Bright N-Soliton Solutions to the Vector Hirota Equation from Nonlinear Optics with Symbolic Computation

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Under investigation in this paper is the vector Hirota (VH) equation which governs the simultaneous propagation of multiple interacting femtosecond pulses in a certain type of coupled optical waveguides. By the Nth iterated Darboux transformation starting from the zero potential, the VH equation is found to admit the bright N-soliton solutions in terms of the multi-component Wronskian. Asymptotic formulae of the bright N-soliton solutions are derived for any given set of spectral parameters, which allows us to directly analyze the collision dynamics of VH solitons. Via symbolic computation, some collision properties possessed by the two- and three-soliton solutions are revealed from four aspects: the asymptotic patterns of the colliding solitons, parametric conditions for the amplitude-preserving collisions, phase shifts induced by the vector-soliton collisions, and soliton state changes described by the generalized linear fractional transformations.

Key words: Vector Solitons; Vector Hirota Equation; Multi-Component Wronskian; Darboux Transformation; Symbolic Computation.
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1. Introduction

Vector solitons (VSs) consist of two or more components (modes) that mutually self-trap in a nonlinear medium [1]. Manakov [2] has firstly suggested VSs comprised of two orthogonally polarized components in a nonlinear Kerr medium under the assumption that the self-phase modulation is identical to the cross-phase modulation. VSs have been experimentally observed in planar wave guides [3], birefringent optical fibers [4], photorefractive materials [5], and fiber laser resonators [6]. It has been shown that the VSs can be coupled in different combinations of the bright and dark solitons [7, 8]. For example, the two-component VSs admit the bright–bright, dark–dark, and bright–dark coupled pairs [8].

As a prototype model for the VSs, the vector nonlinear Schrödinger (VNLS) equation [1, 2, 9],

$$\mathrm{i}\,\mathbf{u}_z + \mathbf{u}_{tt} + 2\sigma||\mathbf{u}||^2\mathbf{u} = \mathbf{0}, \quad \mathbf{u} = (u_1, u_2, \dots, u_m), \qquad (1)$$

is completely integrable [2, 9, 10] and has physical relevance in soliton communication systems [1], optical waveguide systems [1, 3], multi-component Bose–Einstein condensates [11], and so on, where $\sigma = \pm 1$ respectively denote the focusing and defocusing cases, u_j represents the jth optical field ($1 \le j \le m$), z and t respectively denote the direction of propagation and retarded time, and $||\cdot||$ is the Euclidean norm. Due to the multi-component structure, the VNLS solitons with internal degrees of freedom possess more complicated collision properties than the scalar ones although their collisions are also considered to be elastic in the sense that the total energy of each colliding soliton is conserved [12 – 14]. Bright VNLS solitons can exhibit the amplitude-preserving collisions with no energy exchange among all the components and amplitude-changing collisions along with energy exchange among the components, depending on the pre-collision soliton parameters [12 – 15]. Recently, amplitude-changing collisions have been observed for the spatial Manakov-like solitons in the Kerr and Kerr-like media [16] and temporal ones in the linearly birefringent optical fibers [4]. Such collisions have the potential applications in implementing the all-optical digital computation by virtue of some virtual logic gates [16, 17].

Based on the work of [12–14, 17], an *indirect* method of analyzing the collisions of bright VNLS solitons can be summarized as follows: (i) An N-soliton collision can be decomposed into $N(N-1)/2$ pairwise collisions because the multi-soliton collision process has been proved to be pairwise and independent of the order in which the collisions occur [12]; (ii) As (1) admits an $SU(m)$ symmetry, the pairwise soliton collisions with $m>2$ components can be reduced to those in the two-component case by a unitary transformation [14]; (iii) Two-soliton collisions of the two-component VNLS solitons are described by a couple of linear fractional transformations (LFTs), by which the operators defined can form a Möbius transformation group [17]. On the other hand, it has been found that the bright N-soliton solutions to the focusing VNLS equation can be represented in terms of the multi-component Wronskian [18, 19]. Moreover, an algebraic procedure has been derived in [18, 19], which can be used to *directly* analyze the collisions of bright VNLS solitons for any given N and m. It is emphasized that the method used in [18, 19] does not require the collisions of VSs to be pairwise, that is to say, such method can be used to directly analyze the simultaneous collisions among three or more VSs.

In this paper, we would like to apply the method used in [18, 19] to a generalized higher-order VNLS equation, i.e., the vector Hirota (VH) equation,

$$\begin{aligned}&\mathrm{i}\,\mathbf{u}_z+\frac{1}{2}\mathbf{u}_{tt}+||\mathbf{u}||^2\mathbf{u}+\mathrm{i}\,\varepsilon\mathbf{u}_{ttt}+3\,\mathrm{i}\,\varepsilon||\mathbf{u}||^2\mathbf{u}_t\\&+3\,\mathrm{i}\,\varepsilon(\mathbf{u}^*\cdot\mathbf{u}_t)\mathbf{u}=\mathbf{0},\;\;\mathbf{u}=(u_1,u_2,\ldots,u_m),\end{aligned}\tag{2}$$

which can be used to describe the propagation dynamics of multiple interacting femtosecond pulses in a certain type of coupled optical waveguides [20], where the parameter ε is the ratio of the width of the spectra to the carrier frequency, and the asterisk stands for complex conjugate. Ultra-short soliton pulses described by (2) might be capable of increasing the transmission capacity of information systems in the form of the wavelength division multiplexing network which can handle more channels with the minimum frequency difference [1].

Equation (2) is also a completely-integrable model [20–22] and possesses the Painlevé property [23], bright N-Soliton solutions [23], bilinear Bäcklund transformation (BT) [24], and Lax pair-based BT [25]. From the perspective of the inverse scattering transform, [26] has reported the 'inelastic'[1] two- and three-soliton collisions occurring in (2). However, some problems have still not been uncovered for the soliton solutions to (2): (i) What is the determinant representation of the general bright N-soliton solutions? (ii) Is the method in [18, 19] applicable to the collision dynamics of VH solitons? (iii) What are the parametric conditions for the occurrence of the amplitude-preserving and amplitude-changing collisions? (iv) Can we explicitly give the formulae for describing the soliton phase shifts and state changes in the collision process?

With symbolic computation [27–32], the structure of this paper will be arranged as follows: In Section 2, we will convert (2) into the vector complex modified Korteweg-de Vries (VCMKdV) equation and construct the Nth iterated Darboux transformation (DT). In Section 3, by virtue of Cramer's rule, we will give the $(m+1)$-component Wronskian representation of the bright N-soliton solutions to (2). In Section 4, we will derive the asymptotic expressions of the bright N-soliton solutions and reveal some properties of the two- and three-soliton collisions. In Section 5, we will address the conclusions.

2. Nth Iterated DT

Through the transformations

$$\begin{aligned}&\mathbf{u}(t,z)=\mathbf{q}(T,Z)\,\mathrm{e}^{\mathrm{i}\left(\frac{t}{6\varepsilon}-\frac{z}{108\varepsilon^2}\right)},\\&Z=\varepsilon z,\;\;T=t-\frac{z}{12\varepsilon},\end{aligned}\tag{3}$$

(2) can be transformed into the following VCMKdV equation:

$$\begin{aligned}&\mathbf{q}_Z+\mathbf{q}_{TTT}+3\,||\mathbf{q}||^2\mathbf{q}_T+3\,(\mathbf{q}^*\cdot\mathbf{q}_T)\mathbf{q}=\mathbf{0},\\&\mathbf{q}=(q_1,q_2,\ldots,q_m),\end{aligned}\tag{4}$$

whose Lax pair is in the $(m+1)\times(m+1)$-matrix form [2, 10]

$$\Psi_T=U(\lambda)\Psi=(\lambda\,U_0+U_1)\,\Psi,$$

[1] The authors think it is more correct to use the term 'amplitude-changing' instead of 'inelastic' because of the vector nature of (2).

$$\Psi_Z = V(\lambda)\Psi = (\lambda^3 V_0 + \lambda^2 V_1 + \lambda V_2 + V_3)\Psi, \tag{5}$$

with

$$U_0 = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & -E_m \end{pmatrix}, \quad U_1 = \begin{pmatrix} 0 & \mathbf{q} \\ -\mathbf{q}^\dagger & \mathbf{0} \end{pmatrix},$$
$$V_0 = -4U_0, \quad V_1 = -4U_1,$$
$$V_2 = -2\begin{pmatrix} \mathbf{q}\mathbf{q}^\dagger & \mathbf{q}_T \\ \mathbf{q}_T^\dagger & \mathbf{q}^\dagger\mathbf{q} \end{pmatrix},$$
$$V_3 = \begin{pmatrix} \mathbf{q}\mathbf{q}_T^\dagger - \mathbf{q}_T\mathbf{q}^\dagger & -\mathbf{q}_{TT} - 2\mathbf{q}\mathbf{q}^\dagger\mathbf{q} \\ \mathbf{q}_{TT}^\dagger + 2\mathbf{q}^\dagger\mathbf{q}\mathbf{q}^\dagger & \mathbf{q}^\dagger\mathbf{q}_T - \mathbf{q}_T^\dagger\mathbf{q} \end{pmatrix},$$

where $\Psi = (\psi_1, \psi_2, \ldots, \psi_{m+1})^{\mathrm{T}}$ (T represents the transpose of a vector) is the vector eigenfunction, λ is the spectral parameter, E_m is the $m\times m$ identity matrix, and the dagger denotes the Hermitian conjugate (i.e., transpose and conjugate). One can check that the compatibility condition $\Psi_{TZ} = \Psi_{ZT}$ is exactly equivalent to (4).

