

# General Solutions for the Unsteady Flow of Second-Grade Fluids over an Infinite Plate that Applies Arbitrary Shear to the Fluid

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General solutions corresponding to the unsteady motion of second-grade fluids induced by an infinite plate that applies a shear stress  $f(t)$  to the fluid are established. These solutions can immediately be reduced to the similar solutions for Newtonian fluids. They can be used to obtain known solutions from the literature or any other solution of this type by specifying the function  $f(\cdot)$ . Furthermore, in view of a simple remark, general solutions for the flow due to a moving plate can be developed.

*Key words:* General Solutions; Second-Grade Fluids; Infinite Plate; Shear Stress.

## 1. Introduction

The classical viscous Newtonian fluid model cannot describe flows of many polymeric liquids and biological fluids, and so various non-Newtonian fluid models have been proposed to describe them. Some of the non-Newtonian fluids, especially dilute polymeric solutions as well as some biological fluids, can well enough be described by non-Newtonian fluids of differential type. The simplest differential-type fluid is the incompressible fluid of second grade. It has been widely used as a first approximation to explain the normal stress differences. Although this model has been used to study a variety of flow problems, there is some controversy concerning the nature of the material moduli that characterize the fluid. Thus, any additional results that can help to clarify its status and usage would be welcome, especially in view of its extensive use.

The flow of a second-grade fluid over an infinite plate, with suitable boundary and initial conditions, has been investigated by many authors. It can be realized if the plate is moving in its plane or applies a tangential shear stress to the fluid. In the second case, unlike the usual no slip condition, a boundary condition on the shear stress is used. This is very important as in some problems, what is specified is the force applied on the boundary. It is also important to bear in mind that the ‘no slip’ boundary condition may not be necessarily

applicable to flows of polymeric fluids that can slip or slide on the boundary. Thus, the shear stress boundary condition is particularly meaningful. To the best of our knowledge, the first exact solutions for motions of non-Newtonian fluids in which the shear stress is given on the boundary are those of Waters and King [1] and Bandelli et al. [2]. Meanwhile, other exact solutions for different motions of viscous and second-grade fluids have been established [3–9].

The purpose of this note is to provide general solutions for the unsteady motion of a second-grade fluid induced by an infinite plate that applies a shear stress  $f(t)$  to the fluid. In addition to being a study of a general time-dependent problem, it leads to exact solutions. Such solutions are uncommon in the literature and they provide an important check for numerical methods that are used to study flows of such fluids in a complex domain. For generality, the solutions are firstly established for the motion between two parallel walls perpendicular to the plate. These solutions, in the absence of the side walls, reduce to the similar solutions over an infinite plate. In order to illustrate their importance, some special cases are considered and known solutions from the literature are recovered. Finally, relying on an immediate consequence of the governing equations, an important relation with the motion over a moving plate is brought to light.

## 2. Flow Between Side Walls Perpendicular to a Plate

Consider an incompressible second-grade fluid at rest occupying the space above an infinite plate perpendicular to the  $y$ -axis and between two side walls situated in the planes  $z = 0$  and  $z = d$  of a fixed Cartesian coordinate system  $x, y$ , and  $z$ . At time  $t = 0^+$  the plate is pulled with the time-dependent shear stress  $f(t)$  along the  $x$ -axis and  $f(0) = 0$ . Owing to the shear the fluid is gradually moved and its velocity is of the form

$$\mathbf{v} = \mathbf{v}(y, z, t) = u(y, z, t) \mathbf{i}, \quad (1)$$

where  $\mathbf{i}$  is a unit vector along the  $x$ -direction. For such a flow the constraint of incompressibility is automatically satisfied while the governing equation is given by [5, 6]

$$\frac{\partial u(y, z, t)}{\partial t} = \left( \nu + \alpha \frac{\partial}{\partial t} \right) \left( \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) u(y, z, t);$$

$y, t > 0$  and  $z \in (0, d)$ , (2)

where  $\nu$  is the kinematic viscosity and  $\alpha = \alpha_1/\rho$  ( $\alpha_1$  is a material constant and  $\rho$  is the density of the fluid). The appropriate initial and boundary conditions are

$$u(y, z, 0) = 0 \text{ for } y > 0 \text{ and } z \in [0, d], \quad (3a)$$

$$\tau(0, z, t) = \left( \mu + \alpha_1 \frac{\partial}{\partial t} \right) \frac{\partial u(y, z, t)}{\partial y} \Big|_{y=0} = f(t) \text{ for } z \in (0, d) \text{ and } t > 0, \quad (3b)$$

$$u(y, 0, t) = u(y, d, t) = 0 \text{ for } y, t > 0; \\ u(y, z, t) \rightarrow 0 \text{ as } y \rightarrow \infty. \quad (3c)$$

In (3b)  $\mu = \rho\nu$  is the dynamic viscosity of the fluid and  $\tau(y, z, t) = S_{xy}(y, z, t)$  is one of the non-trivial shear stresses.

