Application of Differential Transformation Method for Numerical Computation of Regularized Long Wave Equation

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In this article, the differential transformation method (DTM) is utilized for finding the solution of the regularized long wave (RLW) equation. Not only the exact solutions have been achieved by the known forms of the series solutions, but also for the finite terms of series, and the corresponding numerical approximations have been computed.

Key words: Regularized Long Wave Equation; Differential Transformation Method; Nonlinear Partial Differential Equations; Closed-Form Solution.
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1. Introduction

The regularized long wave (RLW) equation was originally introduced to describe the behaviour of the undular bore by Peregrine [1]. It is an alternative description of nonlinear dispersive waves to the more usual Korteweg–de Vries (KdV) equation. Various techniques have been developed to obtain the numerical solution of this equation. For example, approximate solutions based on finite difference [2], Runge–Kutta [3], Galerkin [4], and finite element methods [5] have been proposed. Authors of [6] used a cubic B-spline function to develop a collocation method to solve the RLW, and Ali [7] proposed a Chebyshev collocation spectral method. In this paper, we use the differential transformation method (DTM) for numerical study of the following equation:

$$u_t + u_x + \lambda u u_x - \mu u_{xxt} = \varphi(x,t), \qquad (1)$$

where λ and μ are positive parameters, and the subscripts x and t denote differentiation. If $\varphi(x,t) = 0$, then (1) is known as the RLW equation.

The concept of the DTM was first proposed by Zhou [8], who solved linear and nonlinear problems in electrical circuit problems. Chen and Ho [9] developed this method for partial differential equations, and Ayaz [10] applied it to the system of differential equations. During recent years, many authors used this method for solving various types of equations. For example, this method has been used for differential algebraic equations [11], partial differential equations [12], fractional differential equations [13], Volterra integral equations [14], and difference equations [15]. Shahmorad et al. developed the DTM to fractional-order integro-differential equations with nonlocal boundary conditions [16]. Borhanifar and Abazari applied this method for the Schrödinger equation [17]. Authors of [18, 19] used it for an approximate solution of the Hantavirus infection model and Emden–Fowler type differential equations.

In [20 – 22], this method has been utilized for the Kuramoto–Sivashinsky, telegraph, and Kawahara equations with supplementary conditions. There exist similar problems. For example, authors of [23, 24] presented several matrix formulation methods for solving some equation with a boundary integral condition. Authors of [25, 26] solved problems by using the homotopy method. Similar problems can be found in [27, 28].

2. The Two-Dimensional Differential Transform

The basic definitions and operations of the one-dimensional differential transform (DT) are introduced in [8–10]. In order to speed up the convergence rate and improve the accuracy of calculation, the entire domain of t needs to be split into sub-domains [13, 15].

Now, we introduce the basic definition of the two-dimensional differential transform. To this end, consider a function of two variables, $w(x,t)$, and suppose that it can be represented as a product of two single-variable functions, i.e., $w(x,t) = f(x)g(t)$. Based on the properties of the one-dimensional differential transform, the function $w(x,t)$ can be represented as

$$w(x,t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} W(i,j)x^i t^j\,, \tag{2}$$

where $W(i,j)$ is called the spectrum of $w(x,y)$. Further, we introduce the basic definitions and operations of the two-dimensional DT as follows [10].

Definition 1. If $w(x,t)$ is analytic and continuously differentiable with respect to time t in the domain of interest, then

$$W(h,k) = \frac{1}{k!h!}\left[\frac{\partial^{k+h}}{\partial x^k \partial t^h} w(x,t)\right]_{x=x_0,\ t=t_0}, \tag{3}$$

where the spectrum function $W(k,h)$ is the transformed function, which is also called the T-function. $w(x,y)$ is the original function and $W(k,h)$ the transformed one. Then we define the differential inverse transform of $W(k,h)$ as

$$w(x,t) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} W(k,h)(x-x_0)^k (t-t_0)^h. \tag{4}$$

Using (4) in (3), when $x_0 = 0$ and $t_0 = 0$, we have

$$\begin{aligned} w(x,t) &= \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} \frac{1}{k!h!}\left[\frac{\partial^{k+h}}{\partial x^k \partial t^h} w(x,t)\right]_{x_0=0,\ t_0=0} x^k t^h \\ &= \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} W(k,h)x^k t^h. \end{aligned} \tag{5}$$

From the above definitions and (4) and (5), we can obtain some of the fundamental mathematical operations performed by the two-dimensional differential transform in Table 1.