Since the Lax pair of (4) has been obtained, we would like to adopt the DT method [33] to find the determinant representation of soliton solutions. It is known that the DT is such a kind of gauge transformation that leaves the form of a Lax pair invariant, and is in general comprised of the eigenfunction and potential transformations [33]. We assume that the Nth iterated eigenfunction transformation for (5) be of the form

$$\Psi[N] = \Gamma[N](\lambda)\Psi, \tag{6}$$

in which $\Psi[N] = (\psi_1[N], \psi_2[N], \ldots, \psi_{m+1}[N])^{\mathrm{T}}$ is the Nth iterated eigenfunction that satisfies $\Psi_T[N] = U[N](\lambda)\Psi[N]$ and $\Psi_Z[N] = V[N](\lambda)\Psi[N]$ with $U[N](\lambda)$ and $V[N](\lambda)$ being the same as $U(\lambda)$ and $V(\lambda)$ except that $\mathbf{q}$ is replaced by the Nth iterated potential $\mathbf{q}[N] = (q_1[N], \ldots, q_m[N])$, and $\Gamma[N](\lambda)$ is the undetermined Nth iterated Darboux matrix,

$$\Gamma[N](\lambda) = \begin{pmatrix} A[N](\lambda) & B_1[N](\lambda) & \cdots & B_m[N](\lambda) \\ C_1[N](\lambda) & D_{11}[N](\lambda) & \cdots & D_{1m}[N](\lambda) \\ \vdots & \vdots & \ddots & \vdots \\ C_m[N](\lambda) & D_{m1}[N](\lambda) & \cdots & D_{mm}[N](\lambda) \end{pmatrix}, \tag{7}$$

with

$$A[N](\lambda) = \lambda^N - \sum_{n=0}^{N-1} \alpha_n \lambda^n,$$
$$B_j[N](\lambda) = \sum_{n=0}^{N-1} \beta_{jn}(-\lambda)^n, \tag{8}$$
$$C_i[N](\lambda) = -\sum_{n=0}^{N-1} \gamma_{in}\lambda^n,$$

$$D_{ii}[N](\lambda) = \lambda^N + \sum_{n=0}^{N-1} \delta_{ii}^{(n)}(-\lambda)^n,$$
$$D_{ij}[N](\lambda) = \sum_{n=0}^{N-1} \delta_{ij}^{(n)}(-\lambda)^n \quad (i \neq j), \tag{9}$$

where β_{jn}, γ_{in}, and $\delta_{ij}^{(n)}$ $(1 \le i,j \le m;\ 0 \le n \le N-1)$ are the functions of T and Z to be determined.

Suppose that $\Psi_{0k} = (f_k, g_k^{(1)}, \ldots, g_k^{(m)})^{\mathrm{T}}$ is the general solution of (5) which corresponds to $\lambda = \lambda_k$ $(1 \le k \le N)$. We generate a sequence of vector functions $\{\Psi_{jk}\}_{j=1}^m$ which are orthogonal to Ψ_{0k}, with $\Psi_{jk} = (-g_k^{(j)*}, \overbrace{0,\ldots,0}^{j-1}, f_k^*, \overbrace{0,\ldots,0}^{m-j})^{\mathrm{T}}$ for $1 \le j \le m$. Since $\{\Psi_{0k}\}_{k=1}^N$ are linearly independent of one another, the functions α_n, β_{jn}, γ_{in}, and $\delta_{ij}^{(n)}$ $(1 \le i,j \le m;\ 0 \le n \le N-1)$ can be uniquely determined by requiring that

$$\Gamma[N](\lambda_k)\Psi_{0k} = \mathbf{0}, \quad \Gamma[N](-\lambda_k^*)\Psi_{jk} = \mathbf{0} \quad (1 \le k \le N;\ 1 \le j \le m), \tag{10}$$

which can be expanded as

$$f_k A[N](\lambda_k) + \sum_{n=1}^m g_k^{(n)} B_n[N](\lambda_k) = 0,$$
$$f_k^* B_j[N](-\lambda_k^*) - A[N](-\lambda_k^*) g_k^{(j)*} = 0, \tag{11}$$
$$f_k C_i[N](\lambda_k) + \sum_{n=1}^m g_k^{(n)} D_{in}(\lambda_k) = 0,$$
$$f_k^* D_{ij}[N](-\lambda_k^*) - C_i[N](-\lambda_k^*) g_k^{(j)*} = 0, \tag{12}$$

where $1 \le i,j \le m$ and $1 \le k \le N$.

With the Nth iterated Darboux matrix $\Gamma[N](\lambda)$ determined by (11) and (12), one can verify that the spatial- and temporal-flow invariant conditions

$$\Gamma_T[N](\lambda) + \Gamma[N](\lambda)U(\lambda) = U[N](\lambda)\Gamma[N](\lambda), \tag{13a}$$
$$\Gamma_Z[N](\lambda) + \Gamma[N](\lambda)V(\lambda) = V[N](\lambda)\Gamma[N](\lambda), \tag{13b}$$

are satisfied under the Nth iterated potential transformations:

$$\mathbf{q}[N]=\mathbf{q}+2\,(-1)^N\mathbf{b}_{N-1},\ \ \mathbf{q}^*[N]=\mathbf{q}^*+2\,\mathbf{c}_{N-1}, \quad (14)$$

where $\mathbf{b}_{N-1}=(\beta_{1,N-1},\ldots,\beta_{m,N-1})$ and $\mathbf{c}_{N-1}=(\gamma_{1,N-1},\ldots,\gamma_{m,N-1})$ with $(-1)^N\beta_{j,N-1}=\gamma^*_{j,N-1}$ for $1\le j\le m$.

Therefore, (6) and (14) constitute the Nth iterated DT $(\Psi,\mathbf{q})\to(\Psi[N],\mathbf{q}[N])$ for (4) [or equivalently, (2)]. We omit the proof of (13a) and (13b) because the focus of this paper is to analyze the collision dynamics of VH solitons.

3. Bright N-Soliton Solutions in Terms of the $(m+1)$-Component Wronskian

From (11) and (12), Cramer's rule enables us to obtain $\beta_{j,N-1}$ and $\gamma_{j,N-1}$ $(1\le j\le m)$ as

$$\beta_{j,N-1}=(-1)^{jN-1}\frac{\chi_j}{\tau},\qquad \gamma_{j,N-1}=(-1)^{(j-1)N-1}\frac{\bar{\chi}_j}{\tau}, \quad (15)$$

with τ, χ_j, and $\bar{\chi}_j$ being the following $(m+1)$-component Wronskians:

$$\tau=\begin{vmatrix} F_N & -G_N^{(1)} & \cdots & -G_N^{(j)} & \cdots & -G_N^{(m)}\\ G_N^{(1)*} & F_N^* & \cdots & \mathbf{0} & \cdots & \mathbf{0}\\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots\\ G_N^{(j)*} & \mathbf{0} & \cdots & F_N^* & \cdots & \mathbf{0}\\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots\\ G_N^{(m)*} & \mathbf{0} & \cdots & \mathbf{0} & \cdots & F_N^* \end{vmatrix}, \quad (16)$$

