New Computational Dynamics for Magnetohydrodynamics Flow over a Nonlinear Stretching Sheet

Majid Khan and Muhammad Asif Gondal

Department of Sciences and Humanities, National University of Computer & Emerging Sciences, Islamabad, Pakistan

Reprint requests to M. K.; E-mail: mk.cfd1@gmail.com

Z. Naturforsch. **67a,** 262 – 266 (2012) / DOI: 10.5560/ZNA.2012-0019
Received September 7, 2011 / revised January 30, 2011

The main idea of the present article is to introduce a new computational technique, explicitly, the modified Laplace Padé decomposition method (MLPDM) which is a recipe of Laplace transformation, decomposition technique, and rational polynomial to offer new solution mechanism of magnetohydrodynamics (MHD) flow of an steady viscous, incompressible nonlinear stretching sheet. A good harmony among the attained solution and the exact solution has been verified.

Key words: Modified Laplace Decomposition Method; Rational Polynomial; MHD Flow; Approximate Solution.

1. Introduction

The decomposition techniques have been revealed to solve powerfully, effortlessly, and precisely a enormous category of linear and nonlinear ordinary, partial, deterministic or stochastic differential equations [1 – 8]. The technique is extremely sound to physical problems because it does not involve needless linearization, perturbation, and assumption which may amend the problem being solved. Khan et al. [9 – 11], Gondal and Khan [12], and Khan und Hussain [13] used this technique to solve nonlinear foam drainage and boundary layer equations. Yusufoglu [14] apply this process for the solution of the Duffing equation. Kiymaz [15] suggest a procedure for finding solutions of initial value problems by means of Laplace Adomian decomposition. Hosseinzadeh et al. [16] and Jafari et al. [17] make use of this method to solve nonlinear Klein–Gordon and fractional differential equations. In recent times, Hussain and Khan [18] and Khan and Gondal [19] present a new consistent alteration in the Laplace decomposition method. This analytical technique basically demonstrates how the Laplace transform can be utilized to find the solution of nonlinear differential equations by operating the decomposition method and rational polynomials [20]. To the best of author's information, no effort has been made to take advantage of this method for solving magnetohydrodynamic flow equations on a semi infinite domain [21 – 23].

The paper is prepared as follows. In Section 2, fundamental concepts of MLDM are presented. Section 3 includes the important idea of rational polynomial approximation. Section 4 contains the governing equations. The concluding remark is given in the last section.

2. Formulation of Modified Laplace Decomposition Method

We start with the equation $K(w(x)) = h(x)$, where K denotes a general nonlinear operator containing both linear and nonlinear operators. The linear term is decomposed into $L+Q$, where L is the maximum-order linear operator and Q is a linear operator with order less than the order of highest-order linear operator. Therefore, the equation can be written as [14]

$$Lw + Qw + Nw = h(x), \qquad (1)$$

where Nw specifies the nonlinear term. Employing the Laplace transformation to (1), we find

$$\Lambda\left[Lw + Qw + Nw = h(x)\right]. \qquad (2)$$

By means of the derivative property of Laplace transform, (2) becomes

$$k^n \Lambda[w] - \sum_{i=1}^{n} k^{i-1} w^{(n-i)}(0) + \Lambda[Qw] + \Lambda[Nu] = \Lambda[h(x)]. \qquad (3)$$

Using the inverse Laplace transform on both sides of (3), we get

$$w = H(x) - \Lambda^{-1}\left[\frac{1}{s^n}[\Lambda[Nw] + \Lambda[Qw]]\right]. \quad (4)$$

The Laplace decomposition method supposes that the solution w can be extended into an infinite series as

$$w = \sum_{j=0}^{\infty} w_j . \quad (5)$$

Moreover, the nonlinear term Nw can be written as

$$Nw = \sum_{j=0}^{\infty} A_j(w_0, w_1, w_2, \ldots, w_j), \quad (6)$$

where A_m are Adomian polynomials [1]. By invoking (5) – (6) in (4), the solution can be written as

$$\sum_{j=0}^{\infty} w_j(x) = H(x) - \Lambda^{-1}\left[\frac{1}{s^n}\left[\Lambda\left[\sum_{j=0}^{\infty} A_j\right] + \Lambda\left[R\left(\sum_{j=0}^{\infty} w_j\right)\right]\right]\right]. \quad (7)$$