3. Application of the Differential Transformation Method

In this section, we apply the DTM to the presented equation.

Remark 1. The symbol $\otimes$ is used to denote the differential transform version of multiplication.

Consider the equation

$$u_t + u_x + \lambda u u_x - \mu u_{xxt} = \varphi(x,t) \tag{6}$$

with the initial conditions

$$u(x,0) = f(x),\ \ 0 \le x \le L \tag{7}$$

and the boundary conditions

$$u(0,t) = p(t),\ \ 0 < t \le T, \tag{8}$$
$$u_x(0,t) = q(t),\ \ 0 < t \le T. \tag{9}$$

Let $U(k,h)$ be the differential transform of $u(x,t)$. Applying Table 1, (2), and Definition 1 when $x_0 = t_0 = 0$,

Table 1. Operations of the two-dimensional differential transform.

Original function	Transformed function
$w(x,t) = u(x,t) \pm v(x,t)$	$W(k,h) = U(k,h) \pm V(k,h)$
$w(x,t) = cu(x,t)$	$W(k,h) = cU(k,h)$
$w(x,t) = \frac{\partial}{\partial x}u(x,t)$	$W(k,h) = (k+1)U(k+1,h)$
$w(x,t) = \frac{\partial^{r+s}}{\partial x^r \partial t^s}u(x,t)$	$W(k,h) = \frac{(k+r)!(h+s)!}{k!h!}U(k+r,h+s)$
$w(x,t) = u(x,t)v(x,t)$	$W(k,h) = \sum_{r=0}^{k}\sum_{s=0}^{h} U(r,h-s)V(k-r,s)$
$w(x,t) = \frac{\partial}{\partial x}u(x,t)\frac{\partial}{\partial t}v(x,t)$	$W(h,k) = \sum_{r=0}^{k}\sum_{s=0}^{h}(k-r+1)(h-s+1) \times U(k-r+1,s)V(r,h-s+1)$