$$\chi_j=\begin{vmatrix} F_{N+1} & -G_N^{(1)} & \cdots & -G_{N-1}^{(j)} & \cdots & -G_N^{(m)}\\ G_{N+1}^{(1)*} & F_N^* & \cdots & \mathbf{0} & \cdots & \mathbf{0}\\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots\\ G_{N+1}^{(j)*} & \mathbf{0} & \cdots & F_{N-1}^* & \cdots & \mathbf{0}\\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots\\ G_{N+1}^{(m)*} & \mathbf{0} & \cdots & \mathbf{0} & \cdots & F_N^* \end{vmatrix}, \quad (17)$$

$$\bar{\chi}_j=\begin{vmatrix} F_{N-1} & -G_N^{(1)} & \cdots & -G_{N+1}^{(j)} & \cdots & -G_N^{(m)}\\ G_{N-1}^{(1)*} & F_N^* & \cdots & \mathbf{0} & \cdots & \mathbf{0}\\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots\\ G_{N-1}^{(j)*} & \mathbf{0} & \cdots & F_{N+1}^* & \cdots & \mathbf{0}\\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots\\ G_{N-1}^{(m)*} & \mathbf{0} & \cdots & \mathbf{0} & \cdots & F_N^* \end{vmatrix}, \quad (18)$$

where the block matrices F_M and $G_M^{(j)}$ $(1\le j\le m;\, M=N-1,N,\,N+1)$ are given by

$$F_M=\begin{pmatrix} f_1 & \frac{\partial f_1}{\partial T} & \cdots & \frac{\partial^{M-1} f_1}{\partial T^{M-1}}\\ f_2 & \frac{\partial f_2}{\partial T} & \cdots & \frac{\partial^{M-1} f_2}{\partial T^{M-1}}\\ \vdots & \vdots & \ddots & \vdots\\ f_N & \frac{\partial f_N}{\partial T} & \cdots & \frac{\partial^{M-1} f_N}{\partial T^{M-1}} \end{pmatrix},\qquad G_M^{(j)}=\begin{pmatrix} g_1^{(j)} & \frac{\partial g_1^{(j)}}{\partial T} & \cdots & \frac{\partial^{M-1}\partial g_1^{(j)}}{\partial T^{M-1}}\\ g_2^{(j)} & \frac{\partial g_2^{(j)}}{\partial T} & \cdots & \frac{\partial^{M-1}\partial g_2^{(j)}}{\partial T^{M-1}}\\ \vdots & \vdots & \ddots & \vdots\\ g_N^{(j)} & \frac{\partial g_N^{(j)}}{\partial T} & \cdots & \frac{\partial^{M-1}\partial g_N^{(j)}}{\partial T^{M-1}} \end{pmatrix}. \quad (19)$$

With regard to the multi-component Wronskians τ, χ_j, and $\bar{\chi}_j$, we make the following two remarks: First, the function τ is a complex-valued one which can be expanded as $\tau=\Upsilon(m)\,\tau'$ with $\Upsilon(m)=\mathrm{e}^{\sum_{k=1}^{N}(m-1)\theta_k^*}\prod_{1\le k<l\le N}(\lambda_l^*-\lambda_k^*)^{m-1}$, where τ' is a real-valued function and has no zeros for all $(Z,T)\in\mathbb{R}^2$. That is to say, the solution $\mathbf{q}[N]$ and its complex conjugate $\mathbf{q}^*[N]$ have no singularity in the ZT plane, provided that $\lambda_k\neq\lambda_l$ for $1\le k<l\le N$. Second, $\chi_j/\Upsilon(m)$ is complex conjugate to $\bar{\chi}_j/\Upsilon(m)$, which implies that $(-1)^N\beta_{j,N-1}=\gamma^*_{j,N-1}$ $(1\le j\le m)$. We will prove the above two facts by the Laplace expansion theorem and Binet–Cauchy theorem in a separate paper.

With $\mathbf{q}=\mathbf{0}$ and $\lambda=\lambda_k$ $(1\le k\le N)$, the solutions of (5) can be given as

$$(f_k,g_k^{(1)},g_k^{(2)},\ldots,g_k^{(m)})=(a_k\mathrm{e}^{\theta_k},b_k^{(1)}\mathrm{e}^{-\theta_k},b_k^{(2)}\mathrm{e}^{-\theta_k},\ldots,b_k^{(m)}\mathrm{e}^{-\theta_k}), \quad (20)$$

with $\theta_k = \lambda_k T - 4\lambda_k^3 Z$, a_k and $b_k^{(j)}$ $(1 \le j \le m)$ as complex constants. Thus, the first transformation in (14) yields the Nth iterated solution $\mathbf{q}[N]$ to (4) in the $(m+1)$-component Wronskian form

$$\mathbf{q}[N] = -\frac{2}{\tau}\left[\chi_1, \ldots, (-1)^{(j+1)N}\chi_j, \ldots, (-1)^{(m+1)N}\chi_m\right]. \tag{21}$$

For the case $N = 1$, (21) can be expressed as

$$\mathbf{q}[1] = \frac{|a_1|\kappa_1 \mathbf{B}_1^*}{a_1^* ||\mathbf{B}_1||} \mathrm{e}^{(\lambda_1 - \lambda_1^*)T - 4(\lambda_1^3 - \lambda_1^{*3})Z} \cdot \operatorname{sech}\left(\kappa_1 T - \omega_1 Z + \ln\frac{|a_1|}{||\mathbf{B}_1||}\right), \tag{22}$$

which is called the bright one-soliton solutions if $a_1 \neq 0$ and $\mathbf{B}_1 \neq \mathbf{0}$, where $\mathbf{B}_1 = (b_1^{(1)}, b_1^{(2)}, \ldots, b_1^{(m)})$, $\kappa_1 = \lambda_1 + \lambda_1^*$ is the wave number, $\omega_1 = 4\left(\lambda_1^3 + \lambda_1^{*3}\right)$ is the frequency, $\frac{1}{\kappa_1}\ln\frac{|a_1|}{||\mathbf{B}_1||}$ is the initial phase, and the soliton amplitude and velocity are respectively given by

$$\mathbf{A}_1 = \frac{|\kappa_1|}{||\mathbf{B}_1||}\left(|b_1^{(1)}|, |b_1^{(2)}|, \ldots, |b_1^{(m)}|\right)^{\mathrm{T}}, \quad v_1 = 4\left(\lambda_1^2 - |\lambda_1|^2 + \lambda_1^{*2}\right). \tag{23}$$

For $N \ge 2$, (21) can describe the collision dynamics of the bright N-soliton solutions excluding one singular situation $a_k = 0$ and four reducible situations: (i) $\lambda_k = 0$; (ii) $\lambda_k = \lambda_l^*$; (iii) $\mathbf{B}_k = (b_k^{(1)}, b_k^{(2)}, \ldots, b_k^{(m)}) = \mathbf{0}$; (iv) $\lambda_k^2 + \lambda_k^{*2} - |\lambda_k|^2 = \lambda_l^2 + \lambda_l^{*2} - |\lambda_l|^2$ $(k \neq l)$. Here, the word 'reducible' means that the N-soliton solutions will be reduced to the $(N-1)$-soliton solutions for the first three cases, and to the partially coherent solitons [34, 35] or multi-soliton complexes [36] for the last case (see detailed analysis in [18, 19]). Moreover, we can without loss of generality take $a_k = 1$ for $1 \le k \le N$ in (21), which implies that the bright N-soliton solutions to (4) [or equivalently, (2)] are characterized by $(m+1)N$ complex parameters: λ_k and $b_k^{(j)}$ $(1 \le k \le N; 1 \le j \le m)$. We note that the bright soliton solutions obtained by the Hirota method [23], the ones in terms of the double Wronskian [24], and the ones via the Lax pair-based BT [25] contain less than $(m+1)N$ free parameters. Therefore, (21) is more general than those obtained in [23–25].

4. Asymptotic Analysis of the Bright N-Soliton Solutions

Since the bright N-soliton solutions to (2) also admit the $(m+1)$-component Wronskian representation, in this section we will follow the way in [18, 19] to derive the expressions of asymptotic solitons for the bright N-soliton solutions with any given set of the spectral parameters $\{\lambda_k\}_{k=1}^N$. Then, we will use the method in [18, 19] to analyze the two- and three-soliton collisions. For convenience, throughout this section we define the notations $[n] := \{1, 2, \ldots, n\}$, $[k, n] := \{k, k+1, \ldots, n\}$, and use $|\cdot|$ to denote the number of elements of a set.

