

N-dimensional B-spline surface estimated by lofting for locally improving IRI

Research Article

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Abstract:

N-dimensional surfaces are defined by the tensor product of B-spline basis functions. To estimate the unknown control points of these B-spline surfaces, the lofting method also called skinning method by cross-sectional curve fits is applied. It is shown by an analytical proof and numerically confirmed by the example of a four-dimensional surface that the results of the lofting method agree with the ones of the simultaneous estimation of the unknown control points. The numerical complexity for estimating v^n control points by the lofting method is $O(v^{n+1})$ while it results in $O(v^{3n})$ for the simultaneous estimation. It is also shown that a B-spline surface estimated by a simultaneous estimation can be extended to higher dimensions by the lofting method, thus saving computer time.

An application of this method is the local improvement of the International Reference Ionosphere (IRI), e.g. by the slant total electron content (STEC) obtained by dual-frequency observations of the Global Navigation Satellite System (GNSS). Three-dimensional B-spline surfaces at different time epochs have to be determined by the simultaneous estimation of the control points for this improvement. A four-dimensional representation in space and time of the electron density of the ionosphere is desirable. It can be obtained by the lofting method. This takes less computer time than determining the four-dimensional surface solely by a simultaneous estimation.

Keywords:

NURBS • B-spline • N-dimensional surface • Lofting method • International Reference Ionosphere (IRI) • STEC data

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Received 9 November 2010; accepted 1 December 2010

1. Introduction

The electron density of the ionosphere causes a delay in the electromagnetic signals emitted by a satellite and received at an observation site, e.g. on the surface of the Earth. Knowledge of the electron density is therefore mandatory for correcting the delays. Dual-frequency observations of the Global Navigation Satellite System (GNSS), like the Global Positioning System (GPS), can be used to determine the slant total electron content (STEC), which is the integral of the electron density along the path of the signal between the transmitter and the receiver. Other space-

geodetic observation techniques provide further information on ionospheric parameters. Satellite altimetry, for instance, yields the vertical total electron content (VTEC) over the oceans (Brunini et al., 2005).

To calculate ionospheric parameters such as the electron density or VTEC from space-geodetic observations, an appropriate parameterization has to be chosen. Today the existing ionospheric models, e.g. the VTEC models of the International GNSS Service (IGS) (Dow et al., 2009), are generally reported globally by spherical harmonic expansions with a temporal resolution of 2 hours (Hernández-Pajarez et al., 2009). The International Reference Ionosphere (IRI) is an empirical climatological ionospheric model which allows the calculation of ionospheric parameters at any point at any time (Bilitza and Reinisch, 2008). Since at present IRI does not consider space-geodetic observations, it can be used as a reference model for improving ionospheric parameters from GNSS dual-frequency

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observations. However, the ground stations with satellite receivers are not evenly distributed over the surface of the Earth. Some parts are densely covered by receivers. A local representation of the electron density or the VTEC by localizing base functions with compact support is therefore appropriate.

Schmidt (2007) proposed a regional B-spline modelling. The tensor product of the B-splines is used for a three-dimensional representation of the electron density of the ionosphere at different time epochs. STEC observations are applied to determine corrections to the electron density of IRI at different time epochs for points in a spatial grid defined by latitude, longitude and height. Numerical studies using simulated STEC data computed by IRI are presented by Zeilhofer et al. (2009). The unknown parameters are simultaneously estimated by least-squares adjustment. Gaps exist in the data and to avoid a singular matrix of normal equations, prior information by IRI data is introduced. Instead of the three-dimensional B-spline surface at different time epochs, a four-dimensional representation in space and time has been suggested, Schmidt et al. (2008).

A two-dimensional representation of a surface by the tensor product of two B-splines is a well known task of reverse engineering. Generally, nonuniform rational B-splines (NURBS) are applied, Pieg and Tiller (1997), Farin and Hansford (2000), Rogers (2001). Three-dimensional coordinates of points on the surface of manufactured objects are measured and approximated by a NURBS surface for computer aided design, cf. Varady et al. (1997). A nonrational B-spline surface includes the representation of a NURBS surface by introducing homogeneous coordinates. We will therefore work in the following only with B-spline surfaces. The representation of a B-spline surface by the tensor product leads to a linear relation between the given coordinates and the unknown control points, after the knots of the B-spline basis functions have been selected and the location parameters of the measured points have been determined. The linear relation gives the observation equations of a linear model where the control points are simultaneously estimated by least-squares adjustment, Farin and Hansford (2000) p. 186, Rogers (2001) p. 193.

