On the Hybrid of Fourier Transform and Adomian Decomposition Method for the Solution of Nonlinear Cauchy Problems of the Reaction-Diffusion Equation

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The physical science importance of the Cauchy problem of the reaction-diffusion equation appears in the modelling of a wide variety of nonlinear systems in physics, chemistry, ecology, biology, and engineering. A hybrid of Fourier transform and Adomian decomposition method (FTADM) is developed for solving the nonlinear non-homogeneous partial differential equations of the Cauchy problem of reaction-diffusion. The results of the FTADM and the ADM are compared with the exact solution. The comparison reveals that for the same components of the recursive sequences, the errors associated with the FTADM are much lesser than those of the ADM. We show that as time increases the results of the FTADM approaches 1 with only six recursive terms. This is in agreement with the physical property of the density-dependent nonlinear diffusion of the Cauchy problem which is also in agreement with the exact solution.

The monotonic and very rapid convergence of the results of the FTADM towards the exact solution is shown to be much faster than that of the ADM.

Key words: Cauchy Reaction-Diffusion; Fourier Transformation; Adomian Decomposition Method; Non-Homogeneous Partial Differential Equation.

1. Introduction

Cauchy problems of the reaction-diffusion equation have a distinct importance in physical science in modelling nonlinear systems. Also spatial effects in ecology are modelled by the Cauchy problems of the reaction-diffusion equation. Different types of ecological phenomena such as the minimal patch size necessary to sustain a population, wave fronts propagation of biological invasions, and the formation of spatial patterns in the distributions of populations are supported and analyzed by nonlinear Cauchy problems of the reaction-diffusion model. The nonlinear Cauchy problems of reaction-diffusion equation are also used in the modelling of nonlinear chemical reactions in combustion phenomena. The nonlinear interactions between the convection and dispersion generating solitary waves, compactions, are studied with aid of the nonlinear Cauchy problems of the reaction-diffusion model. Moreover, the complexity of the nonlinear nature of these models devoted the researcher's attention to the approximate solutions obtained by semi-analytical methods [1 – 19]. Recently, Wazwaz and other researchers [10 – 20] pioneered a modification of the Adomian decomposition method (ADM). The basic idea of the modified Adomian decomposition method is to accelerate the convergence of the series solution arising from the method [20]. However, the solutions of these problems using the Adomian decomposition method and other semi-analytical methods are valid only in the one-directional problem domain, either in time or in space. In other words, the unsatisfied boundary conditions in the solutions of the ADM and other semi-analytical methods play no role in the final results [1 – 21]. The basic motivation of the present work is to develop a new modified ADM to overcome the deficiency caused due to the validity of the solution in a small range of problem domain because the boundary conditions are satisfied only in one dimension [1–21] when using the semi-analytical methods such as ADM. The new modified ADM is developed by combining the Fourier transform (FT) and ADM, where all conditions are satisfied over the entire range of time and space problem domains. In the present work, three different non-homogeneous linear and nonlinear partial differential equations, the Cauchy problems of the reaction-diffusion equation, are solved using the new modified ADM, the so called FTADM. The closed form solutions for the three partial differential equations which are the same as the exact solutions of the problems are obtained. Furthermore, the trends of very rapid convergence of the results toward the exact solutions have been demonstrated.

2. Basic Idea of FTADM

The general forms of one-dimensional nonlinear partial differential equations are considered for illustrating the basic idea of the FTADM. Consider the following differential equation:

$$E(u(x,t)) = 0, \ x \geq 0, \ t \geq 0. \tag{1}$$

Usually, the operator E can be decomposed into two parts, the linear operator L and the nonlinear operator N,

$$L(u(x,t)) + N(u(x,t)) = g(x). \tag{2}$$

Taking the Fourier transform from both sides of (2), we get

$$F\{L(u(x,t))\} + F\{N(u(x,t))\} = F(g(x)), \tag{3}$$

where the symbol F denotes the Fourier transform. Using the concept of Adomian decomposition method [18, 19], the unknown function $u(x,t)$ of the linear operator L in (9) can be decomposed by a series solution as [19–21]

$$\begin{aligned} u &= \sum_{n=0}^{\infty} u_n, \\ L(u(x,t)) &= L\left(\sum_{n=0}^{\infty} u_n\right). \end{aligned} \tag{4}$$

For the nonlinear operator N in (2), we use the Taylor series expansion to expand the nonlinear operator $N(u(x,t))$ around $u_0 = u(x_0,t_0)$ as

$$N(u(x,t)) = \sum_{n=0}^{\infty} \frac{1}{n!}(u-u_0)^n N^{(n)}(u_0), \tag{5}$$

where the superscript n indicates the order of derivative with respect to the dependent variable u. Substituting $u = \sum_{n=0}^{\infty} u_n$ into (5) and rearranging terms, we get

$$\begin{aligned} N(u(x,t)) &= N(u_0) + \left(u_1 N'(u_0)\right) + \bigg(u_2 N'(u_0) \\ &+ \frac{1}{2!}u_1^2 N''(u_0)\bigg) + \bigg(u_3 N'(u_0) + u_1 u_2 N''(u_0) \\ &+ \frac{1}{3!}u_1^3 N'''(u_0)\bigg) + \left(u_4 N'(u_0) + \left(\frac{1}{2!}u_2^2 + u_1 u_3\right)\right. \\ &\left. \cdot N''(u_0) + \frac{1}{2!}u_1^2 u_2 N'''(u_0) + \frac{1}{4!}u_1^4 N^{(iv)}(u_0)\right) + \cdots. \end{aligned} \tag{6}$$

