[2] J. D. Cole, Quart. Appl. Math. **9**, 225 (1951).
[3] S. E. Esipov, Phys. Rev. E. **52**, 3711 (1995).
[4] J. Nee and J. Duan, Appl. Math. Lett. **11**, 57 (1998).
[5] J. D. Fletcher, Int. J. Numer. Meth. Fluids **3**, 213 (1983).
[6] P. C. Jain and D. N. Holla, Int. J. Numer. Meth. Eng. **12**, 213 (1978).

[7] A. R. Bahadir, Appl. Math. Comput. **137**, 131 (2003).
[8] C. J. Fletcher, J. Comput. Phys. **51**, 159 (1983).
[9] S. F. Radwan, J. Comput. Appl. Math. **174**, 383 (2004).
[10] M. A. Abdou and A. A. Soliman, J. Comput. Appl. Math. **181**, 245 (2005).
[11] D. Kaya, Int. J. Math. Math. Sci. **27**, 675 (2001).
[12] M. Dehghan, A. Hamidi, and M. Shakourifar, Appl. Math. Comput. **189**, 1034 (2007).
[13] A. Y. A. Kelleci, Numer. Meth. Part. D. E. **27**, 982 (2011).
[14] C. Lanczos, J. Math. Phys. **17**, 123 (1938).
[15] M. Dehghan and A. Saadatmandi, Comput. Math. Appl. **52**, 933 (2006).
[16] A. Saadatmandi and M. Dehghan, Commun. Numer. Meth. Eng. **24**, 1467 (2008).
[17] T. Tajvidi, M. Razzaghi, and M. Dehghan, Chaos Soliton. Fract. **35**, 59 (2008).
[18] M. Razzaghi, S. Oppenheimer, and F. Ahmad, J. Quant. Spectrosc. Ra. **72**, 439 (2002).
[19] A. Saadatmandi and M. Dehghan, Comput. Math. Appl. **59**, 1326 (2010).
[20] A. Saadatmandi and M. Dehghan, Comput. Math. Appl. **62**, 1135 (2011).
[21] E. H. Doha, A. H. Bhrawy, and S. S. Ezz-Eldien, Appl. Math. Modell. **35**, 5662 (2011).