Trace Formulae for Matrix Integro-Differential Operators

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In this paper, we consider the eigenvalue problems for matrix integro-differential operators with separated boundary conditions on the finite interval and find new trace formulae for the matrix integro-differential operators.

Key words: Matrix Integro-Differential Operator; Eigenvalue; Trace formula.
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1. Introduction

In this paper, we will find trace formulae for the following matrix integro-differential operators $L(Q,M;h,H)$:

$$-Y''(x)+Q(x)Y(x)+\int_0^x M(x,t)Y(t)\,\mathrm{d}t=\lambda Y(x),\quad x\in(0,\pi), \tag{1}$$

with boundary conditions

$$Y'(0)-hY(0)=0 \tag{2}$$

and

$$Y'(\pi)+HY(\pi)=0, \tag{3}$$

where λ is a spectral parameter, $Y(x)=[y_k(x)]_{k=\overline{1,d}}$ is a column vector, $Q(x)$ and $M(x,t)$ are $d\times d$ real symmetric matrix-valued functions, and h and H are $d\times d$ real symmetric constant matrices. $M(x,t)$ is an integrable function on the set $D_0\overset{\text{def}}{=}\{(x,t):0\le t\le x\le\pi,$ $x,t\in\mathbb{R}\}$, $Q\in C^1[0,\pi]$, where $C^1[0,\pi]$ denotes a set whose element is a continuously differentiable function on $[0,\pi]$. In particular, $h=\infty$ in (2) means the Dirichlet boundary condition $Y(0)=0$, and $H=\infty$ in (3) means the Dirichlet boundary condition $Y(\pi)=0$.

For the matrix Sturm–Liouville equation (when $M=0$ in (1)) properties of spectral characteristics were provided in [1 – 4], and asymptotics of eigenvalues for the integro-differential operator with $d=1$ in (1) were given in [5 – 9].

Gelfand and Levitan [10] discussed the Sturm–Liouville problem

$$-y''(x)+q(x)y(x)=\lambda y(x),\ \ x\in(0,\pi), \tag{4}$$

with the Neumann boundary conditions

$$y'(0)=y'(\pi)=0 \tag{5}$$

and obtained the remarkable formula for the regularized trace as follows:

$$\sum_{n=0}^{\infty}\left[\lambda_n-n^2-\frac{1}{\pi}\int_0^\pi q(x)\,\mathrm{d}x\right]=\frac{1}{4}[q(0)+q(\pi)]-\frac{1}{2\pi}\int_0^\pi q(x)\,\mathrm{d}x,$$

where $q\in C^1[0,\pi]$ and λ_n $(n=0,1,2,\dots)$ are the eigenvalues of the Sturm–Liouville problem (4) and (5). For the Sturm–Liouville problem (4) with Dirichlet boundary conditions and the eigenvalues λ_n $(n=1,2,\dots)$, they got the following formula:

$$\sum_{n=1}^{\infty}\left[\lambda_n-n^2-\frac{1}{\pi}\int_0^\pi q(x)\,\mathrm{d}x\right]=-\frac{1}{4}[q(0)+q(\pi)]+\frac{1}{2\pi}\int_0^\pi q(x)\,\mathrm{d}x.$$

Since then, this kind of trace formulae for differential operators was found by a number of authors (see references). A trace formula of a differential operator has many applications in the inverse problem, in the numerical calculation of eigenvalues, in the theory of integrable systems, etc. Sadovnichii and Podol'skii [11] stated several sharp methods to trace formulae of second-order operators, high-order operators as well as partial differential operators.

Using Rouché's theorem for operator-valued functions in [12], we can suitably locate the eigenvalues of $L(Q,M;h,H)$ and find a precise description for the formula of the square root of the large eigenvalues up to the $o(\frac{1}{n})$-term, which are similar to the results in [1, 2]:

(i) Let $\lambda_n^{(j)}(j=\overline{1,d};\ n=0,1,2,\ldots)$ be eigenvalues of the operator $L(Q,M;h,H)$, then $\lambda_n^{(j)}$ satisfy the asymptotic formula

$$\rho_n^{(j)} \overset{\text{def}}{=} \sqrt{\lambda_n^{(j)}} = n + \frac{\omega_1^{(j)}}{n\pi} + \frac{\kappa_{1,n}^{(j)}}{n},$$

where $\omega_1^{(j)}$ are the characteristic values of the $d\times d$ real symmetric matrix $\omega_1 = h+H+\frac{1}{2}\int_0^\pi Q(x)\,\mathrm{d}x$ and $\sum_n |\kappa_{1,n}^{(j)}|^2 < \infty$.

