# Unsteady Flow of a Power–Law Fluid past a Shrinking Sheet with Mass Transfer

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The unsteady two-dimensional boundary layer flow past a shrinking sheet in a non-Newtonian power-law fluid is investigated. The governing partial differential equations are transformed into a nonlinear ordinary differential equation using a similarity transformation before being solved numerically by the Runge–Kutta–Fehlberg method and the NAG Fortran library subroutine DO2HAF with shooting technique. The results obtained by both methods are in good agreement. It is found that dual solutions exist for a certain range of the unsteadiness parameter and the suction parameter. Moreover, by increasing the power-law index $n$, the skin friction coefficient is enhanced.

*Key words:* Unsteady Flow; Power-Law Fluid; Boundary Layer; Heat Transfer; Shrinking Sheet; Dual Solutions.

## 1. Introduction

The flow induced by a shrinking sheet is different from that induced by moving and stretching sheets as formulated by Sakiadis [1] and Crane [2], respectively. The difference is found in the velocity on the boundary which is for a shrinking sheet towards a fixed point. The stretching sheet would induce a far field suction towards the sheet, while the shrinking sheet would cause a velocity away from the sheet. Thus from physical grounds, vorticity of the shrinking sheet is not confined within the boundary layer, and the flow is unlikely to exist unless adequate suction on the boundary is imposed [3]. Compared to the stretching sheet case, there are only a few publications on the shrinking sheet, such as Fang et al. [4], who reported the theoretical estimation of the solution domain and solved the Blasius equation with associated boundary conditions using a numerical technique. Further, Fang et al. [5] solved analytically the slip flow over a permeable shrinking surface in a viscous fluid, which is modelled using the newly proposed second-order slip flow. On the other hand, Fang and Zhang [6] and Hayat et al. [7] have included the magnetohydrodynamic (MHD) effect to the steady shrinking sheet problems and solved the problems analytically. They found that the solution exists only if adequate suction on the boundary is imposed, and multiple solutions exist for a certain parameter domain.

The shrinking sheet problem was also extended to a power-law velocity, a prescribed power-law temperature, and to other fluids [8 – 13]. Fang et al. [14] investigated also an unsteady shrinking sheet with mass transfer in a viscous fluid. It is worth mentioning that Schowalter [15] and Acrivos et al. [16] were the first who have studied theoretically the boundary layer flow of a non-Newtonian fluid past a fixed flat plate. A non-Newtonian fluid is a fluid where the relation between the shear stress and the strain rate is nonlinear. Examples of non-Newtonian fluids are molten plastics, polymer solutions, dyes, varnishes, suspensions, adhesives, paints, greases, paper pulp, and biological fluids like blood [17]. Andersson et al. [18] examined numerically the flow of a thin liquid film of a non-Newtonian power-law fluid due to an unsteady stretching surface. Later, Wang and Pop [19] studied the similar problem using the homotopy analysis method (HAM). They found that the numerical and analytical solutions are in very good agreement.

Motivated by the above investigations, the objective of the present study is to extend the work of Fang et al. [14] to a non-Newtonian power-law fluid. It should be mentioned that the steady two-dimensional boundary layer flow in a non-Newtonian pseudoplastic fluid was analyzed by Zheng et al. [20].

## 2. Problem Formulation

Consider a two-dimensional laminar flow of an unsteady incompressible fluid obeying the power-law model over a permeable shrinking sheet in a quiescent fluid. The sheet velocity is $u_w(x,t)$, and the wall mass velocity through the sheet is $v_w(x,t)$ with $v_w(x,t) < 0$ for suction and $v_w(x,t) > 0$ for injection. Both $u_w(x,t)$ and $v_w(x,t)$ will be defined later. Here $x$ is the coordinate measured along the shrinking sheet in the direction of motion, $y$ is the coordinate perpendicular to it, and $t$ is the time. The basic equations governing the resulting boundary layer flow are [18]

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{1}{\rho}\left(\frac{\partial \tau_{xy}}{\partial y}\right), \tag{2}$$

where $u$ and $v$ are the velocity components along the $x$- and $y$-directions, respectively, $\rho$ is the fluid density, and $\tau_{xy}$ represents the shear stress. In the present problem, we have $\partial u/\partial y > 0$; this gives the shear stress as

$$\tau_{xy} = K\left(\frac{\partial u}{\partial y}\right)^n, \tag{3}$$

where $K$ is called the consistency coefficient and $n$ is the power-law index. For the particular parameter value $n = 1$, one can retrieve the Newtonian fluid model with a dynamic coefficient of viscosity $K$. As $n$ deviates from unity, deviations from Newtonian behaviour occur. It should be noted that $0 < n < 1$ and $n > 1$ correspond to pseudoplastic and dilatant fluids, respectively.