4.1. Asymptotic Formulae

For a given set of the spectral parameters $\{\lambda_k\}_{k=1}^N$, we assume that $\mu_k = (\lambda_k + \lambda_k^*)$, $\nu_k = \mathrm{i}(\lambda_k^* - \lambda_k)$ and $r_k = 4(\lambda_k^2 + \lambda_k^{*2} - |\lambda_k|^2) = \mu_k^2 - 3\nu_k^2$ for $1 \le k \le N$. Without loss of generality, the spectral parameters $\{\lambda_k\}_{k=1}^N$ can be ordered according to the relation $r_1 < r_2 < \ldots < r_N$ because (21) is required to be irreducible. Before applying the method in [18, 19] to (21), we give the following three important properties as the basis for us to derive the asymptotic expressions for this bright N-soliton solutions as $Z \to \mp\infty$.

The first is for the limiting states of $\mathrm{Re}(\theta_k) = \frac{1}{2}\mu_k(T - r_k Z)$ as $Z \to \mp\infty$ for $1 \le k \le N$. If $\mathrm{Re}(\theta_n) \sim 0$ in the TZ-plane as $Z \to \mp\infty$ for some $n \in [N]$, we can make use of the relation

$$\mu_n \mathrm{Re}(\theta_k) = \mu_k \mathrm{Re}(\theta_n) + \frac{1}{2}\mu_n\mu_k(r_n - r_k)Z, \tag{24}$$

to determine that, (i) as $Z \to -\infty$, $\mathrm{Re}(\theta_k) \sim -\infty$ for $k \in \mathcal{B}_n^{(\mathrm{I})} \cup \mathcal{B}_n^{(\mathrm{II})}$ and $\mathrm{Re}(\theta_k) \sim +\infty$ for $k \in \mathcal{B}_n^{(\mathrm{III})} \cup \mathcal{B}_n^{(\mathrm{IV})}$; (ii) as $Z \to +\infty$, $\mathrm{Re}(\theta_k) \sim +\infty$ for $k \in \mathcal{B}_n^{(\mathrm{I})} \cup \mathcal{B}_n^{(\mathrm{II})}$ and $\mathrm{Re}(\theta_k) \sim -\infty$ for $k \in \mathcal{B}_n^{(\mathrm{III})} \cup \mathcal{B}_n^{(\mathrm{IV})}$. Here, the sets $\mathcal{B}_n^{(\mathrm{I})}$, $\mathcal{B}_n^{(\mathrm{II})}$, $\mathcal{B}_n^{(\mathrm{III})}$, and $\mathcal{B}_n^{(\mathrm{IV})}$ are defined as $\mathcal{B}_n^{(\mathrm{I})} = \{l | \mu_l > 0 \text{ and } l \in [n-1]\}$, $\mathcal{B}_n^{(\mathrm{II})} = \{l | \mu_l < 0 \text{ and } l \in [n+1, N]\}$, $\mathcal{B}_n^{(\mathrm{III})} = \{l | \mu_l < 0 \text{ and } l \in [n-1]\}$, and $\mathcal{B}_n^{(\mathrm{IV})} = \{l | \mu_l > 0 \text{ and } l \in [n+1, N]\}$.

The second is for the linear relation of phase combinations in the expansions of τ and χ_j $(1 \le j \le m)$. The expansions of τ and χ_j $(1 \le j \le m)$ are the sums of exponential terms with the exponents respectively as the linear phase combinations $\vartheta_\tau = \sum_{k=1}^N \left(c_k^{(\mathrm{I})}\theta_k + d_k^{(\mathrm{I})}\theta_k^*\right)$ and $\vartheta_{\chi_j} = \sum_{k=1}^N \left(c_k^{(\mathrm{II})}\theta_k + d_k^{(\mathrm{II})}\theta_k^*\right)$, where $c_k^{(\mathrm{I})}, c_k^{(\mathrm{II})} \in$

$\{-1,1\}$, $d_k^{(\mathrm{I})}, d_k^{(\mathrm{II})} \in \{m-2, m\}$, $\left|\{k|c_k^{(\mathrm{I})} = -1\}\right| = \left|\{k|d_k^{(\mathrm{I})} = m-2\}\right|$, $\left|\{k|c_k^{(\mathrm{I})} = 1\}\right| = \left|\{k|d_k^{(\mathrm{I})} = m\}\right|$, $\left|\{k|c_k^{(\mathrm{II})} = -1\}\right| = \left|\{k|d_k^{(\mathrm{II})} = m-2\}\right| - 1$, $\left|\{k|c_k^{(\mathrm{II})} = 1\}\right| = \left|\{k|d_k^{(\mathrm{II})} = m\}\right| + 1$.

The third is for the asymptotically-dominating (AD) behaviour of τ and χ_j $(1 \le j \le m)$ as $Z \to \mp\infty$. If $\mathrm{Re}(\theta_n) \sim 0$ in the TZ-plane as $Z \to \mp\infty$ for some $n \in [N]$, then we have $\mathrm{e}^{\vartheta_\tau - \vartheta_{\tau,l}^{\mathrm{AD}}} \sim 0$ or $O(1)$ $(l = 1,2)$ and $\mathrm{e}^{\vartheta_{\chi_j} - \vartheta_{\chi_j}^{\mathrm{AD}}} \sim 0$ or $O(1)$ as $Z \to \mp\infty$, with $\Theta_{\tau,1}^{\mp} = \theta_n + m\theta_n^* + \Xi_n^{\mp}$ and $\Theta_{\tau,2}^{\mp} = (m-2)\theta_n^* - \theta_n + \Xi_n^{\mp}$ as the linear phase combinations associated with the AD terms in the expansion of τ as $Z \to \mp\infty$, and $\Theta_{\chi_j}^{\mp} = \theta_n + (m-2)\theta_n^* + \Xi_n^{\mp}$ as the one associated with the AD term in the expansion of χ_j $(1 \le j \le m)$ as $Z \to \mp\infty$, where $\Xi_n^{\mp} = \sum_{k \ne n} \left[(m-1)\theta_k^* + \sigma_{kn}^{\mp}(\theta_k + \theta_k^*)\right]$, and $\sigma_{kn}^{\mp}$'s are given by

$$\sigma_{kn}^{-} = \begin{cases} -1, & \text{for } k \in \mathcal{B}_n^{(\mathrm{I})} \cup \mathcal{B}_n^{(\mathrm{II})}, \\ 1, & \text{for } k \in \mathcal{B}_n^{(\mathrm{III})} \cup \mathcal{B}_n^{(\mathrm{IV})}, \\ 0, & \text{for } k = n, \end{cases} \qquad \sigma_{kn}^{+} = \begin{cases} 1, & \text{for } k \in \mathcal{B}_n^{(\mathrm{I})} \cup \mathcal{B}_n^{(\mathrm{II})}, \\ -1, & \text{for } k \in \mathcal{B}_n^{(\mathrm{III})} \cup \mathcal{B}_n^{(\mathrm{IV})}, \\ 0, & \text{for } k = n. \end{cases} \tag{25}$$

Therefore, for any given set of the spectral parameters $\{\lambda_k\}_{k=1}^N$, we can obtain the expressions for the nth asymptotic soliton $(1 \le n \le N)$ of (21) as $Z \to \mp\infty$ as follows:

$$\mathbf{S}_n^{\mp} = (s_{1n}^{\mp}, \ldots, s_{jn}^{\mp}, \ldots, s_{mn}^{\mp}) = \frac{-\mathrm{e}^{\theta_n - \theta_n^*}}{2\sqrt{c_n^{\mp} d_n^{\mp}}} \cdot \left[\mathrm{e}_{1n}^{\mp}, \ldots, (-1)^{(j+1)N} \mathrm{e}_{jn}^{\mp}, \ldots, (-1)^{(m+1)N} \mathrm{e}_{mn}^{\mp}\right] \cdot \operatorname{sech}\left(\theta_n + \theta_n^* + \ln\sqrt{\frac{c_n^{\mp}}{d_n^{\mp}}}\right), \tag{26}$$

where $c_n^{\mp}$ and $d_n^{\mp}$ respectively correspond to the coefficients of AD terms associated with $\Theta_{\tau,1}^{\mp}$ and $\Theta_{\tau,2}^{\mp}$ in the expansion of τ, and $e_{nj}^{\mp}$ is the coefficient of AD term associated with $\Theta_{\chi_j}^{\mp}$ in the expansion of χ_j $(1 \le j \le m)$.