(ii) Let $\lambda_n^{(j)}(j=\overline{1,d};\ n=1,2,\ldots)$ be eigenvalues of the operator $L(Q,M;h,\infty)$, then $\lambda_n^{(j)}$ satisfy the asymptotic formula

$$\rho_n^{(j)} \overset{\text{def}}{=} \sqrt{\lambda_n^{(j)}} = n - \frac{1}{2} + \frac{\omega_2^{(j)}}{\left(n-\frac{1}{2}\right)\pi} + \frac{\kappa_{2,n}^{(j)}}{n},$$

where $\omega_2^{(j)}$ are the characteristic values of the $d\times d$ real symmetric matrix $\omega_2 = h+\frac{1}{2}\int_0^\pi Q(x)\,\mathrm{d}x$ and $\sum_n |\kappa_{2,n}^{(j)}|^2 < \infty$.

(iii) Let $\lambda_n^{(j)}(j=\overline{1,d};\ n=1,2,\ldots)$ be eigenvalues of the operator $L(Q,M;\infty,H)$, then $\lambda_n^{(j)}$ satisfy the asymptotic formula

$$\rho_n^{(j)} \overset{\text{def}}{=} \sqrt{\lambda_n^{(j)}} = n - \frac{1}{2} + \frac{\omega_3^{(j)}}{\left(n-\frac{1}{2}\right)\pi} + \frac{\kappa_{3,n}^{(j)}}{n},$$

where $\omega_3^{(j)}$ are the characteristic values of the $d\times d$ real symmetric matrix $\omega_3 = H+\frac{1}{2}\int_0^\pi Q(x)\,\mathrm{d}x$ and $\sum_n |\kappa_{3,n}^{(j)}|^2 < \infty$.

(iv) Let $\lambda_n^{(j)}(j=\overline{1,d};\ n=1,2,\ldots)$ be eigenvalues of the operator $L(Q,M;\infty,\infty)$, then $\lambda_n^{(j)}$ satisfy the asymptotic formula

$$\rho_n^{(j)} \overset{\text{def}}{=} \sqrt{\lambda_n^{(j)}} = n + \frac{\omega_4^{(j)}}{n\pi} + \frac{\kappa_{4,n}^{(j)}}{n},$$

where $\omega_4^{(j)}$ are the characteristic values of the $d\times d$ real symmetric matrix $\omega_4 = \frac{1}{2}\int_0^\pi Q(x)\,\mathrm{d}x$ and $\sum_n |\kappa_{4,n}^{(j)}|^2 < \infty$.

However, some trace formulae for the matrix integro-differential operator $L(Q,M;h,H)$ have never been considered before. In this paper, we shall discuss the eigenvalue problem for the operator $L(Q,M;h,H)$ and find new trace formulae.

2. Result

For simplicity A_{ij} denotes the entry of matrix A at the ith row and jth column and $\operatorname{tr} A$ denotes the trace of the matrix A; I_d is a $d\times d$ identity matrix and 0_d is a $d\times d$ zero matrix.

Theorem 1.

(i) For the operator $L(Q,M;h,H)$: let $\lambda_n^{(j)}(j=\overline{1,d};\ n=0,1,2,\ldots)$ be eigenvalues of the operator $L(Q,M;h,H)$, then we have the trace formula

$$\sum_{n=0}^{\infty}\left[\sum_{j=1}^{d}\left(\lambda_n^{(j)}-n^2\right)-\frac{2}{\pi}\operatorname{tr}\omega_1\right] = \frac{1}{4}\operatorname{tr}\left(Q(0)+Q(\pi)\right) - \frac{1}{\pi}\operatorname{tr}\omega_1 + \frac{1}{2}\int_0^\pi \operatorname{tr} M(t,t)\,\mathrm{d}t - \frac{1}{2}\operatorname{tr}\left(h^2+H^2\right). \tag{6}$$