Combining (2) and (3), we have

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{K}{\rho}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)^n, \tag{4}$$

which is subjected to the boundary conditions

$$\begin{aligned} &v = v_w(x,t),\ \ u = u_w(x,t)\ \text{ at }\ y = 0,\\ &u \to 0\ \text{ as }\ y \to \infty. \end{aligned} \tag{5}$$

We assume now that $u_w(x,t)$ is given by

$$u_w(x,t) = \frac{-cx}{1-\alpha t}, \tag{6}$$

where $c$ ($c > 0$) and $\alpha$ are constants, both having dimensions time$^{-1}$; $\alpha$ is a parameter showing the unsteadiness of the problem.

Equations (1) and (4) have the following similarity solution [18]:

$$\begin{aligned} u &= \frac{cx}{1-\alpha t} f'(\eta),\\ v &= -\left(\frac{c^{1-2n}}{K/\rho}\right)^{-1/(1+n)}\\ &\cdot x^{(n-1)/(n+1)} (1-\alpha t)^{(1-2n)/(1+n)}\\ &\cdot \left[\left(\frac{2n}{1+n}\right) f(\eta) + \left(\frac{1-n}{1+n}\right)\eta f'(\eta)\right], \end{aligned} \tag{7}$$

where the similarity variable $\eta$ is given by

$$\begin{aligned} \eta &= \left(\frac{c^{2-n}}{K/\rho}\right)^{1/(1+n)} x^{(1-n)/(1+n)}\\ &\cdot (1-\alpha t)^{(n-2)/(1+n)} y, \end{aligned} \tag{8}$$

and primes denote differentiation with respect to $\eta$. Thus, we assume that $v_w(x,t)$ has the form

$$\begin{aligned} v_w(x,t) &= -\left(\frac{c^{1-2n}}{K/\rho}\right)^{-1/(1+n)} x^{(n-1)/(n+1)}\\ &\cdot (1-\alpha t)^{(1-2n)/(1+n)} \left(\frac{2n}{1+n}\right) s, \end{aligned} \tag{9}$$

where $s = f(0)$ is the wall mass flux parameter with $s > 0$ for suction and $s < 0$ for injection. Using (7) and (8), (1) and (4) reduce to the following ordinary differential equation:

$$\begin{aligned} &n\left(f''\right)^{n-1} f''' + \left(\frac{2n}{1+n}\right) f f'' - f'^2\\ &- \beta\left[f' + \left(\frac{2-n}{1+n}\right)\eta f''\right] = 0, \end{aligned} \tag{10}$$

subject to the boundary conditions

$$f(0) = s,\ \ f'(0) = -1,\ \ f'(\infty) = 0. \tag{11}$$

Here $\beta = \alpha/c$ is a dimensionless parameter, which measures the flow unsteadiness. For the present paper, we assume a decelerating sheet with $\beta \le 0$.

A physical quantity of interest is the skin friction coefficient $C_f = \tau_{xy}/(\rho u_w^2)$, which can be shown that it is given by

$$\mathrm{Re}_x^{1/(1+n)} C_f = [f''(0)]^n, \quad (12)$$

where $\tau_{xy}$ is given by (3) and $\mathrm{Re}_x = u_w^{2-n} x^n/(K/\rho)^n$ is the generalized local Reynolds number.

It is worth mentioning that for $n = 1$ (Newtonian fluid), (10) reduces to (6) of the paper by Fang et al. [14], and when $n = 1$ and $\beta = 0$, (10) reduces to the problem discussed by Miklavčič and Wang [3].

## 3. Results and Discussion

The transformed equation (10) subjected to the boundary conditions (11) was integrated numerically using the Runge–Kutta–Fehlberg method and NAG routine DO2HAF with shooting technique to obtain the missing slopes $f''(0)$ for some values of the governing parameters, namely unsteadiness parameter $\beta$, suction parameter $s$, and the power-law index $n$. The results of these two methods are in very good agreement, as shown in Table 1. Thus, this lends confidence to the accuracy of the numerical results to be reported subsequently. In this study, we have considered the unsteady flow past a shrinking sheet immersed in power-law fluids, where $0 < n < 1$ represents pseudoplastic fluids, $n > 1$ for dilatant fluids, and $n = 1$ corresponds to a Newtonian fluid. Figure 1 shows the variations of the skin friction coefficient $[f''(0)]^n$ with $s$ for different values of $n$ when $\beta = -0.5$. The figure indicates that it is possible to obtain dual solutions for a certain range of $s$, with upper and lower branch solutions. It can be seen that $[f''(0)]^n$ increases with an increase in $n$ for the upper branch solutions, and the opposite behaviour is observed for the lower branch solutions. The range of $s$ for which the solution exists increases with $n$. Moreover, stronger suction is necessary for the solution to exist for $0 < n < 1$ (pseudoplastic fluids) compared to that of $n > 1$ (dilatant fluids).