In order to solve this initial and boundary value problem, we use the Fourier transforms [10, 11]. Consequently, multiplying (2) by  $\sqrt{2/\pi} \cos(y\xi) \sin(\lambda_n z)$ , integrating the result with respect to  $y$  from 0 to  $\infty$  and  $z$  from 0 to  $d$ , respectively, and taking into account the conditions (3), we obtain

$$\frac{\partial u_{cn}(\xi, t)}{\partial t} + \frac{\nu(\xi^2 + \lambda_n^2)}{1 + \alpha(\xi^2 + \lambda_n^2)} u_{cn}(\xi, t) \\ = -\sqrt{\frac{2}{\pi}} \frac{f(t)}{\rho \lambda_n} \frac{1 - (-1)^n}{1 + \alpha(\xi^2 + \lambda_n^2)}; \quad \xi, t > 0, \quad (4)$$

where  $\lambda_n = n\pi/d$  and the double Fourier sine and cosine transform  $u_{cn}(\xi, t)$  of  $u(y, z, t)$  must satisfy the initial condition

$$u_{cn}(\xi, 0) = 0 \text{ for } \xi > 0. \quad (5)$$

Inverting this result by means of the Fourier inversion formulae [10, 11], setting  $d = 2h$ , and changing the origin of the coordinate system to the middle of the channel, we can write the velocity field  $u(y, z, t)$  in the suitable form

$$u(y, z, t) = \frac{4}{\rho \pi h} \sum_{n=1}^{\infty} \frac{(-1)^n \cos(\mu_m z)}{\mu_m} \\ \cdot \int_0^{\infty} \frac{\cos(y\xi)}{1 + \alpha(\xi^2 + \mu_m^2)} \\ \cdot \int_0^t f(s) \exp \left[ -\frac{\nu(\xi^2 + \mu_m^2)(t-s)}{1 + \alpha(\xi^2 + \mu_m^2)} \right] ds d\xi, \quad (6)$$

where  $\mu_m = (2n-1)\pi/(2h)$ .

In order to determine the shear stress in planes parallel to the bottom wall, as well as the shear stress on the side walls, the expressions of the non-trivial shear stresses are needed. The first of these, for instance, has the form

$$\tau(y, z, t) = -\frac{2f(t)}{h} \sum_{n=1}^{\infty} \frac{(-1)^n \cos(\mu_m z)}{\mu_m} \\ \cdot \left\{ e^{-\mu_m y} - \frac{2}{\pi} \int_0^{\infty} \frac{\xi \sin(y\xi)}{(\xi^2 + \mu_m^2)[1 + \alpha(\xi^2 + \mu_m^2)]} d\xi \right\} \\ - \frac{4\nu}{\pi h} \sum_{n=1}^{\infty} \frac{(-1)^n \cos(\mu_m z)}{\mu_m} \int_0^{\infty} \frac{\xi \sin(y\xi)}{[1 + \alpha(\xi^2 + \mu_m^2)]^2} \\ \cdot \int_0^t f(s) \exp \left[ -\frac{\nu(\xi^2 + \mu_m^2)(t-s)}{1 + \alpha(\xi^2 + \mu_m^2)} \right] ds d\xi. \quad (7)$$