(ii) For the operator $L(Q,M;h,\infty)$: let $\lambda_n^{(j)}(j=\overline{1,d};\ n=1,2,\ldots)$ be eigenvalues of the operator $L(Q,M;h,\infty)$, then we have the trace formula

$$\sum_{n=1}^{\infty}\left[\sum_{j=1}^{d}\left(\lambda_n^{(j)}-\left(n-\frac{1}{2}\right)^2\right)-\frac{2}{\pi}\operatorname{tr}\omega_2\right] = \frac{1}{4}\operatorname{tr}\left(Q(0)-Q(\pi)\right) + \frac{1}{2}\int_0^\pi \operatorname{tr} M(t,t)\,\mathrm{d}t - \frac{1}{2}\operatorname{tr} h^2. \tag{7}$$

(iii) For the operator $L(Q,M;\infty,h)$: let $\lambda_n^{(j)}(j=\overline{1,d};\ n=1,2,\ldots)$ be eigenvalues of the operator $L(Q,M;\infty,h)$, then we have the trace formula

$$\sum_{n=1}^{\infty}\left[\sum_{j=1}^{d}\left(\lambda_n^{(j)}-\left(n-\frac{1}{2}\right)^2\right)-\frac{2}{\pi}\operatorname{tr}\omega_3\right] = \frac{1}{4}\operatorname{tr}\left(Q(\pi)-Q(0)\right) + \frac{1}{2}\int_0^\pi \operatorname{tr} M(t,t)\,\mathrm{d}t - \frac{1}{2}\operatorname{tr} H^2. \tag{8}$$

(iv) For the operator $L(Q,M;\infty,\infty)$: let $\lambda_n^{(j)}(j=\overline{1,d};\ n=1,2,\ldots)$ be eigenvalues of the operator

$L(Q,M;\infty,\infty)$, then we have the trace formula

$$\sum_{n=1}^{\infty}\left[\sum_{j=1}^{d}\left(\lambda_n^{(j)}-n^2\right)-\frac{2}{\pi}\operatorname{tr}\omega_4\right] = -\frac{1}{4}\operatorname{tr}(Q(0)+Q(\pi))+\frac{1}{\pi}\operatorname{tr}\omega_4+\frac{1}{2}\int_0^{\pi}\operatorname{tr}M(t,t)\,\mathrm{d}t. \tag{9}$$

3. Proof

We only give the proof for (6) in Theorem 1. Analogously, we can also prove that (7)–(9) in Theorem 1 hold.

Let $\Phi(x,\lambda)$ be a solution of (1) satisfying $\Phi(0,\lambda)=I_d$, $\Phi'(0,\lambda)=h$, then we have

$$\Phi(x,\lambda)=\cos(\rho x)I_d+\frac{h}{\rho}\sin(\rho x) +\int_0^x\left[\frac{\sin\rho(x-t)}{\rho}Q(t) +\int_t^x M(\xi,t)\frac{\sin\rho(x-\xi)}{\rho}d\xi\right]\Phi(t,\lambda)\,\mathrm{d}t. \tag{10}$$

Using integration by parts and the iterative method, we can compute

$$\Phi(x,\lambda)=\cos(\rho x)I_d+\left[h+\frac{1}{2}\int_0^x Q(t)\,\mathrm{d}t\right]\frac{\sin(\rho x)}{\rho} +\left\{\frac{Q(x)-Q(0)}{4}-\frac{1}{2}\int_0^x[Q(t)h+M(t,t)]\,\mathrm{d}t -\frac{1}{8}\left(\int_0^x Q(t)\,\mathrm{d}t\right)^2\right\}\frac{\cos(\rho x)}{\rho^2}+o\left(\frac{\mathrm{e}^{\tau x}}{|\rho|^2}\right), \tag{11}$$

where $\lambda=\rho^2$, $\tau=|\mathrm{Im}\rho|$. From (11), we obtain

$$\Phi'(x,\lambda)=-\rho\sin(\rho x)I_d +\left[h+\frac{1}{2}\int_0^x Q(t)\,\mathrm{d}t\right]\cos(\rho x)+\left\{\frac{Q(x)+Q(0)}{4} +\frac{1}{2}\int_0^x[Q(t)h+M(t,t)]\,\mathrm{d}t +\frac{1}{8}\left(\int_0^x Q(t)\,\mathrm{d}t\right)^2\right\}\frac{\sin(\rho x)}{\rho}+o\left(\frac{\mathrm{e}^{\tau x}}{|\rho|}\right). \tag{12}$$