Equation (6) can be rewritten as the series expansion of the Adomian polynomial A_n as follows:

$$N(u(x,t)) = \sum_{n=0}^{\infty} A_n, \tag{7}$$

where the Adomian polynomials A_n are defined as

$$A_n = \frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\left[N\left(\sum_{i=0}^{n} \lambda^i u_i\right)\right]_{\lambda=0}. \tag{8}$$

Substituting (7) and (4) into (3), we obtain:

$$F\left\{L\left(\sum_{i=0}^{\infty} u_i\right)\right\} + F\left\{\sum_{i=0}^{\infty} A_i\right\} = F(g(x)), \tag{9}$$

where the first five Adomian polynomials are

$$\begin{aligned} A_0 &= N(u_0), \\ A_1 &= u_1 N'(u_0), \\ A_2 &= u_2 N'(u_0) + \frac{1}{2!}u_1^2 N''(u_0), \\ A_3 &= u_3 N'(u_0) + u_1 u_2 N''(u_0) + \frac{1}{3!}u_1^3 N'''(u_0), \\ A_4 &= u_4 N'(u_0) + \left(\frac{1}{2!}u_2^2 + u_1 u_3\right) N''(u_0) \\ &\quad + \frac{1}{2!}u_1^2 u_2 N'''(u_0) + \frac{1}{4!}u_1^4 N^{(iv)}(u_0). \end{aligned} \tag{10}$$

Equation (9) can be rewritten in the following form:

$$\sum_{i=0}^{\infty} F\{L(u_i)\} + \sum_{i=0}^{\infty} F\{A_i\} = F\{g\}. \tag{11}$$

Using (11), we introduce the recursive relation as

$$\begin{aligned} &F\{L(u_0)\} = F\{g\}, \\ &\sum_{i=1}^{\infty} F\{L(u_i)\} + \sum_{i=0}^{\infty} F\{A_i\} = 0. \end{aligned} \tag{12}$$

The recursive equation (12) can be rewritten as

$$\begin{aligned} &F\{L(u_0)\} = F\{g\},\\ &F\{L(u_1)\} + F\{A_0\} = 0,\\ &F\{L(u_2)\} + F\{A_1\} = 0,\\ &F\{L(u_3)\} + F\{A_2\} = 0,\\ &F\{L(u_k)\} + F\{A_{k-1}\} = 0. \end{aligned} \tag{13}$$

Using the Maple package, the first part of (13) gives the value of $F\{u_0\}$. Then applying the inverse Fourier transform to $F\{u_0\}$ gives the value of u_0 that will define the Adomian polynomial A_0 using the first part of (10). In the second part of (13) using the Adomian polynomial A_0 will enable us to evaluate the value of $F\{u_1\}$. Then applying the inverse Fourier transform to $F\{u_1\}$ gives the value of u_1 that will define the Adomian polynomial A_1 using the second part of (10) and so on. This in turn will lead to the complete evaluation of the components of u_k, $k \geq 0$, upon using different corresponding parts of (13) and (10).

3. Case Study of the Cauchy Problem of Reaction-Diffusion

We solve three one-dimensional transient and non-homogeneous partial differential equations, the Cauchy problem of reaction-diffusion, to demonstrate the effectiveness and the validity of the presented method FTADM in the entire range of problem domain. The Cauchy problem of the reaction-diffusion equation expresses the mathematical model of the influence of the chemical reaction which the substances transforms into each other and the diffusion which the substances disperses over a surface in space. This equation has wide applications in chemical engineering, biology, geology, ecology, and physics. The Cauchy problem of the reaction-diffusion equation in the one-dimensional and time-dependent case is written as [13, 21]

$$u_t(x,t) = Du_{xx}(x,t) + r(x,t)u(x,t), \tag{14}$$

where $u(x,t)$ is the concentration of the substances, $r(x,t)$ the reaction parameter at position x and time t, and D is the diffusion coefficient. Equation (14) is solved subject to the following initial and boundary conditions:

$$u(x,0) = f(x), \tag{15}$$

$$u(0,t) = g_0(t), \quad u_x(0,t) = g_1(t). \tag{16}$$

Example 1. The Kolmogorov–Petrovskii–Piskunov (KPP) equation is obtained by taking $D = 1$, $r(x,t) = 2t$ in (14) as follows:

$$\begin{aligned} &u_t = u_{xx} + 2tu, \quad x \geq 0, \quad t \geq 0,\\ &u(x,0) = \mathrm{e}^{-x},\\ &u(0,t) = \mathrm{e}^{t+t^2}, \quad u_x(0,t) = \mathrm{e}^{t+t^2}. \end{aligned} \tag{17}$$

The Fourier transform of (17) is

$$\begin{aligned} &\hat{u}_t(\omega,t) + (\omega^2 - 2t)\hat{u}(\omega,t) + \mathrm{e}^{t+t^2}(\mathrm{i}\omega + 1) = 0,\\ &\hat{u}(\omega,0) = 1/(1+\mathrm{i}\omega). \end{aligned} \tag{18}$$

Substituting the recursive equation (12) into (18), we get

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\sum_{n=0}^{\infty}\hat{u}_n(\omega,t) + (\omega^2 - 2t)\sum_{n=0}^{\infty}\hat{u}_n(\omega,t)\\ &+ \mathrm{e}^{t+t^2}(\mathrm{i}\omega + 1) = 0,\\ &\hat{u}(\omega,0) = 1/(1+\mathrm{i}\omega). \end{aligned} \tag{19}$$