4.2. Two-Soliton Collisions

For the two-soliton collisions described by (21) with $N = 2$, we can employ (26) to derive two different asymptotic expressions for the nth colliding soliton $(n = 1,2)$ as follows:

$$\mathbf{S}_n^{(i)} = \frac{\mu_n \mathbf{A}_n^{(i)}}{||\mathbf{A}_n^{(i)}||} \mathrm{e}^{\theta_n - \theta_n^*} \operatorname{sech}\left[\mu_n(T - r_n Z) + \ln \Delta_n^{(i)}\right] \quad (i = 1,2), \tag{27}$$

with

$$\Delta_n^{(1)} = \frac{|\lambda_n - \lambda_{3-n}|^2}{||\mathbf{A}_n^{(1)}||},$$

$$\Delta_n^{(2)} = \frac{|\lambda_{3-n} - \lambda_n^*|^2 ||\mathbf{B}_{3-n}||^2}{||\mathbf{A}_n^{(2)}||},$$

$$\mathbf{A}_n^{(1)} = (a_{1n}^{(1)}, \ldots, a_{mn}^{(1)}) = (\lambda_n - \lambda_{3-n})(\lambda_{3-n} - \lambda_n^*)\mathbf{B}_n^*,$$

$$\mathbf{A}_n^{(2)} = (a_{1n}^{(2)}, \ldots, a_{mn}^{(2)}) = (\lambda_{3-n}^* - \lambda_n) \cdot \left[(\lambda_n^* - \lambda_{3-n})||\mathbf{B}_{3-n}||^2 \mathbf{B}_n^* + (\lambda_{3-n} - \lambda_{3-n}^*)(\mathbf{B}_n^* \cdot \mathbf{B}_{3-n})\mathbf{B}_{3-n}^*\right],$$

where the superscript '(i)' represents the state of each asymptotic soliton, and the dot denotes the product of vectors.

It is easy to find that $\mathbf{A}_n^{(i)}$ and $\Delta_n^{(i)}$ $(1 \le n \le 2; 1 \le i \le 2)$ in (27) are exactly the same as those in the asymptotic expressions of the two-soliton solutions to the focusing VNLS equation [19]. Accordingly, the two-soliton collisions described by (21) with $N = 2$ also admit the following properties:

(i) According to the signs of μ_1 and μ_2, (21) with $N = 2$ is found to admit four kinds of asymptotic patterns as $Z \to \mp\infty$, which are shown in Table 1. Velocities and total energies of each colliding soliton are the same before and after collision, i.e., $v_n^- = v_n^+ = r_n$, $E_n^- = E_n^+ = 2|\mu_n|$. The parametric conditions $r_1 r_2 > 0$ and $r_1 r_2 < 0$, respectively, correspond to the overtaking and head-on collisions between two solitons.

(ii) Either

$$\mathbf{B}_1 \parallel \mathbf{B}_2 \ \text{ or } \ \mathbf{B}_1 \perp \mathbf{B}_2^*, \tag{28}$$

Table 1. Asymptotic patterns of (21) with $N = 2$ under four different parametric conditions.

Parametric conditions	Asymptotic solitons ($Z \to -\infty$)	Asymptotic solitons ($Z \to +\infty$)
$\mu_1 > 0, \mu_2 > 0$	$\mathbf{S}_1^- = \mathbf{S}_1^{(1)}, \mathbf{S}_2^- = \mathbf{S}_2^{(2)}$	$\mathbf{S}_1^+ = \mathbf{S}_1^{(2)}, \mathbf{S}_2^+ = \mathbf{S}_2^{(1)}$
$\mu_1 < 0, \mu_2 > 0$	$\mathbf{S}_1^- = \mathbf{S}_1^{(1)}, \mathbf{S}_2^- = \mathbf{S}_2^{(1)}$	$\mathbf{S}_1^+ = \mathbf{S}_1^{(2)}, \mathbf{S}_2^+ = \mathbf{S}_2^{(2)}$
$\mu_1 > 0, \mu_2 < 0$	$\mathbf{S}_1^- = \mathbf{S}_1^{(2)}, \mathbf{S}_2^- = \mathbf{S}_2^{(2)}$	$\mathbf{S}_1^+ = \mathbf{S}_1^{(1)}, \mathbf{S}_2^+ = \mathbf{S}_2^{(1)}$
$\mu_1 < 0, \mu_2 < 0$	$\mathbf{S}_1^- = \mathbf{S}_1^{(2)}, \mathbf{S}_2^- = \mathbf{S}_2^{(1)}$	$\mathbf{S}_1^+ = \mathbf{S}_1^{(1)}, \mathbf{S}_2^+ = \mathbf{S}_2^{(2)}$

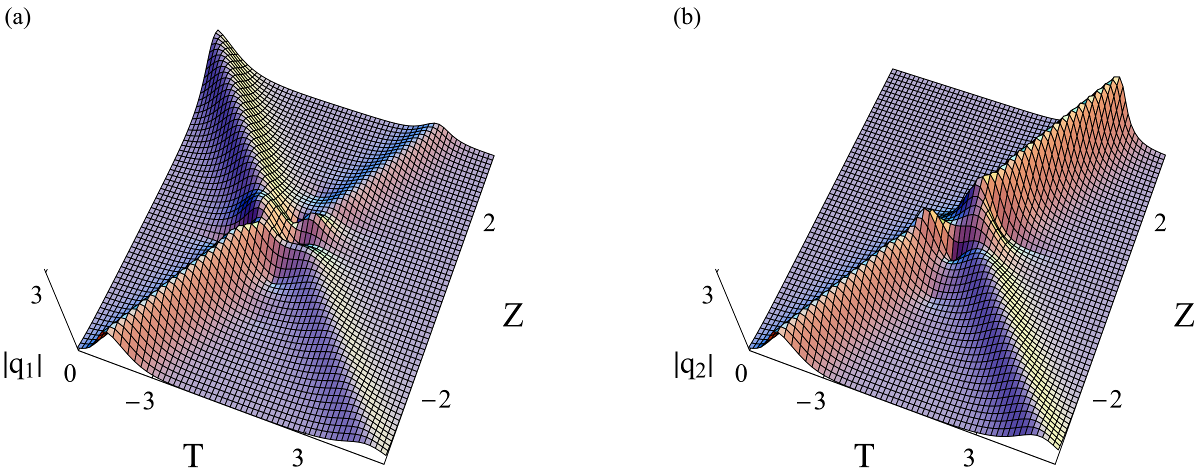

Fig. 1 (colour online). Amplitude-changing collision between two bright vector solitons with $m=2$, where the related parameters are chosen as $\mathbf{B}_1=(1,0)$, $\mathbf{B}_2=(-1,1)$, $\lambda_1=-\frac{3}{2}+\mathrm{i}$, and $\lambda_2=-2+\mathrm{i}$. Note that the second component of the second colliding soliton vanishes after the collision.

being satisfied, the amplitudes for all the components are preserved in the two-soliton collision process; otherwise, the amplitudes for some or all components will change after collision along with the energy exchange among the relevant components (see Figs. 1a and 1b). It is mentioned that the bright two-soliton solutions obtained in [23, 24] are subject to the condition $\mathbf{B}_1 \parallel \mathbf{B}_2$, so they just exhibit the amplitude-preserving collisions. In particular, the amplitude of the jth component ($1 \le j \le m$) for the nth colliding soliton becomes zero if $a_{jn}^{(i)}=0$ for $\mathbf{S}_n^{(i)}$ ($i=1,2$), as displayed in Figure 1a.

(iii) Given the parameters λ_1 and λ_2, the phase shift of the nth colliding soliton is dependent on the angle ϕ_n between the vectors $\mathbf{B}_n^*$ and $\mathbf{B}_{3-n}$ in the explicit form

$$\begin{aligned}\Phi_n^{(1\leftrightarrow 2)} &= \left|\ln\frac{\Delta_n^{(2)}}{\Delta_n^{(1)}}\right| \\ &= \left|\ln\frac{|\lambda_n-\lambda_{3-n}^*|^2}{|\lambda_n-\lambda_{3-n}|\left[|\lambda_n-\lambda_{3-n}^*|^2+(\lambda_n-\lambda_n^*)(\lambda_{3-n}-\lambda_{3-n}^*)|\cos\phi_n|^2\right]^{\frac{1}{2}}}\right|,\end{aligned} \tag{29}$$

where $\Phi_n^{(1\leftrightarrow 2)}$ reaches the maximum or minimum value at $\phi_n=0$ ($\mathbf{B}_n^* \parallel \mathbf{B}_{3-n}$) or $\phi_n=\frac{\pi}{2}$ ($\mathbf{B}_n^* \perp \mathbf{B}_{3-n}$).