Taking  $\alpha \rightarrow 0$  into above relations, the similar solutions

$$u_N(y, z, t) = \frac{4}{\rho \pi h} \sum_{n=1}^{\infty} \frac{(-1)^n \cos(\mu_m z)}{\mu_m} \int_0^{\infty} \cos(y\xi) \\ \cdot \int_0^t f(s) e^{-\nu(\xi^2 + \mu_m^2)(t-s)} ds d\xi, \quad (8)$$

$$\tau_N(y, z, t) = -\frac{4\nu}{\pi h} \sum_{n=1}^{\infty} \frac{(-1)^n \cos(\mu_m z)}{\mu_m} \int_0^{\infty} \xi \sin(y\xi) \cdot \int_0^t f(s) e^{-\nu(\xi^2 + \mu_m^2)(t-s)} ds d\xi, \tag{9}$$

corresponding to a Newtonian fluid performing the same motion, are obtained. In view of the entry 5 of Table 4 from [11] and its immediate consequence

$$\int_0^{\infty} \xi \sin(y\xi) e^{-\nu t \xi^2} d\xi = \frac{y}{4\nu t} \sqrt{\frac{\pi}{\nu t}} \exp\left(-\frac{y^2}{4\nu t}\right),$$

the Solutions (8) and (9) can be written under the simplified forms

$$u_N(y, z, t) = \frac{2}{\rho h \sqrt{\nu \pi}} \sum_{n=1}^{\infty} \frac{(-1)^n \cos(\mu_m z)}{\mu_m} \cdot \int_0^t \frac{f(t-s)}{\sqrt{s}} \exp\left(-\frac{y^2}{4\nu s} - \nu \mu_m^2 s\right) ds, \tag{10}$$

$$\tau_N(y, z, t) = -\frac{y}{h \sqrt{\nu \pi}} \sum_{n=1}^{\infty} \frac{(-1)^n \cos(\mu_m z)}{\mu_m} \cdot \int_0^t \frac{f(t-s)}{s \sqrt{s}} \exp\left(-\frac{y^2}{4\nu s} - \nu \mu_m^2 s\right) ds. \tag{11}$$

Integrating by parts the last integrals from (8) and (9) and using the entries 6 and 7 of Tables 4 and 5 from [11], the Newtonian solutions can also be written in equivalent forms

$$u_N(y, z, t) = \frac{2f(t)}{\mu h} \sum_{n=1}^{\infty} \frac{(-1)^n \cos(\mu_m z)}{\mu_m^2} \cdot e^{-\mu_m y} - \frac{1}{\mu h} \sum_{n=1}^{\infty} \frac{(-1)^n \cos(\mu_m z)}{\mu_m^2} \cdot \int_0^t f'(t-s) \left\{ e^{-\mu_m y} \operatorname{Erfc}\left(\mu_m \sqrt{\nu s} - \frac{y}{2\sqrt{\nu s}}\right) + e^{\mu_m y} \operatorname{Erfc}\left(\mu_m \sqrt{\nu s} + \frac{y}{2\sqrt{\nu s}}\right) \right\} ds; \mu_m \neq 0, \tag{12}$$

$$\tau_N(y, z, t) = -\frac{2f(t)}{h} \sum_{n=1}^{\infty} \frac{(-1)^n \cos(\mu_m z)}{\mu_m} \cdot e^{-\mu_m y} + \frac{1}{h} \sum_{n=1}^{\infty} \frac{(-1)^n \cos(\mu_m z)}{\mu_m} \cdot \int_0^t f'(t-s) \left\{ e^{-\mu_m y} \operatorname{Erfc}\left(\mu_m \sqrt{\nu s} - \frac{y}{2\sqrt{\nu s}}\right) - e^{\mu_m y} \operatorname{Erfc}\left(\mu_m \sqrt{\nu s} + \frac{y}{2\sqrt{\nu s}}\right) \right\} ds, \tag{13}$$

in terms of the complementary error function of Gauss  $\operatorname{Erfc}(\cdot)$  which can be obtained from tables [12].

To the best of our knowledge, the general Solutions (6) and (7) for second-grade fluids, as well as the Solutions (8)–(13) for Newtonian fluids, are new in the literature and their value for theory and practice can be significant. They can provide exact solutions for different motions with physical relevance of these fluids. In order to bring to light the theoretical importance of these general solutions, some known solutions from the literature will be recovered as limiting cases.