The recurrence relation is given by the following mathematical expression:

$$w_0(x) = H(x), \quad (8)$$

$$w_{j+1}(x) = -\Lambda^{-1}\left[\frac{1}{s^n}\left[\Lambda\left[\sum_{j=0}^{\infty} A_j\right] + \Lambda\left[R\left(\sum_{j=0}^{\infty} w_j\right)\right]\right]\right], \quad j \geq 0, \quad (9)$$

where $H(x)$ represents the term which is obtained from utilizing prescribed conditions and the nonhomogeneous part in (1). The modified Laplace decomposition scheme recommend that the function $H(x)$ in (8) break into two parts,

$$H(x) = H_0(x) + H_1(x), \quad (10)$$

where $H_0(x)$ is allocated to the zeroth-order solution and the remaining part $H_1(x)$ is allocated to the first-order solution. Using that assumption, we reformulate (8) – (9) for the modified Laplace decomposition method as [23]

$$\begin{aligned} w_0(x) &= H_0(x), \\ w_1(x) &= H_1(x) - \Lambda^{-1}\left[\frac{1}{s^n}[\Lambda[A_0] + \Lambda[R(w_0)]]\right], \\ w_{j+1}(x) &= -\Lambda^{-1}\left[\frac{1}{s^n}\left[\Lambda\left[\sum_{j=0}^{\infty} A_j\right] + \Lambda\left[R\left(\sum_{j=0}^{\infty} w_j\right)\right]\right]\right], \quad j \geq 1. \end{aligned} \quad (11)$$

3. Rational Approximation

A rational approximation to function $w(y)$ on $[a,b]$ is the quotient of two polynomials $S_k(y)$ and $T_n(y)$ of degrees k and n, respectively. We make use of the notation $P_{k,n}(y)$ to indicate this proportion of two polynomials [19, 20]:

$$P_{k,n}(y) = \frac{S_k(y)}{T_n(y)}. \quad (12)$$

The power series expansion of $w(y)$ in terms of y is given as

$$w(y) = \sum_{i=0}^{\infty} c_i y^i, \quad (13)$$

$$w(y) = \frac{S_k(y)}{T_n(y)} + O(y^{k+n+1}). \quad (14)$$

We imposed the normalization condition to polynomiasl in the demomenator which is given below:

$$T_0(y) = t_0 = 1. \quad (15)$$

Expanding the polynomials $S_k(y)$ and $T_n(y)$ in power series in terms of y of order k and n, gives

$$\begin{aligned} S_k(y) &= s_0 + s_1 y + s_2 y^2 + \ldots + s_k y^k, \\ T_n(y) &= 1 + t_1 y + t_2 y^2 + \ldots + t_n y^n. \end{aligned} \quad (16)$$

Utilizing (16) in (14), we have

$$\sum_{i=0}^{\infty} c_i y^i = \frac{s_0 + s_1 y + s_2 y^2 + \ldots + s_k y^k}{1 + t_1 y + t_2 y^2 + \ldots + t_n y^n} + O(y^{k+n+1}). \quad (17)$$

$$\begin{aligned} &(1 + t_1 y + t_2 y^2 + \ldots + t_n y^n)(c_0 + c_1 y + c_2 y^2 + \ldots) \\ &= s_0 + s_1 y + s_2 y^2 + \ldots + s_k y^k + O(y^{k+n+1}). \end{aligned} \quad (18)$$

From (18), we arrive a linear system of equations:

$$\begin{aligned} &c_0 = s_0\,, \\ &c_1 + c_0 t_1 = s_1\,, \\ &c_2 + c_1 t_1 + c_0 t_2 = s_2\,, \\ &\vdots \\ &c_k + c_{k-1} t_1 + c_0 t_k = s_k\,, \end{aligned} \tag{19}$$

and

$$\begin{aligned} &c_{k+1} + c_k t_1 + \ldots + c_{k-n+1} t_n = 0\,, \\ &c_{k+2} + c_{k+1} t_1 + \ldots + c_{k-n+2} t_n = 0\,, \\ &\vdots \\ &c_{k+n} + c_{k+n-1} t_1 + c_k t_n = 0\,. \end{aligned} \tag{20}$$

From (20), we find t_i, $1 \le i \le n$. The values of $t_1, t_2, \ldots, t_k$ be used in (19) give the unknown values of quantities $s_0, s_1, s_2, \ldots, s_k$, respectively.