(iv) We use the vectors $\rho_n^{(i)}=\left(\rho_{1n}^{(i)},\rho_{2n}^{(i)},\ldots,\rho_{mn}^{(i)}\right)=\left(1,\frac{a_{2n}^{(i)}}{a_{1n}^{(i)}},\ldots,\frac{a_{mn}^{(i)}}{a_{1n}^{(i)}}\right)$ ($i=1,2$) to stand for the two asymptotic states of the nth colliding soliton as $Z\to\mp\infty$. Change of the nth colliding soliton from the state '1' to '2' can be described by the following generalized LFTs:

$$\begin{aligned}\rho_n^{(2)} &= \mathcal{T}_n^{(1\to 2)}\left[\rho_n^{(1)}\right] \\ &= \frac{\left|\left|\rho_{3-n}^{(1)}\right|\right|^2\rho_n^{(1)}+h_n\left(\rho_{3-n}^{(1)*}\cdot\rho_n^{(1)}\right)\rho_{3-n}^{(1)}}{\left|\left|\rho_{3-n}^{(1)}\right|\right|^2+h_n\left(\rho_{3-n}^{(1)*}\cdot\rho_n^{(1)}\right)} \\ &\quad \left(h_n=\frac{\lambda_{3-n}-\lambda_{3-n}^*}{\lambda_n^*-\lambda_{3-n}}\right),\end{aligned} \tag{30}$$

where the operators $\mathcal{T}_n^{(1\to 2)}$ ($n=1,2$) directly reflect the state changes undergone by two colliding solitons which contain m components, *without reducing the m-component soliton collisions to the two-component ones by a unitary transformation* [14].

4.3. Three-Soliton Collisions

For the three-soliton collisions described by (21) with $N=3$, we use (26) to find that the nth colliding soliton ($1\le n\le 3$) has four different asymptotic expressions as follows:

$$\mathbf{S}_n^{(i)}=\frac{\mu_n\mathbf{A}_n^{(i)}}{||\mathbf{A}_n^{(i)}||}\,\mathrm{e}^{\theta_n-\theta_n^*}\operatorname{sech}\left[\mu_n(T-r_nZ)+\ln\Delta_n^{(i)}\right] \quad (1\le i\le 4), \tag{31}$$

with

$$\Delta_n^{(1)}=\frac{|\lambda_n-\lambda_k|^2|\lambda_n-\lambda_{\tilde{k}}|^2}{||\mathbf{A}_n^{(1)}||},$$

$$\Delta_n^{(2)}=\frac{|\lambda_n-\lambda_{\tilde{k}}|^2|\lambda_n-\lambda_k^*|^2\times||\mathbf{B}_k||^2}{||\mathbf{A}_n^{(2)}||}\quad (k<\tilde{k}),$$

$$\Delta_n^{(3)}=\frac{|\lambda_n-\lambda_{\tilde{k}}|^2|\lambda_n-\lambda_k^*|^2\times||\mathbf{B}_k||^2}{||\mathbf{A}_n^{(3)}||}\quad (k>\tilde{k}),$$

$$\Delta_n^{(4)}=\frac{|\lambda_n-\lambda_k^*|^2|\lambda_n-\lambda_{\tilde{k}}^*|^2}{||\mathbf{A}_n^{(4)}||}\big(|\lambda_k-\lambda_{\tilde{k}}^*|^2||\mathbf{B}_k||^2||\mathbf{B}_{\tilde{k}}||^2 +(\lambda_k-\lambda_k^*)(\lambda_{\tilde{k}}-\lambda_{\tilde{k}}^*)|\mathbf{B}_k\cdot\mathbf{B}_{\tilde{k}}^*|^2\big),$$

$$\mathbf{A}_n^{(1)}=\left(a_{1n}^{(1)},\dots,a_{mn}^{(1)}\right)=(\lambda_n-\lambda_k)(\lambda_n-\lambda_{\tilde{k}})\left(\lambda_n^*-\lambda_k\right)\left(\lambda_n^*-\lambda_{\tilde{k}}\right)\mathbf{B}_n^*,$$

$$\mathbf{A}_n^{(2)}=\left(a_{1n}^{(2)},\dots,a_{mn}^{(2)}\right)=\left(\lambda_n-\lambda_{\tilde{k}}\right)\left(\lambda_n^*-\lambda_{\tilde{k}}\right)\left(\lambda_n-\lambda_k^*\right)\cdot\left[(\lambda_k-\lambda_k^*)(\mathbf{B}_k\cdot\mathbf{B}_n^*)\mathbf{B}_k^*+(\lambda_n^*-\lambda_k)||\mathbf{B}_k||^2\,\mathbf{B}_n^*\right]\quad (k<\tilde{k}),$$

$$\mathbf{A}_n^{(3)}=\left(a_{1n}^{(3)},\dots,a_{mn}^{(3)}\right)=\left(\lambda_n-\lambda_{\tilde{k}}\right)\left(\lambda_n^*-\lambda_{\tilde{k}}\right)\left(\lambda_n-\lambda_k^*\right)\cdot\left[(\lambda_k-\lambda_k^*)(\mathbf{B}_k\cdot\mathbf{B}_n^*)\mathbf{B}_k^*+(\lambda_n^*-\lambda_k)||\mathbf{B}_k||^2\,\mathbf{B}_n^*\right]\quad (k>\tilde{k}),$$

$$\begin{aligned}\mathbf{A}_n^{(4)}&=\left(a_{1n}^{(4)},\dots,a_{mn}^{(4)}\right)=(\lambda_n-\lambda_k^*)\left(\lambda_{\tilde{k}}^*-\lambda_n\right)\\&\cdot\big[\Lambda_n^{(\mathrm{I})}||\mathbf{B}_k||^2||\mathbf{B}_{\tilde{k}}||^2\mathbf{B}_n^*-\Lambda_n^{(\mathrm{II})}(\mathbf{B}_k\cdot\mathbf{B}_{\tilde{k}}^*)(\mathbf{B}_{\tilde{k}}\cdot\mathbf{B}_k^*)\mathbf{B}_n^*\\&+\Lambda_n^{(\mathrm{III})}(\mathbf{B}_k\cdot\mathbf{B}_n^*)(\mathbf{B}_{\tilde{k}}\cdot\mathbf{B}_k^*)\mathbf{B}_{\tilde{k}}^*-\Lambda_n^{(\mathrm{IV})}||\mathbf{B}_k||^2(\mathbf{B}_{\tilde{k}}\cdot\mathbf{B}_n^*)\mathbf{B}_{\tilde{k}}^*\\&+\Lambda_n^{(\mathrm{III})}(\mathbf{B}_k\cdot\mathbf{B}_{\tilde{k}}^*)(\mathbf{B}_{\tilde{k}}\cdot\mathbf{B}_n^*)\mathbf{B}_k^*-\Lambda_n^{(\mathrm{IV})}||\mathbf{B}_{\tilde{k}}||^2(\mathbf{B}_k\cdot\mathbf{B}_n^*)\mathbf{B}_k^*\big],\end{aligned}$$

where $k\in\{1,2,3\}$, $k\neq n$, $\tilde{k}=6-k-n$, $\Lambda_n^{(\mathrm{I})}$, $\Lambda_n^{(\mathrm{II})}$, $\Lambda_n^{(\mathrm{III})}$, and $\Lambda_n^{(\mathrm{IV})}$ are given by

$$\Lambda_n^{(\mathrm{I})}=\lambda_k^2\left(\lambda_n^*\lambda_k^*+|\lambda_{\tilde{k}}|^2\right)+\lambda_n^{*2}\left(\lambda_k-\lambda_{\tilde{k}}^*\right)\left(\lambda_{\tilde{k}}-\lambda_k^*\right)+\lambda_{\tilde{k}}^2\left(|\lambda_k|^2+\lambda_n^*\lambda_{\tilde{k}}^*\right),$$

$$\Lambda_n^{(\mathrm{II})}=\lambda_k^2\left(\lambda_{\tilde{k}}\lambda_k^*+\lambda_{\tilde{k}}^*\lambda_n^*\right)+\lambda_n^{*2}\left(\lambda_k-\lambda_k^*\right)\left(\lambda_{\tilde{k}}-\lambda_{\tilde{k}}^*\right)+\lambda_{\tilde{k}}^2\left(\lambda_k\lambda_{\tilde{k}}^*+\lambda_k^*\lambda_n^*\right),$$

$$\Lambda_n^{(\mathrm{III})}=\lambda_{\tilde{k}}^2\left(\lambda_k\lambda_{\tilde{k}}^*+\lambda_n^*\lambda_k^*\right)+\lambda_k^{*2}\left(\lambda_k-\lambda_n^*\right)\left(\lambda_{\tilde{k}}-\lambda_{\tilde{k}}^*\right)+\lambda_k^2\left(\lambda_{\tilde{k}}\lambda_n^*+\lambda_{\tilde{k}}^*\lambda_k^*\right),$$

$$\Lambda_n^{(\mathrm{IV})}=\lambda_{\tilde{k}}^2\left(\lambda_k\lambda_n^*+\lambda_{\tilde{k}}^*\lambda_k^*\right)+\lambda_k^{*2}\left(\lambda_{\tilde{k}}-\lambda_n^*\right)\left(\lambda_k-\lambda_{\tilde{k}}^*\right)+\lambda_k^2\left(|\lambda_{\tilde{k}}|^2+\lambda_n^*\lambda_k^*\right).$$