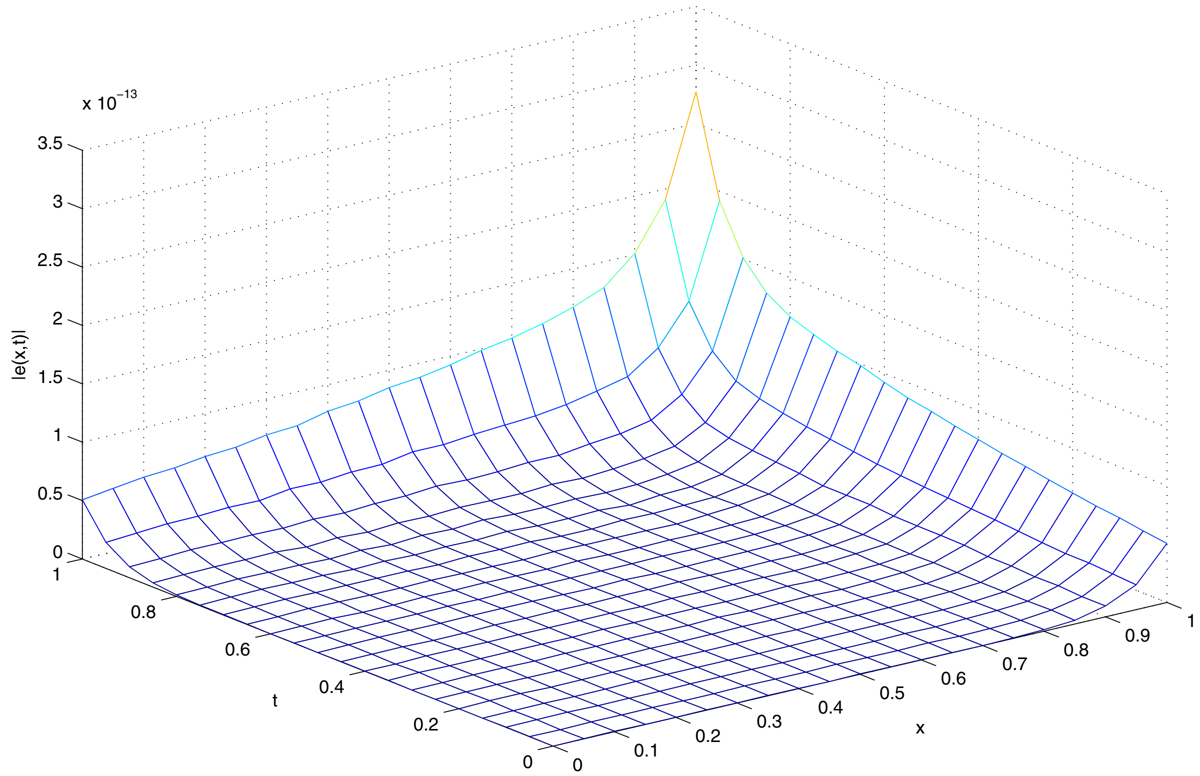

Fig. 1 (colour online). Plot of error function from Example 1 for $n = 15$.

we get the differential version of (6) as follows:

$$transform\\
(h+1)U(k,h+1)+(k+1)U(k+1,h)\\
+\lambda u\otimes u_x\Big|_{x=k,\ t=h}-\mu\frac{(k+2)!}{k!}\frac{(h+1)!}{h!}U(k+2,h+1)\\
=\varphi(k,h). \tag{10}$$

With the initial condition, we get

$$\sum_{k=0}^{\infty}U(k,0)x^k=\sum_{k=0}^{\infty}\frac{f^k(0)}{k!}x^k, \tag{11}$$

which implies

$$U(k,0)=\frac{f^k(0)}{k!},\quad k=0,1,2,\ldots. \tag{12}$$

From the first boundary condition, (8), we have

$$\sum_{h=0}^{\infty}U(0,h)y^h=\sum_{h=0}^{\infty}\frac{p^h(0)}{h!}y^h \tag{13}$$

and from the second boundary condition, (9), we get

$$\sum_{h=0}^{\infty}U(1,h)y^h=\sum_{h=0}^{\infty}\frac{q^h(0)}{h!}y^h. \tag{14}$$

Then from (13) and (14), we have

$$U(0,h)=\frac{p^h(0)}{h!},\quad h=1,2,\ldots, \tag{15}$$

$$U(1,h)=\frac{q^h(0)}{h!},\quad h=1,2,\ldots. \tag{16}$$

Therefore, for $k=0,1,2,\ldots$, the values of $U(k,0)$ and for $h=1,2,\ldots$, the values of $U(0,h)$ and $U(1,h)$ can be obtained from (12), (15), and (16). Using (16), we find the remainder values of U as

$$U(k+2,h+1)=\frac{1}{\mu(h+1)(k+2)(k+1)}\\
\cdot\Big((h+1)U(k,h+1)+\lambda\sum_{r=0}^{k}\sum_{s=0}^{h}(k-r+1)U(r,h-s)$$