The recursive equation deduced from (19) can be written as

$$\begin{aligned} &\hat{u}_{0t}(\omega,t) + \mathrm{e}^{t+t^2}(\mathrm{i}\omega + 1) = 0, \quad \hat{u}_0(\omega,0) = 1/(1+\mathrm{i}\omega),\\ &\hat{u}_{1t}(\omega,t) + (\omega^2 - 2t)\hat{u}_0(\omega,t) = 0, \quad \hat{u}_1(\omega,0) = 0,\\ &\hat{u}_{2t}(\omega,t) + (\omega^2 - 2t)\hat{u}_1(\omega,t) = 0, \quad \hat{u}_2(\omega,0) = 0,\\ &\hat{u}_{3t}(\omega,t) + (\omega^2 - 2t)\hat{u}_2(\omega,t) = 0, \quad \hat{u}_3(\omega,0) = 0, \end{aligned} \tag{20}$$

and so on. Solving the recursive equation (20) and using the Maple package to take the inverse Fourier transform, we obtain the following:

$$\begin{aligned} &u_0(x,t) = \mathrm{e}^{-x},\\ &u_1(x,t) = \mathrm{e}^{-x}(t+t^2),\\ &u_2(x,t) = \mathrm{e}^{-x}(t^2/2 + t^3 + t^4/2),\\ &u_3(x,t) = \mathrm{e}^{-x}(t^3/6 + t^4/2 + t^5/2 + t^6/2), \end{aligned} \tag{21}$$

and so on. Consequently, the solution of (17) in a series form is given by

$$\begin{aligned} u(x,t) = \mathrm{e}^{-x}(&1 + (t+t^2) + (t+t^2)^2/2\\ &+ (t+t^2)^3/6 + \cdots). \end{aligned} \tag{22}$$

The Taylor series expansion for e^t is written as

$$\mathrm{e}^t = \sum_{n=0}^{\infty}\frac{t^n}{n!}. \tag{23}$$

By substituting (23) into (22), thus (22) can ultimately be reduced to

$$u(x,t) = \mathrm{e}^{-x}\,\mathrm{e}^{t+t^2}. \tag{24}$$

Equation (24) is the exact solution of (17).

Example 2. The Kolmogorov–Petrovskii–Piskunov (KPP) equation is obtained by taking $D = 1$, $r(x,t) = -(-1+4x^2)$ in (14):

$$\begin{aligned} &u_t = u_{xx} - (-1+4x^2)u, \ x \geq 0, \ t \geq 0, \\ &u(x,0) = \mathrm{e}^{-x^2}, \\ &u(0,t) = \mathrm{e}^{-t}, \ u_x(0,t) = 0. \end{aligned} \tag{25}$$

By applying the Fourier transform to (25), we obtain the followings:

$$\begin{aligned} &\hat{u}_t + \mathrm{i}\omega\,\mathrm{e}^{-t} + (\omega^2 - 1 - 4\partial^2/\partial\omega^2)\hat{u} = 0, \\ &\hat{u}(\omega,0) = \sqrt{\pi}/2\,\mathrm{e}^{-\frac{\omega^2}{4}}\,\mathrm{erf}\left(\frac{\mathrm{i}\omega}{2}\right), \end{aligned} \tag{26}$$

where $\mathrm{erf}(x)$ is the error function. Substituting the recursive equation (12) into (26), we get

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\sum_{n=0}^{\infty}\hat{u}_n(\omega,t) + (\omega^2 - 1 - 4\partial^2/\partial\omega^2) \\ &\cdot\sum_{n=0}^{\infty}\hat{u}_n(\omega,t) + \mathrm{i}\omega\,\mathrm{e}^{-t} = 0, \\ &\hat{u}(\omega,0) = \sqrt{\pi}/2\,\mathrm{e}^{-\frac{\omega^2}{4}}\,\mathrm{erf}\left(\frac{\mathrm{i}\omega}{2}\right). \end{aligned} \tag{27}$$

The recursive equation deduced from (27) can be written as

$$\begin{aligned} &\hat{u}_{0t}(\omega,t) + \mathrm{i}\omega\,\mathrm{e}^{-t} = 0, \\ &\hat{u}_0(\omega,0) = \sqrt{\pi}/2\,\mathrm{e}^{-\frac{\omega^2}{4}}\,\mathrm{erf}\left(\frac{\mathrm{i}\omega}{2}\right), \\ &\hat{u}_{1t}(\omega,t) + (\omega^2 - 1 - 4\partial^2/\partial\omega^2)\hat{u}_0(\omega,t) = 0, \\ &\hat{u}_1(\omega,0) = 0, \\ &\hat{u}_{2t}(\omega,t) + (\omega^2 - 1 - 4\partial^2/\partial\omega^2)\hat{u}_1(\omega,t) = 0, \\ &\hat{u}_2(\omega,0) = 0, \\ &\hat{u}_{3t}(\omega,t) + (\omega^2 - 1 - 4\partial^2/\partial\omega^2)\hat{u}_2(\omega,t) = 0, \\ &\hat{u}_3(\omega,0) = 0, \end{aligned} \tag{28}$$

and so on. Solving the recursive equation (28) and using the Maple package to take the inverse Fourier transform, we obtain

$$\begin{aligned} &u_0 = \mathrm{e}^{-x^2}, \\ &u_1 = \mathrm{e}^{-x^2}(-t), \\ &u_2 = \mathrm{e}^{-x^2}(t^2/2), \\ &u_3 = \mathrm{e}^{-x^2}(-t^3/6), \end{aligned} \tag{29}$$

and so on. Consequently, the solution of (25) in a series form is given by

$$u(x,t) = \mathrm{e}^{-x^2}(1 - t + (t^2/2) - (t^3/6) + \cdots). \tag{30}$$

The Taylor series expansion for e^{-t} is written as

$$\mathrm{e}^{-t} = \sum_{n=0}^{\infty}(-1)^n t^n/n!. \tag{31}$$

Substituting (31) into (30), the closed form solution of (25) is given by

$$u(x,t) = \mathrm{e}^{-(x^2+t)}. \tag{32}$$

Equation (32) is the exact solution of the problem.