4. Governing Equations

Let us consider a two-dimensional MHD, a steady viscous incompressible fluid over a nonlinear stretching sheet at $y = 0$, and an applied magnetic force $K(x)$ perpendicular to the stretching sheet in order to make the fluid electically conducting. Using boundary layer approximations, the resulting equations take the following form [21 – 23]:

$$\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} = 0\,, \tag{21}$$

$$u_1 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_1}{\partial y} = \nu^* \frac{\partial^2 u_1}{\partial y^2} - \sigma_1 \frac{K_0^2(x)}{\rho_1} u_1\,, \tag{22}$$

where u_1 and u_2 are, respective, velocity fields in x and y-direction, ν^* is the kinematic viscosity, σ_1 the conductivity, and ρ_1 the density of the fluid. We take the assumption that polarization effects are insignificant, consequently, our required magnetic field is of the form

$$K_0(x) = K_0 x^{\frac{n-1}{2}}\,. \tag{23}$$

The boundary conditions are

$$u_1(x,0) = b x^j\,,\; u_2(x,0) = 0\,,\; u_1(x,\infty) = 0\,, \tag{24}$$

where b is the stretching factor. Employing following similarity transformations [23] in order to reduced the partial differential equation (22) into an ordinary differential equation:

$$\begin{aligned} &\mu = \sqrt{\frac{b(j+1)}{2\nu}}\, x^{\frac{j-1}{2}} y\,,\; u_1 = b x^j h'(\mu)\,, \\ &u_2 = -\sqrt{\frac{b\nu(j+1)}{2}}\, x^{\frac{j-1}{2}} \left[h(\mu) + \frac{j-1}{j+1} \mu h'(\mu) \right]. \end{aligned} \tag{25}$$

Then the Navier–Stokes equation (22) and the boundary conditions in (24) take the form

$$\frac{\mathrm{d}^3 h}{\mathrm{d}\mu^3} + h \frac{\mathrm{d}^2 h}{\mathrm{d}\mu^2} - \gamma \left(\frac{\mathrm{d}h}{\mathrm{d}\mu} \right)^2 - \delta_1 \frac{\mathrm{d}h}{\mathrm{d}\mu} = 0\,, \tag{26}$$

$$\begin{aligned} &h(\mu) = 0\,,\; h'(\mu) = 1 \;\text{ as }\; \mu \to 0 \\ &\text{and }\; h'(\mu) = 0 \;\text{ as }\; \mu \to \infty\,, \end{aligned} \tag{27}$$

where

$$\gamma = \frac{2j}{j+1}\,,\quad \delta_1 = \frac{2\sigma_1 K_0^2}{\rho_1 b(1+j)}\,. \tag{28}$$

Utilizing the modified Laplace transform algorithm, we obtain

$$\begin{aligned} &s^3 \Lambda\,[h] - s^2 h(0) - s h'(0) - h''(0) \\ &= \Lambda \left[\gamma h'^2 - h h'' + \delta h' \right]. \end{aligned} \tag{29}$$

By means of given boundary conditions, (29) turns out to be in the following form:

$$s^3 \Lambda\,[h] - s - \tau = \Lambda \left[\gamma h'^2 - h h'' + \delta h' \right], \tag{30}$$

$$\Lambda\,[h] = \frac{s+\tau}{s^3} + \frac{1}{s^3} \Lambda \left[\gamma h'^2 - h h'' + \delta h' \right]. \tag{31}$$

Employing the inverse Laplace transform to (31), we find

$$\begin{aligned} h(\mu) = {} & \mu + \frac{\tau \mu^2}{2} \\ & + \Lambda^{-1} \left[\frac{1}{s^3} \Lambda \left[\gamma h'^2 - h h'' + \delta h' \right] \right]. \end{aligned} \tag{32}$$