In virtue of (11) and (12), we get the characteristic matrix $w(\lambda)$ for the boundary-value problem (1)–(3):

$$w(\lambda)=\Phi'(\pi,\lambda)+H\Phi(\pi,\lambda) =-\rho\sin(\rho\pi)I_d+\omega_1\cos(\rho\pi) +\omega_5\frac{\sin(\rho\pi)}{\rho}+o\left(\frac{\mathrm{e}^{\tau\pi}}{|\rho|}\right), \tag{13}$$

where

$$\omega_1=h+H+\frac{1}{2}\int_0^{\pi}Q(x)\,\mathrm{d}x$$

and

$$\omega_5=\frac{Q(0)+Q(\pi)}{4} +\frac{1}{2}\int_0^{\pi}[Q(t)h+HQ(t)+M(t,t)]\,\mathrm{d}t+Hh +\frac{1}{8}\left(\int_0^{\pi}Q(t)\,\mathrm{d}t\right)^2.$$

The eigenvalues $\lambda_n^{(j)}$ of the operator $L(Q,M;h,H)$ can be located by determining whether the matrix-valued function $w(\lambda)$ is singular or not. We can rewrite

$$w(\lambda)=w_0(\lambda)+\varepsilon(\lambda),$$

where $w_0(\lambda)=(-\rho\sin(\rho\pi))I_d$ and $\varepsilon(\lambda)$ is a remainder. We shall see that $w_0(\lambda)$ has a quite neat and simple form from which we can determine those values $\hat{\lambda}$ making $w_0(\hat{\lambda})$ be singular. By the extension theorem of Rouché's theorem on operator-valued functions in [12], we are getting close to locate the eigenvalues of $L(Q,M;h,H)$.

Let $\Delta_1(\lambda)=o(\mathrm{e}^{\tau\pi}/|\rho|^2)$, then

$$w_0^{-1}(\lambda)w(\lambda)=I_d-\frac{\cot(\rho\pi)}{\rho}\omega_1-\frac{1}{\rho^2}\omega_5 +\frac{\Delta_1(\lambda)}{\sin(\rho\pi)}. \tag{14}$$

Denote $\Gamma_{N_0}\overset{\text{def}}{=}\{\rho:|\rho|=N_0+\frac{1}{2}\}$ and $G_\delta=\{\rho:|\rho-k|\geq\delta,\ k=0,\pm1,\pm2,\ldots\}$, where N_0 is a sufficiently large integer, and $\delta>0$ is sufficiently small. According to [13, p. 7], we obtain

$$|\sin(\rho\pi)|\geq C_\delta\,\mathrm{e}^{\tau\pi},\quad \lambda\in G_\delta,\quad |\rho|\geq\rho^*, \tag{15}$$

where $\rho^*=\rho^*(\delta)$ sufficiently large, and C_δ is a constant with respect to δ. From (15) and $\Delta_1(\lambda)=o(\mathrm{e}^{\tau\pi}/|\rho|^2)$, we get

$$\left|\frac{\Delta_1(\lambda)}{\sin(\rho\pi)}\right|=o\left(\frac{1}{|\rho|^2}\right)\ \text{on}\ \Gamma_{N_0}. \tag{16}$$

Thus we have

$$\Delta(\lambda)\overset{\text{def}}{=}\det[w_0^{-1}(\lambda)w(\lambda)] = \det\left\{I_d-\frac{\cot(\rho\pi)}{\rho}\omega_1-\frac{1}{\rho^2}\omega_5+o\left(\frac{1}{\rho^2}\right)\right\}.$$

Using the Laplace expansion of determinants, we obtain that

$$\Delta(\lambda)=\prod_{i=1}^{d}\left\{1-\frac{\cot(\rho\pi)}{\rho}\omega_{1,ii}-\frac{1}{\rho^2}\omega_{5,ii}+o\left(\frac{1}{\rho^2}\right)\right\} +\frac{a\cot^2(\rho\pi)}{\rho^2}+o\left(\frac{1}{\rho^2}\right),$$

where

$$a=-\sum_{i=1}^{d-1}\sum_{i<j}\omega_{1,ij}\omega_{1,ji}.$$

Moreover, we have

$$\Delta(\lambda)=1-\frac{\cot(\rho\pi)}{\rho}\sum_{i=1}^{d}\omega_{1,ii}-\frac{1}{\rho^2}\sum_{i=1}^{d}\omega_{5,ii} +\frac{\cot^2(\rho\pi)}{\rho^2}\sum_{i=1}^{d-1}\sum_{i<j}\omega_{1,ii}\omega_{1,jj}+\frac{a\cot^2(\rho\pi)}{\rho^2}+o\left(\frac{1}{\rho^2}\right).$$