Tables 1 and 2 show the comparison of the trend of convergence of the results for $S_2(x,t) = \sum_{i=0}^{2} u_i(x,t)$, $S_4(x,t) = \sum_{i=0}^{4} u_i(x,t)$, and $S_6(x,t) = \sum_{i=0}^{6} u_i(x,t)$ using the ADM and FTADM of (17) and (25) towards the exact solution, respectively. The monotonic and very rapid convergence of the solution using the FTADM towards the exact solution is clearly shown when compared to that of the ADM. Tables 1 and 2 also show that the relative errors of the ADM increase as the x-axis coordinates increase, so the ADM solution validity range is restricted to just a short region. On the other hand, results of the FTADM solution are valid for a large range of x-coordinates, and moreover the relative errors of the FTADM results are much lesser than those of the ADM solution.

Example 3. An interesting model for the insect population dispersal is the nonlinear Cauchy problem of the reaction-diffusion equation. Most often the diffusion coefficient term depends on the dependent variable. This is called the density-dependent diffusion and is very important in a wide range of physical sciences especially in the study of the insect population dispersal model. In this example, the density dependent nonlinear Cauchy problem of the reaction-diffusion equation is solved using the FTADM. Consider the following

Table 1. Comparison of the relative errors of the results of $S_2(x,t)=\sum_{i=0}^{2}u_i(x,t)$, $S_4(x,t)=\sum_{i=0}^{4}u_i(x,t)$, and $S_6(x,t)=\sum_{i=0}^{6}u_i(x,t)$ of the ADM and FTADM solution of (17) at each location along the x-axis and at different times.

			Percentage of relative error (%RE)						
			$x=2.5$	$x=3$	$x=4$	$x=6$	$x=7$	$x=9$	$x=11$
$t=0.1$	$S_2(x,t)$	ADM	141	368.5	2339	72535	$3.6\cdot10^{+5}$	$7.5\cdot10^{+6}$	$1.3\cdot10^{+8}$
		FTADM	$2.04\cdot10^{-4}$	$2.04\cdot10^{-4}$	$2.04\cdot10^{-4}$	$2.04\cdot10^{-4}$	$2.04\cdot10^{-4}$	$2.04\cdot10^{-4}$	$2.04\cdot10^{-4}$
	$S_4(x,t)$	ADM	147	401	2955	$1.5\cdot10^{+5}$	$9.9\cdot10^{+5}$	$3.8\cdot10^{+7}$	$1.2\cdot10^{+9}$
		FTADM	$1.2\cdot10^{-7}$	$1.2\cdot10^{-7}$	$1.2\cdot10^{-7}$	$1.2\cdot10^{-7}$	$1.2\cdot10^{-7}$	$1.2\cdot10^{-7}$	$1.2\cdot10^{-7}$
	$S_6(x,t)$	ADM	147	402	2979	$1.6\cdot10^{+5}$	$1.1\cdot10^{+6}$	$6.08\cdot10^{+7}$	$2.8\cdot10^{+9}$
		FTADM	$3.5\cdot10^{-11}$	$3.5\cdot10^{-11}$	$3.5\cdot10^{-11}$	$3.5\cdot10^{-11}$	$3.5\cdot10^{-11}$	$3.5\cdot10^{-11}$	$3.5\cdot10^{-11}$
$t=0.3$	$S_2(x,t)$	ADM	141	368.5	2339	72535	$3.6\cdot10^{+5}$	$7.5\cdot10^{+6}$	$1.3\cdot10^{+8}$
		FTADM	$7.4\cdot10^{-3}$	$7.4\cdot10^{-3}$	$7.4\cdot10^{-3}$	$7.4\cdot10^{-3}$	$7.4\cdot10^{-3}$	$7.4\cdot10^{-3}$	$7.4\cdot10^{-3}$
	$S_4(x,t)$	ADM	147	401	2955	$1.5\cdot10^{+5}$	$9.9\cdot10^{+5}$	$3.8\cdot10^{+7}$	$1.2\cdot10^{+9}$
		FTADM	$5.4\cdot10^{-5}$	$5.4\cdot10^{-5}$	$5.4\cdot10^{-5}$	$5.4\cdot10^{-5}$	$5.4\cdot10^{-5}$	$5.4\cdot10^{-5}$	$5.4\cdot10^{-5}$
	$S_6(x,t)$	ADM	147	402	2979	$1.6\cdot10^{+5}$	$1.1\cdot10^{+6}$	$6.08\cdot10^{+7}$	$2.8\cdot10^{+9}$
		FTADM	$1.9\cdot10^{-7}$	$1.9\cdot10^{-7}$	$1.9\cdot10^{-7}$	$1.9\cdot10^{-7}$	$1.9\cdot10^{-7}$	$1.9\cdot10^{-7}$	$1.9\cdot10^{-7}$
$t=0.5$	$S_2(x,t)$	ADM	141	368.5	2339	72535	$3.6\cdot10^{+5}$	$7.5\cdot10^{+6}$	$1.3\cdot10^{+8}$
		FTADM	$4\cdot10^{-2}$	$4\cdot10^{-2}$	$4\cdot10^{-2}$	$4\cdot10^{-2}$	$4\cdot10^{-2}$	$4\cdot10^{-2}$	$4\cdot10^{-2}$
	$S_4(x,t)$	ADM	147	401	2955	$1.5\cdot10^{=5}$	$9.9\cdot10^{+5}$	$3.8\cdot10^{+7}$	$1.2\cdot10^{+9}$
		FTADM	$1.1\cdot10^{-3}$	$1.1\cdot10^{-3}$	$1.1\cdot10^{-3}$	$1.1\cdot10^{-3}$	$1.1\cdot10^{-3}$	$1.1\cdot10^{-3}$	$1.1\cdot10^{-3}$
	$S_6(x,t)$	ADM	147	402	2979	$1.6\cdot10^{+5}$	$1.1\cdot10^{+6}$	$6.08\cdot10^{+7}$	$2.8\cdot10^{+9}$
		FTADM	$1.3\cdot10^{-5}$	$1.3\cdot10^{-5}$	$1.3\cdot10^{-5}$	$1.3\cdot10^{-5}$	$1.3\cdot10^{-5}$	$1.3\cdot10^{-5}$	$1.3\cdot10^{-5}$