The Laplace decomposition method (LDM) [17] supposes that the function $h(\mu)$ can be decomposed into an infinite sum of sequences given by the expression

$$h(\mu) = \sum_{m=0}^{\infty} h_m(\mu)\,. \tag{33}$$

Invoking (33) into (32), we acquire

$$\sum_{m=0}^{\infty} h_m(\mu) = \mu + \frac{\tau\mu^2}{2} + \Lambda^{-1} \cdot \left[\frac{1}{s^3}\Lambda\left[\gamma\sum_{m=0}^{\infty} A_m(\mu) - \sum_{m=0}^{\infty} B_m(\mu) + \delta h'\right]\right]. \tag{34}$$

From (34), $A_m(\mu)$ and $B_m(\mu)$ indicate Adomian polynomials which correspond to nonlinear terms. Thus the mathematical expression for Adomian polynomials is of the following form:

$$\sum_{m=0}^{\infty} A_m(\mu) = h'^2(\mu), \tag{35}$$

$$\sum_{m=0}^{\infty} B_m(\mu) = h(\mu)h''(\mu). \tag{36}$$

The first few Adomian polynomials are known as follows:

$$\begin{aligned} A_0(\mu) &= h_0'^2(\mu), \\ A_1(\mu) &= 2h_0'(\mu)h_1'(\mu), \\ A_2(\mu) &= h_1'^2(\mu) + 2h_0'(\mu)h_2'(\mu), \\ &\vdots \\ A_m(\mu) &= \sum_{i=0}^{m} h_i'(\mu)h_{m-i}'(\mu). \end{aligned} \tag{37}$$

$$\begin{aligned} B_0(\mu) &= h_0(\mu)h_0''(\mu), \\ B_1(\mu) &= h_0(\mu)h_1''(\mu) + h_1(\mu)h_0''(\mu), \\ B_2(\mu) &= h_0(\mu)h_2''(\mu) + h_1(\mu)h_1''(\mu) + h_2(\mu)h_0''(\mu), \\ &\vdots \\ B_m(\mu) &= \sum_{i=0}^{m} h_i(\mu)h_{m-i}''(\mu). \end{aligned} \tag{38}$$

From (35)–(38), our required modified recursive relation [16] is given below:

$$h_0(\mu) = \mu, \tag{39}$$

$$h_1(\mu) = \frac{\tau\mu^2}{2} + \Lambda^{-1} \cdot \left[\frac{1}{s^3}\Lambda\left[\gamma\sum_{m=0}^{\infty} A_m(\mu) - \sum_{m=0}^{\infty} B_m(\mu) + \delta h_m'\right]\right], \tag{40}$$

Table 1. Comparison of skin friction values $\tau = h''(0)$ with Pavlov's solution [22] and the solution obtained with proposed technique.

δ	γ	[13/13]	[16/16]	[20/20]	Pavlov [22]
1.0	1.0	1.41421	1.41421	1.41421	1.41421
2.0		1.73205	1.73205	1.73205	1.73205
4.0		2.23607	2.23607	2.23607	2.23607
20.0		4.58258	4.58258	4.58258	4.58258
25.0		5.09902	5.09902	5.09902	5.09902
30.0		5.56776	5.56776	5.56776	5.56776

$$h_{m+1}(\mu) = \Lambda^{-1} \cdot \left[\frac{1}{s^3}\Lambda\left[\gamma\sum_{m=0}^{\infty} A_m(\mu) - \sum_{m=0}^{\infty} B_m(\mu) + \delta h_m'\right]\right], \tag{41}$$

$m \geq 1$.

In view of that, the analytical approximate solution of (26) is given by

$$\begin{aligned} h(\mu) &= \mu + \frac{\tau\mu^2}{2} + \frac{\delta\mu^3}{6} - \frac{\mu^3\tau}{6} - \frac{\mu^4\tau}{24} \\ &+ \frac{\delta\mu^4\tau}{24} - \frac{\mu^5\tau^2}{120} + \frac{\mu^3\gamma}{6} + \frac{\mu^4\tau\gamma}{12} + \frac{\mu^5\tau^2\gamma}{60} \\ &- \frac{\delta\mu^5}{60} + \frac{\delta^2\mu^5}{120} + \frac{\mu^6\tau}{240} - \frac{\mu^5\gamma}{60} + \frac{\delta\mu^5\gamma}{40} \\ &- \frac{\mu^6\gamma\tau}{60} + \frac{\mu^6\gamma\tau}{72} + \frac{\mu^7\gamma^2\tau^2}{252} + \ldots. \end{aligned} \tag{42}$$