Expanding $\ln\Delta(\lambda)$ by the Maclaurin formula, we obtain that on Γ_{N_0}

$$\ln\Delta(\lambda)=-\frac{\cot(\rho\pi)}{\rho}\sum_{i=1}^{d}\omega_{1,ii}-\frac{1}{\rho^2}\sum_{i=1}^{d}\omega_{5,ii} +\frac{\cot^2(\rho\pi)}{\rho^2}\sum_{i=1}^{d-1}\sum_{i<j}\omega_{1,ii}\omega_{1,jj}-\frac{\cot^2\rho\pi}{2\rho^2}\left(\sum_{i=1}^{d}\omega_{1,ii}\right)^2 +\frac{a\cot^2(\rho\pi)}{\rho^2}+o\left(\frac{1}{\rho^2}\right).$$

Let $\lambda_n^{(j)}$ $(j=\overline{1,d};\ n=0,1,2,\dots)$ be the zeros of the function $\det w(\lambda)$ and $\mu_n=n$ $(n=0,\pm1,\pm2,\dots)$ be the zeros of the function $\det w_0(\lambda)$. From (14), for sufficiently large N_0, we see that the numbers $\lambda_n^{(j)}$ $(n\le N_0)$ are inside Γ_{N_0} and the numbers $\lambda_n^{(j)}$ $(n>N_0)$ are outside Γ_{N_0}. Obviously, $\mu_n=n$ do not lie on the contour Γ_{N_0}. We have the following identity:

$$\sum_{n=0}^{N_0}\sum_{j=1}^{d}(2\lambda_n-2n^2)=-\frac{1}{2\pi\mathrm{i}}\oint_{\Gamma_{N_0}}2\rho\ln\Delta(\lambda)\,\mathrm{d}\rho. \qquad (17)$$

By calculating residues, we get for sufficiently large N_0

$$\oint_{\Gamma_{N_0}}\cot(\rho\pi)\,\mathrm{d}\rho=2\pi\mathrm{i}\sum_{n=-N_0}^{N_0}\frac{1}{\pi}, \qquad \oint_{\Gamma_{N_0}}\frac{\cot^2(\rho\pi)}{\rho}\,\mathrm{d}\rho=-2\pi\mathrm{i}+o(1), \qquad \oint_{\Gamma_{N_0}}o\left(\frac{1}{|\rho|}\right)\mathrm{d}\rho=o(1). \qquad (18)$$

Substituting (18) into (17), we have

$$-\frac{1}{2\pi\mathrm{i}}\oint_{\Gamma_{N_0}}2\rho\ln\frac{w(\lambda)}{w_0(\lambda)}\,\mathrm{d}\rho=\frac{1}{2}\mathrm{tr}\,(Q(0)+Q(\pi)) +\frac{2(2N_0+1)}{\pi}\mathrm{tr}\,\omega_1+\int_0^\pi\mathrm{tr}\,M(t,t)\,\mathrm{d}t -\mathrm{tr}\,(h^2+H^2)+o(1). \qquad (19)$$

From (17) and (19), we obtain

$$\sum_{n=0}^{N_0}\left[\sum_{j=1}^{d}(\lambda_n-n^2)-\frac{2}{\pi}\mathrm{tr}\,\omega_1\right]=\frac{1}{4}\mathrm{tr}\,(Q(0)+Q(\pi)) -\frac{1}{\pi}\mathrm{tr}\,\omega_1+\frac{1}{2}\int_0^\pi\mathrm{tr}\,M(t,t)\,\mathrm{d}t -\frac{1}{2}\mathrm{tr}\,(h^2+H^2)+o(1). \qquad (20)$$

Letting $N_0\to+\infty$ in (20) yields

$$\sum_{n=0}^{\infty}\left[\sum_{j=1}^{d}(\lambda_n-n^2)-\frac{2}{\pi}\mathrm{tr}\,\omega_1\right]=\frac{1}{4}\mathrm{tr}\,(Q(0)+Q(\pi)) -\frac{1}{\pi}\mathrm{tr}\,\omega_1+\frac{1}{2}\int_0^\pi\mathrm{tr}\,M(t,t)\,\mathrm{d}t-\frac{1}{2}\mathrm{tr}\,(h^2+H^2).$$

This completes the proof of Theorem 1. □

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