Table 2. Comparison of the relative errors of the results of $S_2(x,t)=\sum_{i=0}^{2}u_i(x,t)$, $S_4(x,t)=\sum_{i=0}^{4}u_i(x,t)$, and $S_6(x,t)=\sum_{i=0}^{6}u_i(x,t)$ of the ADM and FTADM solution of (25) at each location along the x-axis and at different times.

			Percentage of relative error (%RE)						
			$x=2.5$	$x=3$	$x=4$	$x=6$	$x=7$	$x=9$	$x=11$
$t=0.1$	$S_2(x,t)$	ADM	6543	$6.1\cdot10^{+5}$	$9.2\cdot10^{+9}$	$1.4\cdot10^{+20}$	$2.2\cdot10^{+26}$	$1.4\cdot10^{+41}$	$1.7\cdot10^{+59}$
		FTADM	$1.7\cdot10^{-4}$	$1.7\cdot10^{-4}$	$1.7\cdot10^{-4}$	$1.7\cdot10^{-4}$	$1.7\cdot10^{-4}$	$1.7\cdot10^{-4}$	$1.7\cdot10^{-4}$
	$S_4(x,t)$	ADM	1631	$9.2\cdot10^{+5}$	$1.9\cdot10^{+11}$	$9.9\cdot10^{+22}$	$5.7\cdot10^{+29}$	$2.8\cdot10^{+45}$	$1.7\cdot10^{+64}$
		FTADM	$9.1\cdot10^{-8}$	$9.1\cdot10^{-8}$	$9.1\cdot10^{-8}$	$9.1\cdot10^{-8}$	$9.1\cdot10^{-8}$	$9.1\cdot10^{-8}$	$9.1\cdot10^{-8}$
	$S_6(x,t)$	ADM	55	$2.07\cdot10^{+5}$	$6.16\cdot10^{+11}$	$1\cdot10^{+25}$	$2.2\cdot10^{+32}$	$8.3\cdot10^{+48}$	$2.6\cdot10^{+68}$
		FTADM	$2.16\cdot10^{-11}$	$2.16\cdot10^{-11}$	$2.16\cdot10^{-11}$	$2.16\cdot10^{-11}$	$2.16\cdot10^{-11}$	$2.16\cdot10^{-11}$	$2.16\cdot10^{-11}$
$t=0.3$	$S_2(x,t)$	ADM	6543	$6.1\cdot10^{+5}$	$9.2\cdot10^{+9}$	$1.4\cdot10^{+20}$	$2.2\cdot10^{+26}$	$1.4\cdot10^{+41}$	$1.7\cdot10^{+59}$
		FTADM	$5.6\cdot10^{-3}$	$5.6\cdot10^{-3}$	$5.6\cdot10^{-3}$	$5.6\cdot10^{-3}$	$5.6\cdot10^{-3}$	$5.6\cdot10^{-3}$	$5.6\cdot10^{-3}$
	$S_4(x,t)$	ADM	1631	$9.3\cdot10^{+5}$	$1.9\cdot10^{+11}$	$9.9\cdot10^{+22}$	$5.7\cdot10^{+29}$	$2.8\cdot10^{+45}$	$1.7\cdot10^{+64}$
		FTADM	$2.6\cdot10^{-5}$	$2.6\cdot10^{-5}$	$2.6\cdot10^{-5}$	$2.6\cdot10^{-5}$	$2.6\cdot10^{-5}$	$2.6\cdot10^{-5}$	$2.6\cdot10^{-5}$
	$S_6(x,t)$	ADM	55	$2.07\cdot10^{+5}$	$6.16\cdot10^{+11}$	$1\cdot10^{+25}$	$2.2\cdot10^{+32}$	$8.3\cdot10^{+48}$	$2.6\cdot10^{+68}$
		FTADM	$5.6\cdot10^{-8}$	$5.6\cdot10^{-8}$	$5.6\cdot10^{-8}$	$5.6\cdot10^{-8}$	$5.6\cdot10^{-8}$	$5.6\cdot10^{-8}$	$5.6\cdot10^{-8}$
$t=0.5$	$S_2(x,t)$	ADM	6543	$6.1\cdot10^{+5}$	$9.2\cdot10^{+9}$	$1.4\cdot10^{+20}$	$2.2\cdot10^{+26}$	$1.4\cdot10^{+41}$	$1.7\cdot10^{+59}$
		FTADM	$3\cdot10^{-2}$	$3\cdot10^{-2}$	$3\cdot10^{-2}$	$3\cdot10^{-2}$	$3\cdot10^{-2}$	$3\cdot10^{-2}$	$3\cdot10^{-2}$
	$S_4(x,t)$	ADM	1631	$9.3\cdot10^{+5}$	$1.9\cdot10^{+11}$	$9.9\cdot10^{+22}$	$5.7\cdot10^{+29}$	$2.8\cdot10^{+45}$	$1.7\cdot10^{+64}$
		FTADM	$3.9\cdot10^{-4}$	$3.9\cdot10^{-4}$	$3.9\cdot10^{-4}$	$3.9\cdot10^{-4}$	$3.9\cdot10^{-4}$	$3.9\cdot10^{-4}$	$3.9\cdot10^{-4}$
	$S_6(x,t)$	ADM	55	$2.07\cdot10^{+5}$	$6.16\cdot10^{+11}$	$1\cdot10^{+25}$	$2.1\cdot10^{+32}$	$8.3\cdot10^{+48}$	$2.6\cdot10^{+68}$
		FTADM	$2.4\cdot10^{-6}$	$2.4\cdot10^{-6}$	$2.4\cdot10^{-6}$	$2.4\cdot10^{-6}$	$2.4\cdot10^{-6}$	$2.4\cdot10^{-6}$	$2.4\cdot10^{-6}$