It should be noted that the asymptotic states $\mathbf{S}_n^{(1)}$ and $\mathbf{S}_n^{(4)}$ are independent of whether $k>\tilde{k}$ or $k<\tilde{k}$, while the asymptotic states $\mathbf{S}_n^{(2)}$ and $\mathbf{S}_n^{(3)}$ can be exchanged to each other if k is in exchange with $\tilde{k}$.

Also, $\mathbf{A}_n^{(i)}$ and $\Delta_n^{(i)}$ in (31) are the same as those in the asymptotic expressions of the three-soliton solutions to the focusing VNLS equation [19]. By analyzing (31), we obtain the following properties for the three-soliton collisions described by (21) with $N=3$:

(i) Depending on the signs of μ_1, μ_2, and μ_3, (21) with $N=3$ can exhibit eight kinds of asymptotic patterns as $Z\to\mp\infty$, which are listed in Table 2. Since the velocities and total energies of each colliding soliton are still the same before and after the collision, the three-soliton collisions are also elastic. If

Parametric conditions	Asymptotic solitons ($Z\to-\infty$)	Asymptotic solitons ($Z\to+\infty$)
$\mu_1>0,\mu_2>0,\mu_3>0$	$\mathbf{S}_1^-=\mathbf{S}_1^{(1)},\mathbf{S}_2^-=\mathbf{S}_2^{(2)},\mathbf{S}_3^-=\mathbf{S}_3^{(4)}$	$\mathbf{S}_1^+=\mathbf{S}_1^{(4)},\mathbf{S}_2^+=\mathbf{S}_2^{(3)},\mathbf{S}_3^+=\mathbf{S}_3^{(1)}$
$\mu_1<0,\mu_2>0,\mu_3>0$	$\mathbf{S}_1^-=\mathbf{S}_1^{(1)},\mathbf{S}_2^-=\mathbf{S}_2^{(1)},\mathbf{S}_3^-=\mathbf{S}_3^{(3)}$	$\mathbf{S}_1^+=\mathbf{S}_1^{(4)},\mathbf{S}_2^+=\mathbf{S}_2^{(4)},\mathbf{S}_3^+=\mathbf{S}_3^{(2)}$
$\mu_1>0,\mu_2<0,\mu_3>0$	$\mathbf{S}_1^-=\mathbf{S}_1^{(2)},\mathbf{S}_2^-=\mathbf{S}_2^{(2)},\mathbf{S}_3^-=\mathbf{S}_3^{(2)}$	$\mathbf{S}_1^+=\mathbf{S}_1^{(3)},\mathbf{S}_2^+=\mathbf{S}_2^{(3)},\mathbf{S}_3^+=\mathbf{S}_3^{(3)}$
$\mu_1>0,\mu_2>0,\mu_3<0$	$\mathbf{S}_1^-=\mathbf{S}_1^{(3)},\mathbf{S}_2^-=\mathbf{S}_2^{(4)},\mathbf{S}_3^-=\mathbf{S}_3^{(4)}$	$\mathbf{S}_1^+=\mathbf{S}_1^{(2)},\mathbf{S}_2^+=\mathbf{S}_2^{(1)},\mathbf{S}_3^+=\mathbf{S}_3^{(1)}$
$\mu_1<0,\mu_2<0,\mu_3>0$	$\mathbf{S}_1^+=\mathbf{S}_1^{(2)},\mathbf{S}_2^+=\mathbf{S}_2^{(1)},\mathbf{S}_3^+=\mathbf{S}_3^{(1)}$	$\mathbf{S}_1^-=\mathbf{S}_1^{(3)},\mathbf{S}_2^-=\mathbf{S}_2^{(4)},\mathbf{S}_3^-=\mathbf{S}_3^{(4)}$
$\mu_1<0,\mu_2>0,\mu_3<0$	$\mathbf{S}_1^-=\mathbf{S}_1^{(3)},\mathbf{S}_2^-=\mathbf{S}_2^{(3)},\mathbf{S}_3^-=\mathbf{S}_3^{(3)}$	$\mathbf{S}_1^+=\mathbf{S}_1^{(2)},\mathbf{S}_2^+=\mathbf{S}_2^{(2)},\mathbf{S}_3^+=\mathbf{S}_3^{(2)}$
$\mu_1>0,\mu_2<0,\mu_3<0$	$\mathbf{S}_1^-=\mathbf{S}_1^{(4)},\mathbf{S}_2^-=\mathbf{S}_2^{(4)},\mathbf{S}_3^-=\mathbf{S}_3^{(2)}$	$\mathbf{S}_1^+=\mathbf{S}_1^{(1)},\mathbf{S}_2^+=\mathbf{S}_2^{(1)},\mathbf{S}_3^+=\mathbf{S}_3^{(3)}$
$\mu_1<0,\mu_2<0,\mu_3<0$	$\mathbf{S}_1^-=\mathbf{S}_1^{(4)},\mathbf{S}_2^-=\mathbf{S}_2^{(3)},\mathbf{S}_3^-=\mathbf{S}_3^{(1)}$	$\mathbf{S}_1^+=\mathbf{S}_1^{(1)},\mathbf{S}_2^+=\mathbf{S}_2^{(2)},\mathbf{S}_3^+=\mathbf{S}_3^{(3)}$

Table 2. Asymptotic patterns of (21) with $N=3$ under eight different parametric conditions.

$r_1, r_2, r_3 > 0$ or $r_1, r_2, r_3 < 0$, then the overtaking collisions take place among three solitons; for other parametric choices, the three-soliton collisions are combined of the overtaking and head-on collisions.

(ii) Only under the condition

$$\mathbf{B}_n \parallel \mathbf{B}_k \;\text{ or }\; \mathbf{B}_n \perp \mathbf{B}_k^*, \;\; (1 \le k \le 3; k \ne n), \tag{32}$$

the amplitude-preserving collisions occur for all components of three colliding solitons; or else, the amplitude-changing collisions take place along with the energy exchange among some or all the components. In particular, the jth component ($1 \le j \le m$) for the nth colliding soliton vanishes if $a_{jn}^{(i)} = 0$ for $\mathbf{S}_n^{(i)}$ ($1 \le i \le 4$).