2.1. Case  $f(t) = ft^a$  ( $a > 0$ ): the Plate Applies an Accelerated Shear to the Fluid

Putting  $f(t) = ft^a$  into (6) and (7), the corresponding Solutions (3.12) and (3.14) from [6] are recovered. The solutions corresponding to  $a = 2, 3, \dots, n$ , as it was proved in [6], can be written as simple or multiple integrals of  $u_1(y, z, t)$  and  $\tau_1(y, z, t)$ . The similar solutions for Newtonian fluids are immediately obtained from any one of (8) and (9), (10) and (11) or (12) and (13). By setting  $f(t) = ft$  in (13), for instance, we obtain the shear stress

$$\tau_{1N}(y, z, t) = -\frac{2ft}{h} \sum_{n=1}^{\infty} \frac{(-1)^n \cos(\mu_m z)}{\mu_m} \cdot e^{-\mu_m y} + \frac{f}{h} \sum_{n=1}^{\infty} \frac{(-1)^n \cos(\mu_m z)}{\mu_m} \cdot \int_0^t \left\{ e^{-\mu_m y} \operatorname{Erfc}\left(\mu_m \sqrt{\nu s} - \frac{y}{2\sqrt{\nu s}}\right) - e^{\mu_m y} \operatorname{Erfc}\left(\mu_m \sqrt{\nu s} + \frac{y}{2\sqrt{\nu s}}\right) \right\} ds. \tag{14}$$

Further, unlike the next two cases, this motion is unsteady and remains unsteady.

### 2.2. Flow Due to an Oscillating Shear Stress

By now setting  $f(t) = f \sin(\omega t)$  into (6)–(9), the corresponding solutions obtained in [7] and [9] are recovered. The velocity field for second-grade fluids

$$u_s(y, z, t) = \frac{4f}{\mu\pi h} \sum_{n=1}^{\infty} \frac{(-1)^n \cos(\mu_m z)}{\mu_m} \cdot \left\{ \sin(\omega t) \int_0^{\infty} \frac{(\xi^2 + \mu_m^2) \cos(y\xi)}{(\xi^2 + \mu_m^2)^2 + (\omega/\nu)^2 [1 + \alpha(\xi^2 + \mu_m^2)]^2} d\xi \right. \\ \left. - \frac{\omega}{\nu} \cos(\omega t) \int_0^{\infty} \frac{[1 + \alpha(\xi^2 + \mu_m^2)] \cos(y\xi)}{(\xi^2 + \mu_m^2)^2 + (\omega/\nu)^2 [1 + \alpha(\xi^2 + \mu_m^2)]^2} d\xi \right. \\ \left. + \frac{\omega}{\nu} \int_0^{\infty} \frac{[1 + \alpha(\xi^2 + \mu_m^2)] \cos(y\xi)}{(\xi^2 + \mu_m^2)^2 + (\omega/\nu)^2 [1 + \alpha(\xi^2 + \mu_m^2)]^2} \right. \\ \left. \cdot \exp \left[ -\frac{\nu(\xi^2 + \mu_m^2)t}{1 + \alpha(\xi^2 + \mu_m^2)} \right] d\xi \right\}, \quad (15)$$

is identical to that given by [9, Eq. (23)]. It is presented as a sum of steady-state and transient solutions and describes the motion of the fluid some time after its initiation. After this time, when the transients disappear, it tends to the steady-state solution that is periodic in time and independent of the initial condition. However, it satisfies the boundary conditions and the governing equation. An important problem regarding the technical relevance of starting solutions is to find the approximate time after which the fluid is moving according to the steady-state solutions. More exactly, in practice, it is necessary to find the required time to reach the steady-state.

### 2.3. Case $f(t) = fH(t)$ : Flow Due to a Plate that Applies a Constant Shear to the Fluid

In this case, as well as for  $f(t) = fH(t)\cos(\omega t)$ , where  $f$  is a constant and  $H(\cdot)$  is the Heaviside step function, the solution is obtained following the same way as in [13]. However, it is worth pointing out that the corresponding solutions can also be obtained from the general Solutions (6) and (7). Taking  $f(t) = fH(t)$  into (6), for instance, the corresponding velocity field

$u_0(y, z, t)$  takes the simplified form [6, Eq. (3.16)]

$$u_0(y, z, t) = \frac{2f}{\mu h} \sum_{n=1}^{\infty} \frac{(-1)^n \cos(\mu_m z)}{\mu_m} \cdot \left\{ \frac{e^{-\mu_m y}}{\mu_m} - \frac{2}{\pi} \int_0^{\infty} \frac{\cos(y\xi)}{(\xi^2 + \mu_m^2)} \right. \\ \left. \cdot \exp \left[ -\frac{\nu(\xi^2 + \mu_m^2)t}{1 + \alpha(\xi^2 + \mu_m^2)} \right] d\xi \right\}, \quad (16)$$

which is equivalent to the result obtained by Yao and Liu [5, Sect. 4]. By now setting  $\alpha = 0$  in (16), the solution (16) from [4] is recovered. Of course, this last solution is equivalent to the velocity field