Figure 2 displays the variations of $[f''(0)]^n$ with $\beta$ for different values of $n$ when $s = 2.1$. We noticed that $[f''(0)]^n$ increases with $n$ for the upper branch solutions, whereas the opposite trend occurs for the lower branch solutions. Moreover, for a particular value of $n$, $[f''(0)]^n$ increases with $\beta$ for both of the solution branches. Furthermore, the range of $\beta$ for which the solutions exist increases with a decrease in $n$. Fang et al. [14] studied the similar problem for a viscous fluid and obtained dual solutions for large negative values of $\beta$. As discussed by Merkin [21], Ridha [22], Ishak et al. [23], Harris et al. [24], and very recently by Postelnicu and Pop [25], the lower solution branch is

Table 1. Values of $[f''(0)]^n$ for different values of $n$ and $\beta$.

| $n$ | $\beta$ | $[f''(0)]^n$ NAG routine upper | NAG routine lower | Runge–Kutta–Fehlberg upper | Runge–Kutta–Fehlberg lower |
|---|---|---|---|---|---|
| 1 | 0 | 1.3702 | 0.7298 | 1.3702 | 0.7298 |
| | −0.5 | 1.0933 | 0.4262 | 1.0933 | 0.4262 |
| | −1 | 0.7849 | 0.2106 | 0.7849 | 0.2103 |
| 1.1 | 0 | 1.5409 | 0.5343 | 1.5408 | 0.5343 |
| | −0.5 | 1.2882 | 0.2107 | 1.2882 | 0.2106 |
| | −1 | 0.9826 | 0.0726 | 0.9828 | 0.0721 |

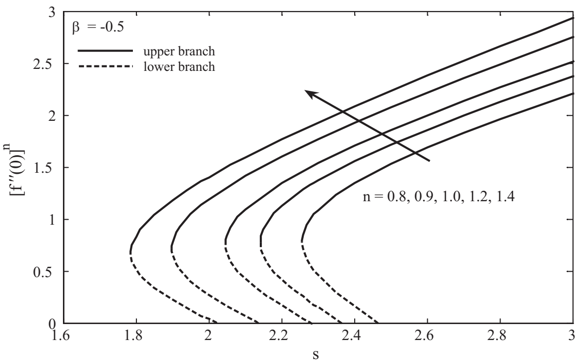


Fig. 1. Variation of the skin friction coefficient $[f''(0)]^n$ with $s$ for different values of $n$ when $\beta = -0.5$.

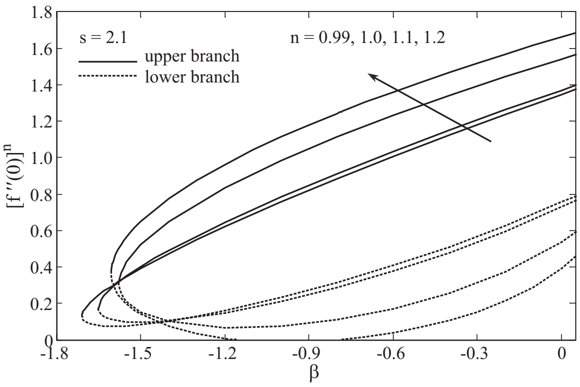


Fig. 2. Variation of the skin friction coefficient $[f''(0)]^n$ with $\beta$ for different values of $n$ when $s = 2.1$.

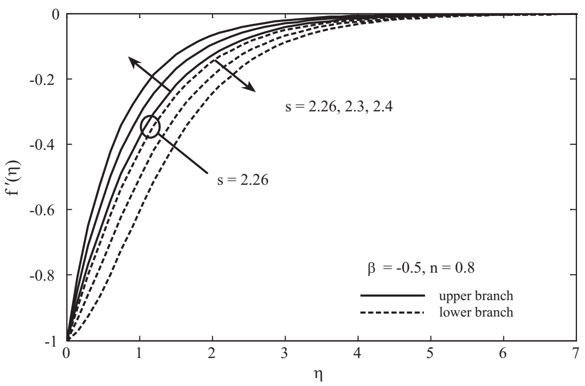


Fig. 3. Velocity profiles $f'(\eta)$ for different values of $s$ when $\beta = -0.5$ and $n = 0.8$.