(iii) $\mathbf{S}_n^{(1)}$ and $\mathbf{S}_n^{(4)}$ always appear in pairs to be the asymptotic states of the nth colliding soliton as $Z \to \mp\infty$, so do $\mathbf{S}_n^{(2)}$ and $\mathbf{S}_n^{(3)}$. Thus, the explicit formulae of the phase shifts for the nth colliding soliton, which changes from the state '2' to '3' and from '1' to '4', are respectively given by

$$\begin{aligned}\Phi_n^{(2\leftrightarrow 3)} &= \left|\ln\frac{\Delta_n^{(3)}}{\Delta_n^{(2)}}\right| = \left|\ln\frac{|\lambda_n-\lambda_k||\lambda_n-\lambda_{\tilde k}^*|^2}{|\lambda_n-\lambda_{\tilde k}||\lambda_n-\lambda_k^*|^2}\right. \\ &\quad\cdot\left.\left\{\frac{|(\lambda_k-\lambda_k^*)\cos\phi_{(n,k)}|^2+|\lambda_n^*-\lambda_k|^2+2\,\mathrm{Re}[(\lambda_k-\lambda_k^*)(\lambda_n-\lambda_k^*)\cos\phi_{(n,k)}\cos\phi_{(k,n)}]}{|(\lambda_{\tilde k}-\lambda_{\tilde k}^*)\cos\phi_{(n,\tilde k)}|^2+|\lambda_n^*-\lambda_{\tilde k}|^2+2\,\mathrm{Re}[(\lambda_{\tilde k}-\lambda_{\tilde k}^*)(\lambda_n-\lambda_{\tilde k}^*)\cos\phi_{(n,\tilde k)}\cos\phi_{(\tilde k,n)}]}\right\}^{\frac12}\right|,\end{aligned} \tag{33}$$

$$\begin{aligned}\Phi_n^{(1\leftrightarrow 4)} &= \left|\ln\frac{\Delta_n^{(4)}}{\Delta_n^{(1)}}\right| = \left|\ln\frac{|\lambda_n-\lambda_k^*|^2|\lambda_n-\lambda_{\tilde k}^*|^2}{|\lambda_n-\lambda_k||\lambda_n-\lambda_{\tilde k}|}\right. \\ &\quad\cdot\left.\frac{|\lambda_k-\lambda_{\tilde k}^*|^2+(\lambda_k-\lambda_k^*)(\lambda_{\tilde k}-\lambda_{\tilde k}^*)|\cos\phi_{(k,\tilde k)}|^2}{\left[\sum_{l\in\{n,k,\tilde k\}}|\eta_l|^2+2\,\mathrm{Re}\big(\eta_n^*\eta_k\cos\phi_{(n,k)}+\eta_k^*\eta_{\tilde k}\cos\phi_{(k,\tilde k)}+\eta_{\tilde k}^*\eta_n\cos\phi_{(\tilde k,n)}\big)\right]^{\frac12}}\right|,\end{aligned} \tag{34}$$

with

$$\begin{aligned}\eta_n &= \Lambda_n^{(\mathrm{I})}-\Lambda_n^{(\mathrm{II})}\big|\cos\phi_{(k,\tilde k)}\big|^2,\\ \eta_k &= \Lambda_n^{(\mathrm{III})}\cos\phi_{(n,\tilde k)}\cos\phi_{(\tilde k,k)}-\Lambda_n^{(\mathrm{IV})}\cos\phi_{(n,k)},\\ \eta_{\tilde k} &= \Lambda_n^{(\mathrm{III})}\cos\phi_{(n,k)}\cos\phi_{(k,\tilde k)}-\Lambda_n^{(\mathrm{IV})}\cos\phi_{(n,\tilde k)},\end{aligned}$$

where $\phi_{(l_1,l_2)}$ represents the angle between $\mathbf{B}_{l_1}^*$ and $\mathbf{B}_{l_2}$ ($1 \le l_1, l_2 \le 3; l_1 \ne l_2$).

(iv) We also use the vectors $\rho_n^{(i)} = \big(\rho_{1n}^{(i)}, \rho_{2n}^{(i)}, \ldots, \rho_{mn}^{(i)}\big) = \left(1, \frac{a_{2n}^{(i)}}{a_{1n}^{(i)}}, \ldots, \frac{a_{mn}^{(i)}}{a_{1n}^{(i)}}\right)$ ($1 \le i \le 4$) to stand for the four asymptotic states of the nth colliding soliton as $Z \to \mp\infty$. Therefore, the changes of the nth colliding soliton from the state '1' to other three states obey the following three generalized LFTs:

$$\begin{aligned}\rho_n^{(2)} &= \mathcal{T}_n^{(1\to 2)}\big[\rho_n^{(1)},\rho_k^{(1)}\big]\\ &= \frac{||\rho_k^{(1)}||^2\rho_n^{(1)}+h_{(n,k)}\big(\rho_k^{(1)*}\cdot\rho_n^{(1)}\big)\rho_k^{(1)}}{||\rho_k^{(1)}||^2+h_{(n,k)}\big(\rho_k^{(1)*}\cdot\rho_n^{(1)}\big)}\\ &\quad (k<\tilde k),\end{aligned} \tag{35}$$

$$\begin{aligned}\rho_n^{(3)} &= \mathcal{T}_n^{(1\to 3)}\big[\rho_n^{(1)},\rho_k^{(1)}\big]\\ &= \frac{||\rho_k^{(1)}||^2\rho_n^{(1)}+h_{(n,k)}\big(\rho_k^{(1)*}\cdot\rho_n^{(1)}\big)\rho_k^{(1)}}{||\rho_k^{(1)}||^2+h_{(n,k)}\big(\rho_k^{(1)*}\cdot\rho_n^{(1)}\big)}\\ &\quad (k>\tilde k),\end{aligned} \tag{36}$$

$$\begin{aligned}\rho_n^{(4)} &= \mathcal{T}_n^{(1\to 4)}\big[\rho_n^{(1)},\rho_k^{(1)},\rho_{\tilde k}^{(1)}\big]\\ &= \frac{\zeta_n\rho_n^{(1)*}+\zeta_k\rho_k^{(1)*}+\zeta_{\tilde k}\rho_{\tilde k}^{(1)*}}{\zeta_n+\zeta_k+\zeta_{\tilde k}},\end{aligned} \tag{37}$$

with $h_{(n,k)} = \frac{\lambda_k-\lambda_k^*}{\lambda_n^*-\lambda_k}$, and $\zeta_n, \zeta_k, \zeta_{\tilde k}$ expressed by

$$\begin{aligned}\zeta_n &= \Lambda_n^{(\mathrm{I})}||\rho_k^{(1)}||^2||\rho_{\tilde k}^{(1)}||^2\\ &\quad -\Lambda_n^{(\mathrm{II})}\big(\rho_k^{(1)}\cdot\rho_{\tilde k}^{(1)*}\big)\big(\rho_{\tilde k}^{(1)}\cdot\rho_k^{(1)*}\big),\\ \zeta_k &= \Lambda_n^{(\mathrm{III})}\big(\rho_k^{(1)}\cdot\rho_{\tilde k}^{(1)*}\big)\big(\rho_{\tilde k}^{(1)}\cdot\rho_n^{(1)*}\big)\\ &\quad -\Lambda_n^{(\mathrm{IV})}||\rho_{\tilde k}^{(1)}||^2\big(\rho_k^{(1)}\cdot\rho_n^{(1)*}\big),\\ \zeta_{\tilde k} &= \Lambda_n^{(\mathrm{III})}\big(\rho_k^{(1)}\cdot\rho_n^{(1)*}\big)\big(\rho_{\tilde k}^{(1)}\cdot\rho_k^{(1)*}\big)\\ &\quad -\Lambda_n^{(\mathrm{IV})}||\rho_k^{(1)}||^2\big(\rho_{\tilde k}^{(1)}\cdot\rho_n^{(1)*}\big),\end{aligned}$$

where the operators $\mathcal{T}_n^{(1\to 2)}$, $\mathcal{T}_n^{(1\to 3)}$, and $\mathcal{T}_n^{(1\to 4)}$ $(1 \leq n \leq 3)$ suffice to directly describe the soliton state changes for any three-soliton collision pattern as given in Table 2, *without decomposing it into three pairwise collisions*.

5. Conclusions

Amplitude-changing collisions of VSs along with the energy exchange among the components have attracted certain attention [2–4, 9–19]. Such collisions have some potential applications in nonlinear optics such as the all-optical switching, logic and computation [16, 17]. In this paper, we have applied the method used in [18, 19] to the VH equation, i.e., (2), which models the m interacting femtosecond soliton pulses simultaneously propagating in a certain type of coupled optical waveguides [20]. By using the Nth iterated DT, we have found that the bright N-soliton solutions to (2) can be expressed as the rational fraction of $(m+1)$-component Wronskians in (16) and (17). Note that the bright N-soliton solutions in (21) are more general than those obtained in [23–25] because they contain the $(m+1)N$ complex parameters. We have also analyzed some algebraic properties of the $(m+1)$-component Wronskians in (16) and (17), and derived the asymptotic formulae in (26) of (21) as $Z \to \mp\infty$ for any given set of spectral parameters $\{\lambda_k\}_{k=1}^N$. Furthermore, we have revealed some properties for the two- and three-soliton collisions, including the asymptotic patterns of (21) with $N = 2$ and 3 (see Tables 1 and 2), parametric conditions in (28) and (32) for the preservation of all soliton components in the collision, phase-shift formulae in (29), (33), and (34) induced by the vector-soliton collisions, and generalized LFTs in (30), (35)–(37) for directly describing the state changes undergone by the colliding vector solitons.

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