$$u_{0N}(y, z, t) = \frac{2f}{\mu h} \sum_{n=1}^{\infty} \frac{(-1)^n \cos(\mu_m z)}{\mu_m^2} \cdot e^{-\mu_m y} - \frac{f}{\mu h} \sum_{n=1}^{\infty} \frac{(-1)^n \cos(\mu_m z)}{\mu_m^2} \cdot \left[ e^{-\mu_m y} \operatorname{Erfc} \left( \mu_m \sqrt{\nu t} - \frac{y}{2\sqrt{\nu t}} \right) \right. \\ \left. + e^{\mu_m y} \operatorname{Erfc} \left( \mu_m \sqrt{\nu t} + \frac{y}{2\sqrt{\nu t}} \right) \right], \quad (17)$$

resulting from (12) for  $f'(t) = fH'(t) = f\delta(t)$ , where  $\delta(\cdot)$  is the Dirac delta function. The corresponding shear stress, namely

$$\tau_{0N}(y, z, t) = -\frac{2f}{h} \sum_{n=1}^{\infty} \frac{(-1)^n \cos(\mu_m z)}{\mu_m} \cdot e^{-\mu_m y} + \frac{f}{h} \sum_{n=1}^{\infty} \frac{(-1)^n \cos(\mu_m z)}{\mu_m} \cdot \left\{ e^{-\mu_m y} \operatorname{Erfc} \left( \mu_m \sqrt{\nu t} - \frac{y}{2\sqrt{\nu t}} \right) \right. \\ \left. - e^{\mu_m y} \operatorname{Erfc} \left( \mu_m \sqrt{\nu t} + \frac{y}{2\sqrt{\nu t}} \right) \right\}, \quad (18)$$

is immediately obtained from (13). It is clearly seen from (16), (17), and (18) that for large times the last terms tend to zero. Consequently, this flow also becomes steady and the steady solutions are the same for both types of fluids (Newtonian and second-grade). Furthermore, as it immediately results from (14) and (18),

$$\tau_{1N}(y, z, t) = \int_0^t \tau_{0N}(y, z, s) ds.$$

**3. Limiting case  $h \rightarrow \infty$ : Flow over an Infinite Plate**

In the absence of the side walls, namely when  $h \rightarrow \infty$ , the general Solutions (6)–(9) take the simplified forms

$$u(y,t) = -\frac{2}{\rho\pi} \int_0^\infty \frac{\cos(y\xi)}{1+\alpha\xi^2} \cdot \int_0^t f(s) \exp\left[-\frac{v\xi^2(t-s)}{1+\alpha\xi^2}\right] ds d\xi, \tag{19}$$

$$\begin{aligned} \tau(y,t) = & f(t) - \frac{2}{\pi} f(t) \int_0^\infty \frac{\sin(y\xi)}{\xi(1+\alpha\xi^2)} d\xi \\ & + \frac{2v}{\pi} \int_0^\infty \frac{\xi \sin(y\xi)}{(1+\alpha\xi^2)^2} \\ & \cdot \int_0^t f(s) \exp\left[-\frac{v\xi^2(t-s)}{1+\alpha\xi^2}\right] ds d\xi, \end{aligned} \tag{20}$$

$$u_N(y,t) = -\frac{2}{\rho\pi} \int_0^\infty \cos(y\xi) \cdot \int_0^t f(s) e^{-v\xi^2(t-s)} ds d\xi, \tag{21}$$

and

$$\begin{aligned} \tau_N(y,t) = & \frac{2v}{\pi} \int_0^\infty \xi \sin(y\xi) \\ & \cdot \int_0^t f(s) e^{-v\xi^2(t-s)} ds d\xi, \end{aligned} \tag{22}$$

corresponding to the motion over an infinite plate that applies a shear stress  $f(t)$  to the fluid. The Newtonian solutions, as they result from (10), (11), (13), (21), and the identity

$$\begin{aligned} \int_0^\infty \frac{1 - e^{-v\xi^2 t}}{\xi^2} \cos(y\xi) d\xi = & \sqrt{v\pi t} \exp\left(-\frac{y^2}{4vt}\right) \\ & - \frac{\pi y}{2} \operatorname{Erfc}\left(\frac{y}{2\sqrt{vt}}\right), \end{aligned}$$