Table 1 and Figure 1 evidently elucidate that the current solution technique, namely MLPDM, shows outstanding conformity with the exact solution. This investigation confirms that MLPDM suits for magne-

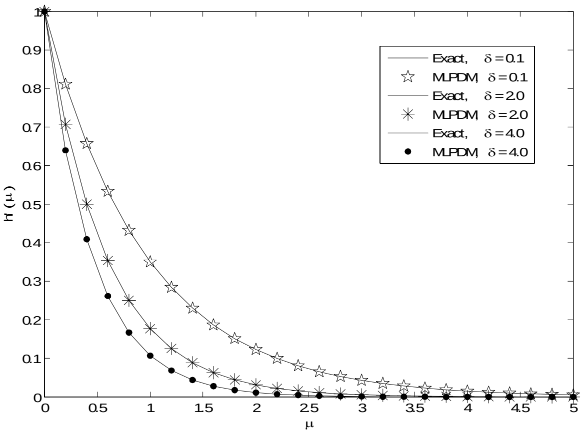

Fig. 1. Comparison of exact and analytical approximate solution obtained with the MLPDM for $\gamma = 1$.

tohydrodynamic boundary layer equations, and with the use of Padé approximants, we speed up the convergence of our suggest procedure.

5. Conclusion

Our principal objective here is to provide the approximate solution of the MHD boundary layer equation by using a modified Laplace Padé decomposition method. The new modified Laplace Padé decomposition method (MLPDM) is a powerful tool to search for solutions of various nonlinear problems. The method overcomes the difficulty in other methods because it is efficient. We derived fast convergent results by combining the series obtained by the modified Laplace decomposition method with the diagonal Padé approximants which shows an excellent agreement with the exact solution.

[1] G. Adomian, Frontier Problem of Physics: The Decomposition Method, Kluwer Academic Publishers, Boston 1994.
[2] H. Jafari and V. D. Gejji, Appl. Math. Comput. **175**, 1 (2006).
[3] J. S. Duan and R. Rach, Appl. Math. Comput. **218**, 2810 (2011).
[4] G. C. Wu, Math. Comput. Model. **54**, 2104 (2011).
[5] A. Hasseine, A. Bellagoun, and H.-J. Bart, Appl. Math. Comput. **218**, 2249 (2011).
[6] Q. Mao, Math. Comput. Model. **54**, 756 (2011).
[7] M. M. Hosseini, J. Comput. Appl. Math. **197**, 373 (2006).
[8] M. M. Hosseini, Appl. Math. Comput. **181**, 1737 (2006).
[9] M. Khan and M. A. Gondal, J. Adv. Res. Sci. Comput. **2**, 35 (2010).
[10] M. Khan and M. A. Gondal, J. Adv. Res. Diff. Eqs. **2**, 53 (2010).
[11] M. Khan and M. A. Gondal, World Appl. Sci. J. **12**, 2309 (2011).
[12] M. A. Gondal and M. Khan, Int. J. Nonlin. Sci. Numer. Simul. **11**, 1145 (2010).
[13] M. Khan and M. Hussain, Numer. Algor. **56**, 211 (2011).
[14] E. Yusufoglu Appl. Math. Comput. **177**, 572 (2006).
[15] O. Kiymaz, Appl. Math. Sci. **3**, 1453 (2009).
[16] H. Hosseinzadeh, H. Jafari, and M. Roohani, World Appl. Sci. J. **8**, 809 (2010).
[17] H. Jafari, C. M. Khalique, and M. Nazari, Appl. Math. Lett. **24**, 1799 (2011).
[18] M. Hussain and M. Khan, Appl. Math. Sci. **4**, 1769 (2010).
[19] M. Khan and M. A. Gondal, Adv. Res. Sci. Comput. **1**, 1 (2011).
[20] G. A. Baker, Essentials of Padé Approximants, Academic Press, London 1975.
[21] T. C. Chaim, Int. J. Eng. Sci. **33**, 429 (1995).
[22] K. B. Pavlov, Magnitnaya Gidrodinamika **4**, 146 (1975).
[23] M. M. Rashidi, Comput. Phys. Commun. **180**, 2210 (2009).