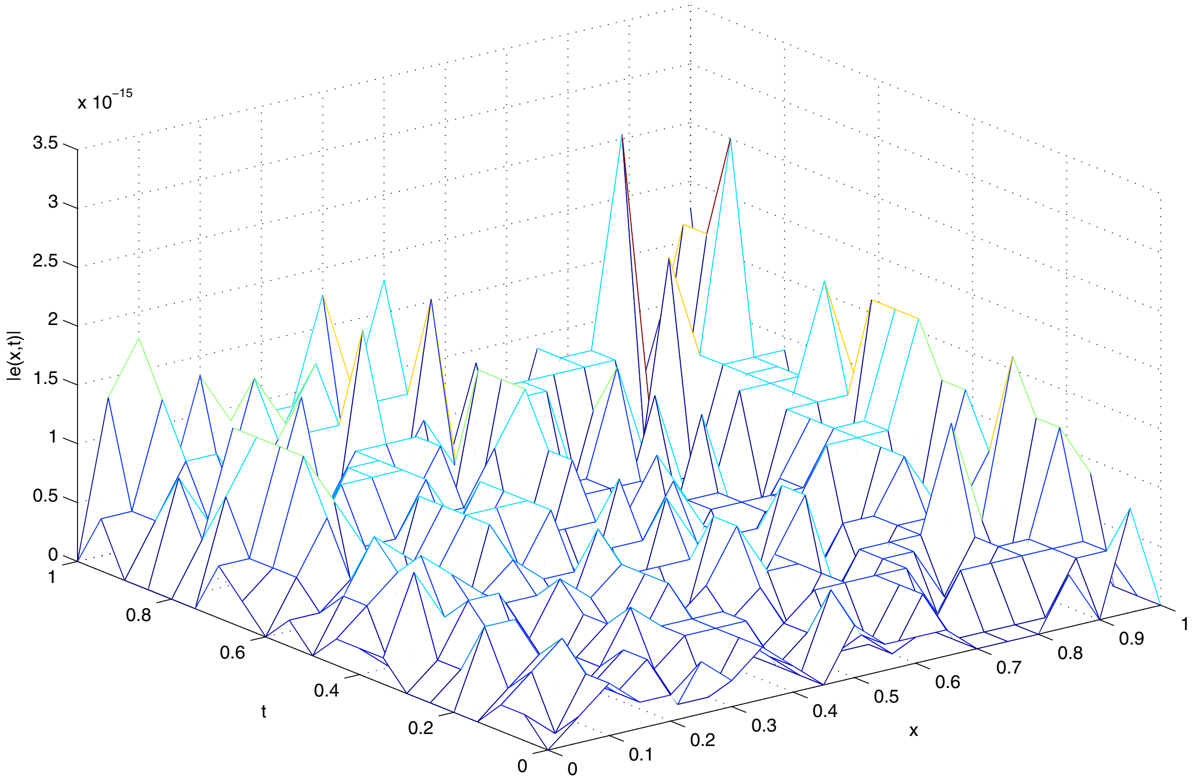

Fig. 2 (colour online). Plot of error function from Example 1 for $n = 20$.

$$\cdot U(k-r+1,s)+(k+1)U(k+1,h)-\varphi(k,h)\Big),$$
$$k=0,1,\ldots,\quad h=0,1,\ldots. \tag{17}$$

4. Numerical Examples

In this section, we applied the presented process and solved two examples. These examples are chosen such that their exact solutions are known. The numerical computations have been done by the software Matlab. Let $e_n(x,t) = u_n(x,t) - u(x,t)$; we calculate the following norm of the error for different values of n:

$$E_\infty = \|e_n(x,t)\|_\infty = \max\{|e_n(x,t)|,\; 0 \le x \le L,\; 0 \le t \le T\}. \tag{18}$$

Example 1. Consider (6)–(9) with

$$f(x)=\exp(x),\quad p(t)=\exp(t),\quad q(t)=\exp(t),$$
$$\lambda=1,\quad \mu=2,\quad \varphi(x,t)=\exp(2x+2t).$$

Applying (12) with the initial conditions of this problem, we get

$$U(k,0)=\frac{1}{k!},\quad k=0,1,\ldots,n, \tag{19}$$

and from (15) and (16), we have

$$U(0,h)=\frac{1}{h!},\quad h=1,2,\ldots,n,$$
$$U(1,h)=\frac{1}{h!},\quad h=1,2,\ldots,n. \tag{20}$$