can also be written in the equivalent forms

$$u_N(y,t) = -\frac{1}{\rho\sqrt{v\pi}} \int_0^t \frac{f(t-s)}{\sqrt{s}} \exp\left(-\frac{y^2}{4vs}\right) ds, \tag{23}$$

$$\tau_N(y,t) = \frac{y}{2\sqrt{v\pi}} \int_0^t \frac{f(t-s)}{s\sqrt{s}} \exp\left(-\frac{y^2}{4vs}\right) ds, \tag{24}$$

respectively,

$$\begin{aligned} u_N(y,t) = & \frac{y}{\mu} \int_0^t f'(t-s) \operatorname{Erfc}\left(\frac{y}{2\sqrt{vs}}\right) ds \\ & - \frac{2}{\mu} \sqrt{\frac{v}{\pi}} \int_0^\infty \sqrt{s} f'(t-s) \exp\left(-\frac{y^2}{4vs}\right) ds, \end{aligned} \tag{25}$$

$$\tau_N(y,t) = \int_0^t f'(t-s) \operatorname{Erfc}\left(\frac{y}{2\sqrt{vs}}\right) ds. \tag{26}$$

If  $f(t)$  is a periodic function, all general solutions that have previously been developed can be written as a sum of steady-state and transient solutions. The Newtonian shear stress (24), for example, can be written as

$$\tau_N(y,t) = \tau_{Ns}(y,t) + \tau_{Nt}(y,t), \tag{27}$$

where

$$\begin{aligned} \tau_{Ns}(y,t) = & \frac{y}{2\sqrt{v\pi}} \int_0^\infty \frac{f(t-s)}{s\sqrt{s}} \exp\left(-\frac{y^2}{4vs}\right) ds, \\ \tau_{Nt}(y,t) = & -\frac{y}{2\sqrt{v\pi}} \int_t^\infty \frac{f(t-s)}{s\sqrt{s}} \exp\left(-\frac{y^2}{4vs}\right) ds. \end{aligned} \tag{28}$$

Choosing  $f(t) = f \sin(\omega t)$  into the last relations, we find that

$$\begin{aligned} \tau_N(y,t) = & \frac{fy}{2\sqrt{v\pi}} \int_0^\infty \frac{\sin[\omega(t-s)]}{s\sqrt{s}} \exp\left(-\frac{y^2}{4vs}\right) ds \\ & - \frac{fy}{2\sqrt{v\pi}} \int_t^\infty \frac{\sin[\omega(t-s)]}{s\sqrt{s}} \exp\left(-\frac{y^2}{4vs}\right) ds. \end{aligned} \tag{29}$$

Under this form, the corresponding boundary condition  $\tau_N(0,t) = f \sin(\omega t)$  seems not to be satisfied. In order to do away with this inconvenience, we shall present the steady-state Solution (28)<sub>1</sub> in a more suitable form. Indeed, making the change of variable  $s = 1/\sigma$  and using the fact that  $\cos x = \cosh(ix)$ ,  $\sin x = -i \sinh(ix)$ , and the known result

$$\int_0^\infty \frac{\exp[-a^2 s - (b^2/4s)]}{s\sqrt{s}} ds = \frac{\sqrt{\pi}}{2a} e^{-ab},$$

we find after lengthy but straightforward computations

$$\begin{aligned} \tau_{Ns}(y,t) = & f \exp\left(-y\sqrt{\frac{\omega}{2v}}\right) \\ & \cdot \sin\left(\omega t - y\sqrt{\frac{\omega}{2v}}\right). \end{aligned} \tag{30}$$

Finally, taking the function  $f(t)$  to be  $f(t) = fH(t)$  or  $f(t) = ft$  in (26), we obtain for the shear stress the simple but elegant expressions

$$\begin{aligned} \tau_{0N}(y,t) &= f \operatorname{Erfc} \left( \frac{y}{2\sqrt{vt}} \right) \quad \text{and} \\ \tau_{1N}(y,t) &= f \int_0^t \operatorname{Erfc} \left( \frac{y}{2\sqrt{vs}} \right) ds, \end{aligned} \quad (31)$$

which are identical as form to  $v_{0n}(y,t)$  and  $v_{1n}(y,t)$  corresponding to the flow due to a flat plate that moves in its plane with the velocities  $VH(t)$  and  $Vt$ , respectively.