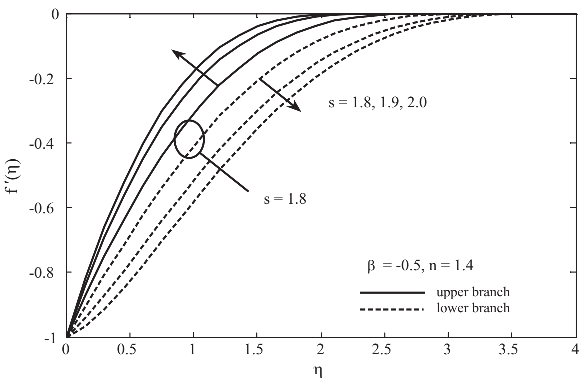


Fig. 4. Velocity profiles $f'(\eta)$ for different values of $s$ when $\beta = -0.5$ and $n = 1.4$.

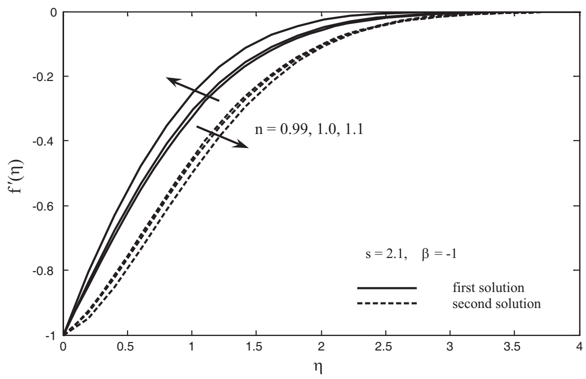


Fig. 5. Velocity profiles $f'(\eta)$ for different values of $n$ when $s = 2.1$ and $\beta = -1$.

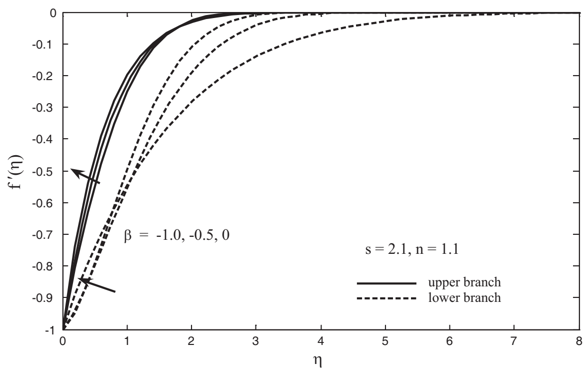


Fig. 6. Velocity profiles $f'(\eta)$ for different values of $\beta$ when $s = 2.1$ and $n = 1.1$.

unstable and not physically realizable. Although such solutions are deprived of physical significance, they are nevertheless of interest so far as the differential equations are concerned. Besides, similar equations may arise in other situations where the corresponding solutions could have more realistic meaning [22].

Figures 3 – 6 illustrate the velocity profiles that support the dual nature of the solutions presented in Figures 1 and 2. Furthermore, it can be seen that the velocity profiles presented in Figures 3 – 6 satisfy the far field boundary conditions (11) asymptotically, thus supporting the validity of the numerical results obtained. Figures 3 and 4 present the velocity profiles for various values of $s$ when the other parameters are fixed. These figures show that the boundary layer thickness decreases with $s$ for the upper branch solutions, which in turn increases the velocity gradient at the surface, hence increases the skin friction coefficient as displayed in Figure 1. However, the opposite behaviours are observed for the lower branch solutions. Moreover, the boundary layer thickness for $n = 0.8$ (pseudoplastic fluid) is greater than that of $n = 1.4$ (dilatant fluid).

Figure 5 presents the velocity profiles for different values of $n$ when $s = 2.1$ and $\beta = -1$. It is seen that the boundary layer thickness decreases with $n$ for the upper branch solutions, in consequence increases the velocity gradient at the surface (skin friction coefficient). This observation is consistent with the graphs presented in Figure 2. However, the opposite trends occur for the lower branch solutions. Finally, the velocity profiles for different values of $\beta$ when $s = 2.1$ and $n = 1.1$ are displayed in Figure 6. It is observed that the velocity gradient at the surface increases with an increase in $\beta$ for both solution branches.

## 4. Conclusions

The unsteady boundary layer flow of a power-law fluid over a shrinking sheet with suction effect at the surface was investigated numerically. The governing partial differential equations were first transformed into a single ordinary differential equation by a similarity transformation, before being solved numerically by the Runge–Kutta–Fehlberg method and the NAG Fortran library subroutine DO2HAF with shooting technique. It was found that the solution exists only if adequate mass suction on the boundary is imposed. Moreover, dual solutions were found to exist only for a certain range of the unsteadiness parameter. Increasing the value of the power-law index $n$ (of the power-law fluid) is to enhance the skin friction coefficient.

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