Now, by using iterative process (17), we have

$$U(2,1)=\frac{1}{2\times 2!}\Big(U(0,1)+U(0,0)U(1,0)+U(1,0)-\varphi(0,0)\Big)=\frac{1}{2!},$$

$$U(3,1)=\frac{1}{2\times 3!}\Big(U(1,1)+2U(0,0)U(2,0)+U(1,0)U(1,0)+2U(2,0)-\varphi(1,0)\Big)=\frac{1}{3!},$$

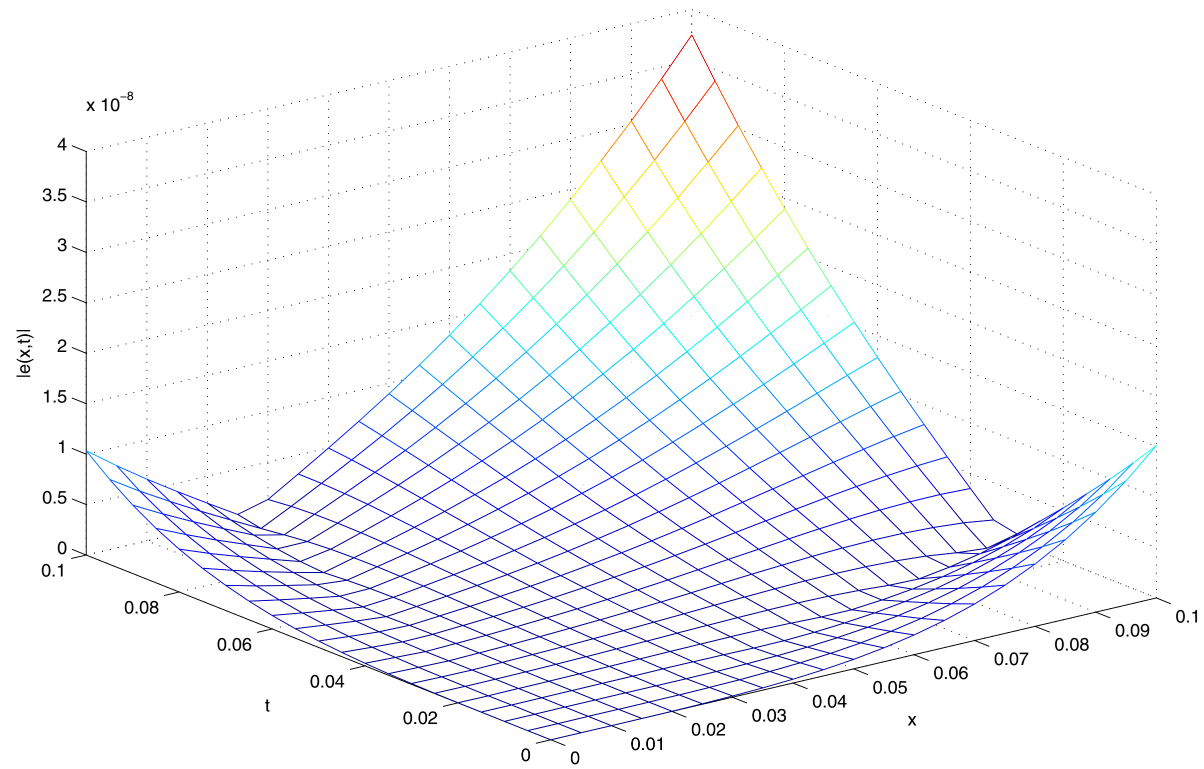

Fig. 3 (colour online). Plot of error function from Example 2 for $n = 2$.