Table 3. Comparison of the relative errors of the results of $S_2(x,t)=\sum_{i=0}^{2}u_i(x,t)$, $S_4(x,t)=\sum_{i=0}^{4}u_i(x,t)$, and $S_6(x,t)=\sum_{i=0}^{6}u_i(x,t)$ of the ADM and FTADM solution of (33) at each location along the x-axis and at different times.

			Percentage of relative error (%RE)						
			$x=2.5$	$x=3$	$x=4$	$x=6$	$x=7$	$x=9$	$x=11$
$t=0.1$	$S_2(x,t)$	ADM	$8\cdot10^{-4}$	$9\cdot10^{-4}$	$1\cdot10^{-3}$	$1.1\cdot10^{-3}$	$1.2\cdot10^{-3}$	$1.2\cdot10^{-3}$	$1.2\cdot10^{-3}$
		FTADM	$4.1\cdot10^{-6}$	$2.7\cdot10^{-6}$	$1.2\cdot10^{-6}$	$2.9\cdot10^{-7}$	$1.4\cdot10^{-7}$	$3.5\cdot10^{-8}$	$8.6\cdot10^{-9}$
	$S_4(x,t)$	ADM	$8.3\cdot10^{-6}$	$9.8\cdot10^{-6}$	$1.1\cdot10^{-5}$	$1.2\cdot10^{-5}$	$1.3\cdot10^{-5}$	$1.3\cdot10^{-5}$	$1.3\cdot10^{-5}$
		FTADM	$5.3\cdot10^{-10}$	$3.5\cdot10^{-10}$	$1.6\cdot10^{-10}$	$3.8\cdot10^{-11}$	$1.8\cdot10^{-11}$	$4.5\cdot10^{-12}$	$1.1\cdot10^{-12}$
	$S_6(x,t)$	ADM	$8\cdot10^{-8}$	$1\cdot10^{-7}$	$1.22\cdot10^{-7}$	$1.37\cdot10^{-7}$	$1.4\cdot10^{-7}$	$1.41\cdot10^{-7}$	$1.42\cdot10^{-7}$
		FTADM	$3\cdot10^{-14}$	$2\cdot10^{-14}$	$9.6\cdot10^{-15}$	$2.2\cdot10^{-15}$	$1.09\cdot10^{-15}$	$2.6\cdot10^{-16}$	$6.4\cdot10^{-17}$
$t=0.3$	$S_2(x,t)$	ADM	$3.2\cdot10^{-2}$	$3.6\cdot10^{-2}$	$3.9\cdot10^{-2}$	$4.2\cdot10^{-2}$	$4.2\cdot10^{-2}$	$4.2\cdot10^{-2}$	$4.3\cdot10^{-2}$
		FTADM	$1\cdot10^{-4}$	$7.2\cdot10^{-5}$	$3.3\cdot10^{-5}$	$7\cdot10^{-6}$	$3\cdot10^{-6}$	$9.3\cdot10^{-7}$	$2.27\cdot10^{-7}$
	$S_4(x,t)$	ADM	$3.3\cdot10^{-3}$	$3.9\cdot10^{-3}$	$4.6\cdot10^{-3}$	$5.1\cdot10^{-3}$	$5.2\cdot10^{-3}$	$5.2\cdot10^{-3}$	$5.2\cdot10^{-3}$
		FTADM	$1.2\cdot10^{-7}$	$8.3\cdot10^{-8}$	$3.8\cdot10^{-8}$	$8.9\cdot10^{-9}$	$4.4\cdot10^{-9}$	$1.1\cdot10^{-9}$	$2.6\cdot10^{-10}$
	$S_6(x,t)$	ADM	$3.5\cdot10^{-4}$	$4.5\cdot10^{-4}$	$5.5\cdot10^{-4}$	$6.2\cdot10^{-4}$	$6.3\cdot10^{-4}$	$6.3\cdot10^{-4}$	$6.4\cdot10^{-4}$
		FTADM	$6.65\cdot10^{-11}$	$4.44\cdot10^{-11}$	$2.07\cdot10^{-11}$	$4.84\cdot10^{-12}$	$2.37\cdot10^{-12}$	$5.74\cdot10^{-13}$	$1.39\cdot10^{-13}$
$t=0.5$	$S_2(x,t)$	ADM	$2.0\cdot10^{-1}$	$2.3\cdot10^{-1}$	$2.5\cdot10^{-1}$	$2.6\cdot10^{-1}$	$2.7\cdot10^{-1}$	$2.7\cdot10^{-1}$	$2.7\cdot10^{-1}$
		FTADM	$4.8\cdot10^{-4}$	$3.2\cdot10^{-4}$	$1.5\cdot10^{-4}$	$3.5\cdot10^{-5}$	$1.7\cdot10^{-5}$	$4\cdot10^{-6}$	$1\cdot10^{-6}$
	$S_4(x,t)$	ADM	$7.3\cdot10^{-2}$	$8.6\cdot10^{-2}$	$1.0\cdot10^{-1}$	$1.1\cdot10^{-1}$	$1.1\cdot10^{-1}$	$1.1\cdot10^{-1}$	$1.1\cdot10^{-1}$
		FTADM	$1.5\cdot10^{-6}$	$1.0\cdot10^{-6}$	$4.8\cdot10^{-7}$	$1.1\cdot10^{-7}$	$5.6\cdot10^{-8}$	$1.3\cdot10^{-8}$	$3.2\cdot10^{-9}$
	$S_6(x,t)$	ADM	$2.6\cdot10^{-2}$	$3.4\cdot10^{-2}$	$4.2\cdot10^{-2}$	$4.7\cdot10^{-2}$	$4.7\cdot10^{-2}$	$4.8\cdot10^{-2}$	$4.8\cdot10^{-2}$
		FTADM	$2.3\cdot10^{-9}$	$1.55\cdot10^{-9}$	$7.27\cdot10^{-10}$	$1.70\cdot10^{-10}$	$8.36\cdot10^{-11}$	$2.02\cdot10^{-11}$	$4.91\cdot10^{-12}$