#### 4. Conclusions

The motion of a second-grade fluid due to an infinite plate that applies a time-dependent shear  $f(t)$  to the fluid is studied by means of integral Fourier transforms. General solutions are firstly obtained for the motion between two infinite parallel walls perpendicular to the plate. These solutions can easily be used to recover different known solutions from the literature or to develop new similar solutions for suitable selections of the function  $f(t)$ . Similar solutions for Newtonian fluids performing the same motion are obtained as special cases of the general solutions. They are also written in simpler forms, (10)–(13), in terms of the elementary function  $\exp(\cdot)$  and the complementary error function  $\operatorname{Erfc}(\cdot)$ .

In the absence of the side walls, namely when the distance between walls tends to infinity, the general solutions take simplified forms like those given by (19)–(26) and correspond to the motion over an infinite plate. If the plate applies an oscillating shear to the fluid, the corresponding solutions can be presented as a sum of steady-state and transient solutions. These solutions describe the motion of the fluid some time after its initiation. After that time, when the transients disappear, they tend to the steady-state solutions that are periodic in time and independent of the initial conditions. However, they satisfy the initial and boundary conditions. Some of the present results can be extended to fluid motions in cylindrical domains [14].

Finally, taking  $f(t) = ft$ ,  $f \sin(\omega t)$  or  $fH(t)$  in (20), we obtain the shear stresses

$$\begin{aligned} \tau(y,t) &= ft - \frac{2f}{v\pi} \int_0^\infty \left\{ 1 - \exp \left( -\frac{v\xi^2 t}{1 + \alpha\xi^2} \right) \right\} \\ &\quad \cdot \frac{\sin(y\xi)}{\xi^3} d\xi, \end{aligned} \quad (32)$$

$$\begin{aligned} \tau(y,t) &= f \sin(\omega t) - \frac{2f}{\pi} \frac{\omega}{v} \cos(\omega t) \\ &\quad \cdot \int_0^\infty \frac{\xi \sin(y\xi)}{\xi^4 + (\omega/v)^2 (1 + \alpha\xi^2)^2} d\xi \\ &\quad - \frac{2f}{\pi} \left( \frac{\omega}{v} \right)^2 \sin(\omega t) \\ &\quad \cdot \int_0^\infty \frac{(1 + \alpha\xi^2) \sin(y\xi)}{\xi [\xi^4 + (\omega/v)^2 (1 + \alpha\xi^2)^2]} d\xi \\ &\quad + \frac{2f}{\pi} \frac{\omega}{v} \int_0^\infty \frac{\xi \sin(y\xi)}{\xi^4 + (\omega/v)^2 (1 + \alpha\xi^2)^2} \\ &\quad \cdot \exp \left( -\frac{v\xi^2 t}{1 + \alpha\xi^2} \right) d\xi, \end{aligned} \quad (33)$$

respectively,

$$\begin{aligned} \tau(y,t) &= fH(t) \left[ 1 - \frac{2}{\pi} \int_0^\infty \frac{\sin(y\xi)}{\xi(1 + \alpha\xi^2)} \right. \\ &\quad \left. \cdot \exp \left( -\frac{v\xi^2 t}{1 + \alpha\xi^2} \right) d\xi \right], \end{aligned} \quad (34)$$

corresponding to the motion due to an infinite plate that, after time  $t = 0$ , applies the shear stresses  $ft$ ,  $f \sin(\omega t)$  or  $fH(t)$  to a second-grade fluid. As form, these expressions are identical to those of the velocity field  $v(y,t)$  (see [15, Eq. (23)], [16, Eq. (3.9)], and [13, Eq. (3)]) corresponding to the motion induced by a plate that moves in its plane with the velocities  $Vt$ ,  $V \sin(\omega t)$  or  $VH(t)$ , respectively. This is not a surprise because a simple analysis shows that the shear stress  $\tau(y,t)$  in such motions of second-grade fluids satisfies the governing equation

$$\frac{\partial \tau(y,t)}{\partial t} = \left( v + \alpha \frac{\partial}{\partial t} \right) \frac{\partial^2 \tau(y,t)}{\partial y^2}, \quad (35)$$

which is identical to that for the velocity  $v(y,t)$  [2, Eq. (2.12)]. Consequently, the velocity field  $v(y,t)$  corresponding to the unsteady motion of a second-grade or Newtonian fluid due to an infinite plate that slides in its plane with a velocity  $V(t)$  ( $= 0$  for  $t \leq 0$ ) is given by anyone of the relations (20), (22), (24), (26) or (27) with  $V(t)$  instead of  $f(t)$ .

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