$$U(4,1) = \frac{1}{4!}, \tag{21}$$

$$\vdots$$

$$U(2,2) = \frac{1}{4\times 2!}\Big(2U(0,2) + U(0,1)U(1,0) + U(0,0)U(1,1) + U(1,1) - \varphi(0,1)\Big) = \frac{1}{2!2!},$$

$$U(3,2) = \frac{1}{2!3!},$$

$$\vdots$$

By continuing this process, we obtain

$$\begin{aligned} u(x,t) &\simeq 1 + t + \frac{1}{2!}t^2 + \ldots + \frac{1}{n!}t^n + \ldots \\ &+ \left(x + xt + \frac{1}{2!}xt^2 + \ldots + \frac{1}{n!}xt^n + \ldots\right) \\ &+ \left(\frac{1}{2!}x^2 + \frac{1}{2!}x^2 t + \frac{1}{2!2!}x^2t^2 + \ldots + \frac{1}{2!n!}x^2t^n + \ldots\right) \\ &+ \ldots, \end{aligned} \tag{22}$$

Table 2. Maximum errors for $x \in [0,1]$, $t \in [0,1]$.

	$n = 5$	$n = 10$	$n = 15$	$n = 20$
E_∞	$8.778 \cdot 10^{-3}$	$1.485 \cdot 10^{-10}$	$2.771 \cdot 10^{-13}$	$1.776 \cdot 10^{-15}$

which is the Taylor expansion of $u(x,t) = \exp(x+t)$ and the exact solution of Example 1. By (21), the closed form of the solution is obtained. The computational results of Example 1 are presented in Table 2, and the plots of the corresponding error functions are shown in Figures 1 and 2.

Example 2. Consider (6)–(9) with

$$\begin{aligned} f(x) &= 3c\ \mathrm{sech}^2(kx), \\ p(t) &= 3c\ \mathrm{sech}^2(-kat), \\ q(t) &= -6c\ \mathrm{sech}^2(-kat) \times k\ \tanh(-kat), \\ \lambda &= \mu = 1, \quad \varphi(x,t) = 0, \end{aligned}$$

where $a = (1+\lambda c)$ is the wave velocity and $k = \frac{1}{2}\left(\frac{\lambda c}{\mu(1+\lambda c)}\right)^{\frac{1}{2}}$ [6]. This equation represents a single