nonlinear Cauchy problem of the differential equation as follows:

$$u_t=(uu_x)_x+u(1-u),\ x\ge0,\ t\ge0,\quad u(x,0)=1-\mathrm{e}^{-\frac{x}{\sqrt2}},\quad u(0,t)=1-\mathrm{e}^{-\frac{t}{2}},\ u_x(0,t)=1/\sqrt2\,\mathrm{e}^{-\frac{t}{2}}. \tag{33}$$

By applying the Fourier transform to (33), we obtain

$$\hat u_t-\hat u+(1+\omega^2/2)F\{u^2\}+(1/\sqrt2)\mathrm{e}^{-\frac{t}{2}}\left(1-\mathrm{e}^{-\frac{t}{2}}\right)+(\mathrm{i}\omega/2)(1-\mathrm{e}^{-\frac{t}{2}})^2=0,\quad \hat u(\omega,0)=\pi\delta(\omega)-\mathrm{i}\sqrt2/\omega(2\mathrm{i}\omega+\sqrt2), \tag{34}$$

where the superscript on the dependent variable u indicates the Fourier transform; $F\{u^2\}$ is the Fourier transform of u^2, and $\delta(\omega)$ is the Dirac delta function. By substituting (12) into (34), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{n=0}^{\infty}\hat u_n(\omega,t)=\sum_{n=0}^{\infty}\hat u_n(\omega,t)-(1+\omega^2)\sum_{n=0}^{\infty}\hat A_n(\omega,t)-\frac{1}{\sqrt2}\mathrm{e}^{-\frac{t}{2}}\left(1-\mathrm{e}^{-\frac{t}{2}}\right)-(\mathrm{i}\omega/2)\left(1-\mathrm{e}^{-\frac{t}{2}}\right)^2,\quad \hat u(\omega,0)=\pi\delta(\omega)-\mathrm{i}\sqrt2/\omega(2\mathrm{i}\omega+\sqrt2). \tag{35}$$

The recursive equation deduced from (35) can be written as

$$\begin{aligned}&\hat u_{0t}(\omega,t)+1/\sqrt2\mathrm{e}^{-\frac{t}{2}}\left(1-\mathrm{e}^{-\frac{t}{2}}\right)+(\mathrm{i}\omega2)\left(1-\mathrm{e}^{-\frac{t}{2}}\right)^2=0,\\&\hat u_0(\omega,0)=\pi\delta(\omega)-\mathrm{i}\sqrt2/\omega(2\mathrm{i}\omega+\sqrt2),\\&\hat u_{1t}(\omega,t)=\hat u_0(\omega,t)-(\omega^2+1)\hat A_0(\omega,t),\\&\hat u_1(\omega,0)=0,\\&\hat u_{2t}(\omega,t)=\hat u_1(\omega,t)-(\omega^2+1)\hat A_1(\omega,t),\\&\hat u_2(\omega,0)=0,\\&\hat u_{3t}(\omega,t)=\hat u_2(\omega,t)-(\omega^2+1)\hat A_2(\omega,t),\\&\hat u_3(\omega,0)=0,\end{aligned} \tag{36}$$

and so on. Solving the recursive equation (36) and using the Maple package to take the inverse Fourier transform, we obtain

$$\begin{aligned}&u_0(\omega,t)=1-\mathrm{e}^{-\frac{x}{\sqrt2}},\\&u_1(\omega,t)=(1/2)t\,\mathrm{e}^{-\frac{x}{\sqrt2}},\\&u_2(\omega,t)=-(1/8)t^2\,\mathrm{e}^{-\frac{x}{\sqrt2}},\\&u_3(\omega,t)=(1/48)t^3\,\mathrm{e}^{-\frac{x}{\sqrt2}},\end{aligned} \tag{37}$$

and so on. Consequently, the solution of (33) in a series form is given by

$$u(x,t) = 1 - \mathrm{e}^{-\frac{x}{\sqrt{2}}} + (1/2)t\,\mathrm{e}^{-\frac{x}{\sqrt{2}}} - (1/8)t^2\,\mathrm{e}^{-\frac{x}{\sqrt{2}}} + (1/48)t^3\,\mathrm{e}^{-\frac{x}{\sqrt{2}}}. \tag{38}$$

The Taylor series expansion for $\left(1 - \mathrm{e}^{-\frac{x+\frac{t}{\sqrt{2}}}{\sqrt{2}}}\right)$ is written as

$$\left(1 - \mathrm{e}^{-\frac{x+\frac{t}{\sqrt{2}}}{\sqrt{2}}}\right) = 1 - \mathrm{e}^{-\frac{x}{\sqrt{2}}} + (1/2)t\,\mathrm{e}^{-\frac{x}{\sqrt{2}}} - (1/8)t^2\,\mathrm{e}^{-\frac{x}{\sqrt{2}}} + (1/48)t^3\,\mathrm{e}^{-\frac{x}{\sqrt{2}}} + \cdots. \tag{39}$$

Substituting (39) into (38), the closed form solution of (33) is given by

$$u(x,t) = 1 - \mathrm{e}^{-\frac{x+\frac{t}{\sqrt{2}}}{\sqrt{2}}}. \tag{40}$$

Equation (40) is the exact solution of (33). Table 3 shows the comparison of the trend of convergence and the relative errors of the results of $S_2(x,t) = \sum_{i=0}^{2} u_i(x,t)$, $S_4(x,t) = \sum_{i=0}^{4} u_i(x,t)$, and $S_6(x,t) = \sum_{i=0}^{6} u_i(x,t)$ of the ADM and FTADM solutions of (33) towards the exact solution at each location along the x-axis and at different times. The trend of very rapid convergence of the solution using the FTADM towards the exact solution is clearly shown when compared to that of the ADM. Table 3 also shows that the relative errors associated with the ADM are increased as one moves along the x-axis, so the results of the ADM solution validity range is restricted to just a short region. On the other hand, for the nonlinear case, the relative errors associated with the FTADM are decreased rapidly as one moves along the x-axis, so the results of the FTADM solution are valid for a wide range of x-axis coordinates, and the relative errors of the FTADM are much lesser than those of the ADM solution. Moreover, for the bounded initial condition, the solution of the nonlinear Cauchy problem of the reaction-diffusion found to be approaching 1 as time approaches infinity, $t \to \infty$. This is an important physical property of the density-dependent nonlinear diffusion of the Cauchy problem. Figure 1 shows that, as time increases, the results of the FTADM approaches 1 with only six recursive terms. This is in agreement with the physical property of the density-dependent nonlinear diffusion of the Cauchy problem which is also in agreement with the exact solution. In Table 4, the root-mean square (RMS) errors of the results for $S_6(x,t) = \sum_{i=0}^{6} u_i(x,t)$ are calculated for ADM and FTADM. The RMS error for the first seven terms of the series solution for the ADM is much greater than that of the FTADM. This means the fast rate of convergence of the FTADM in comparison with the ADM. This in fact shows the effectiveness of the FTADM in handling the nonlinear differential equations in comparison with the ADM.

t	u
0.01	0.5093
0.05	0.5191
0.09	0.5286
0.1	0.5309
0.2	0.55385
0.3	0.5756
0.5	0.6159
0.8	0.6694
1.1	0.7155
1.8	0.7994
3	0.8885
4	0.9233
4.2	0.9257
4.4	0.9263

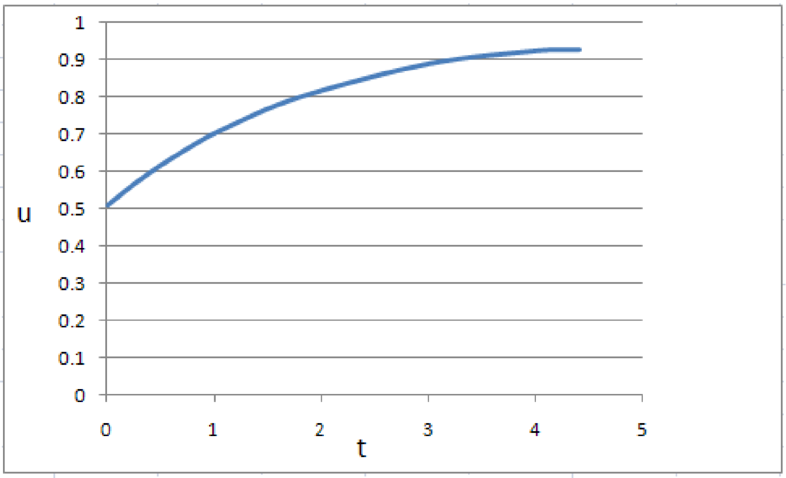

Fig. 1 (colour online). Variations of the results using the FTADM solution of (33) with six recursive terms.

Table 4. Comparison of the RMS errors of the results of $S_6(x,t) = \sum_{i=0}^{6} u_i(x,t)$ of the ADM and FTADM solution of (33) at different times and locations.

	$t = 0.1$	$t = 0.3$	$t = 0.5$
RMS error of the ADM method	$3.03 \cdot 10^{-8}$	$5.6 \cdot 10^{-4}$	$4.25 \cdot 10^{-2}$
RMS error of the FTADM method	$1.4 \cdot 10^{-14}$	$3.1 \cdot 10^{-11}$	$1.09 \cdot 10^{-9}$

4. Conclusions

In this paper, a new effective modification of the ADM, the Fourier transform Adomian decomposition method (FTADM), is proposed. The new modification of the ADM is the combination of the Fourier transform and the Adomian decomposition method. The comparison of the results for the linear and nonlinear Cauchy problems of reaction-diffusion using the FTADM with those of the ADM shows that the errors associated with the FTADM are much lesser than those of the ADM. Table 4 gives the values of the RMS errors of the results for the nonlinear problem for the FTADM and ADM. The values of the RMS errors show the fast rate of convergence of the FTADM in comparison with the ADM. Moreover, for the results of nonlinear Cauchy problem of reaction-diffusion as time approaches infinity, $t \rightarrow \infty$, the solution using the FTADM approaches 1 with only six recursive terms. The very rapid convergence of the results towards the exact solutions using the FTADM indicates that the amount of computational work is much lesser than the computational work required for the previous ADM.

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