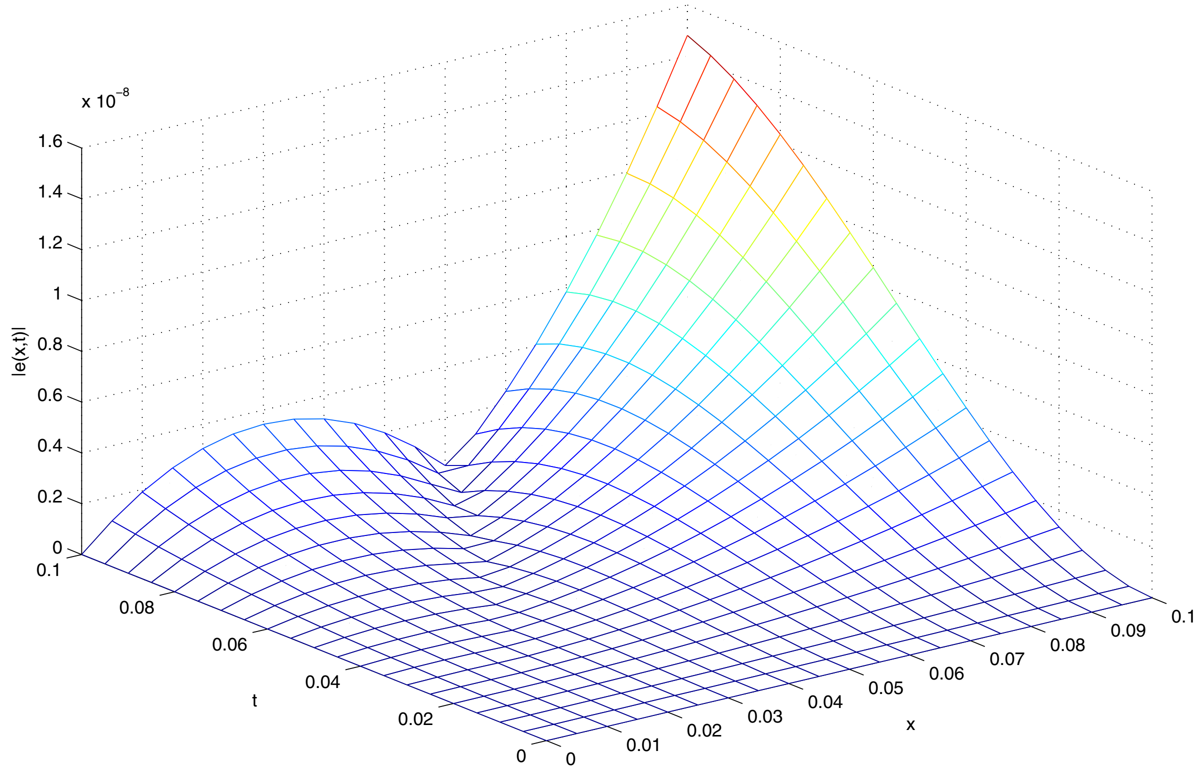

Fig. 4 (colour online). Plot of error function from Example 2 for $n = 4$.

soliton of magnitude $3c$ with the constant speed $c+1$.

By (12), we have

$$U(0,0)=\frac{3}{10},\ U(1,0)=0,\ U(2,0)=-\frac{3}{440},$$
$$U(3,0)=0,\ U(4,0)=\frac{1}{968}, \tag{23}$$
$$U(5,0)=0,\ U(6,0)=-\frac{17}{12\,777\,600},\ldots.$$

From (15) and (16), we obtain

$$U(0,1)=0,\ U(0,2)=-\frac{33}{4000},\ U(0,3)=0,$$
$$U(0,4)=\frac{121}{800\,000},\ U(0,5)=0,$$
$$U(0,6)=-\frac{22627}{960\,000\,000},\ \ldots,\ U(1,1)=\frac{3}{200},$$
$$U(2,1)=0,\ U(3,1)=-\frac{11}{2000},$$
$$U(4,1)=0,\ldots. \tag{24}$$

By the similar process of Example 1, we can find the remainder value of $U(k,h)$ as follows:

$$U(2,1)=\frac{1}{2\times 2!}\Big(U(0,1)+U(0,0)U(1,0)$$
$$+U(1,0)\Big)=0,$$
$$U(3,1)=\frac{1}{2\times 3!}\Big(U(1,1)+2U(0,0)U(2,0)$$
$$U+(1,0)U(1,0)+2U(2,0)\Big)=\frac{97}{8800},$$
$$U(4,1)=0, \tag{25}$$
$$\vdots$$
$$U(2,2)=\frac{3}{8000},$$
$$U(3,2)=0,$$
$$\vdots$$

Table 3. Maximum errors for $x \in [0,1]$, $t \in [0,1]$.

	$n=1$	$n=2$	$n=3$	$n=4$
E_∞	$1.507 \cdot 10^{-4}$	$3.750 \cdot 10^{-8}$	$4.024 \cdot 10^{-8}$	$1.481 \cdot 10^{-8}$

By continuing this process, we obtain

$$u(x,t) = \frac{3}{10} - \frac{3}{440}x^2 - \frac{33}{4000}t^2 + \frac{3}{200}xt + \dots,$$

which it is the Taylor expansion of

$$u(x,t) = (0.3)\,\mathrm{sech}^2\left(\sqrt{\frac{0.1}{4.4}}(x-1.1t)\right)$$

and the exact solution of Example 2. We present the values of E_∞ for $n = 1,2,3,4$ and $c = 0.1$ in Table 3 and the plots of the error functions for $n = 2,4$ in Figures 3 and 4.

5. Conclusions

The basic goal of this work is applying the DTM to obtain the solution of the RLW equation with an initial and two boundary conditions. By using this method, numerical and analytical results are obtained by a simple iterative process. This method reduces the computational difficulties of the other methods. Also we can increase the accuracy of the series solution by increasing the number of terms in the series solution. The strong point of the method proposed in this paper is obtaining a continuous $u(x,t)$ for all values of x and t.

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