A large number of given points generally requires a large number of unknown control points which have to be estimated. If $v \times v$ control points of a two-dimensional surface are simultaneously estimated, the Cholesky factorization of the $v^2 \times v^2$ matrix of normal equations leads to a computational complexity of $O(v^6)$, Koch (2009a). The computational burden for a simultaneous estimation is therefore quite heavy. Nevertheless, Sarkar and Menq (1991) and Lai and Lu (1996), for instance, use it as a part of a nonlinear adjustment in order to estimate the location parameters of the measured points in addition to the control points. Ma and Kruth (1998) determine the weights of the control points first and then simultaneously estimate the control points.

To avoid the computational load of the simultaneous estimation of the control points, the lofting also called skinning method has been proposed, Tiller (1983), Pieg (1991). It consists of

interpolating cross-sectional isoparametric curves for determining the control points of a B-spline surface. Instead of interpolating the cross-sectional curves, the curves can also approximate the measured points by a least-squares fit. This approach has been called approximate lofting by Park (2001). First lofting by curve fits and then as a final step the simultaneous estimation of the control points have been used for a surface reconstruction from direct slicing of point clouds, Yuwen et al. (2006). Yang and Qian (2007) introduced heterogeneous lofting to represent surfaces of objects which are composed of different materials.

The lofting method by cross-sectional curve fits reduces the numerical complexity from $O(v^6)$ for the simultaneous estimation to $O(v^3)$ (Koch, 2009a). However, the lofting method is considered to be an approximation of the simultaneous estimation of the control points of B-spline surfaces, Pieg and Tiller (1997) p. 419. But Koch (2009a) proved that the estimates of the control points by the lofting method using cross-sectional curve fits and by the simultaneous estimation give identical results.

The tensor product for a two-dimensional B-spline surface is readily generalized to three or higher dimensions by adding additional summations and B-spline basis functions. A three-dimensional B-spline surface is needed not only for representing quantities depending on three coordinates, like the electron density mentioned above, but also for deformation analysis or for dynamical problems of reverse engineering where surfaces vary with time (Koch, 2010a). The equivalence of estimating the control points by the lofting method and by the simultaneous estimation also exists for three-dimensional B-spline surfaces (Koch, 2010a). If v^3 unknown parameters have to be estimated, the numerical complexity becomes $O(v^4)$ which computationally is still manageable. This is important if the accuracy of fitting B-spline surfaces is investigated by Monte Carlo simulations (Koch, 2009b). The complexity for the simultaneous estimation increases to $O(v^9)$ which is more than the square of the one of the lofting method. If possible, the simultaneous estimation should be avoided.

For a general approach, B-spline surfaces of n dimensions are introduced here. An analytical proof is given and confirmed by a numerical example of a four-dimensional surface that the estimates of the unknown control points by the lofting method using cross-sectional curve fits are identical with the simultaneous estimates of the control points. The numerical complexity becomes $O(v^{n+1})$ for estimating v^n control points by the lofting method and $O(v^{3n})$ in case of the simultaneous estimation. The lofting method needs compatible isoparametric curves of the surface representation with respect to an identical number of given points, an identical number of knots and an identical degree for the B-spline basis functions. Thus, n coordinates of points in an n -dimensional grid have to be given together with the quantity to be represented by the B-spline surface. Otherwise, the lofting method cannot be applied.

As mentioned above, Schmidt (2007) and Zeilhofer et al. (2009) took STEC data and prior information to determine the control points of

The paper is organized as follows: Section 2 defines the B-spline surface in n dimensions. Section 3 presents the estimation of the unknown control points by the lofting method, Section 4 the simultaneous estimation and the proof of the equivalence of both methods. Section 5 shows the extension of a simultaneously estimated B-spline surface of lower dimensions to higher dimensions by the lofting method. Section 6 gives the example of a four-dimensional surface to numerically confirm the identity of the results by the lofting method and the simultaneous estimation. The paper finishes with conclusions.

2. B-spline surface in n dimensions

A B-spline surface in n dimensions depends on the n parameters, say $\xi_1, \xi_2, \dots, \xi_n$, and is expressed by the tensor product of n B-spline basis functions $N_{i_1 q_1}(\xi_1), N_{i_2 q_2}(\xi_2), \dots, N_{i_n q_n}(\xi_n)$ of degrees $q_1, q_2, \dots, q_n$ with

$$s(\xi_1, \xi_2, \dots, \xi_n) = \sum_{i_1=0}^{m_1} \sum_{i_2=0}^{m_2} \dots \sum_{i_n=0}^{m_n} N_{i_1 q_1}(\xi_1) N_{i_2 q_2}(\xi_2) \dots N_{i_n q_n}(\xi_n) p_{i_1 i_2 \dots i_n} \quad (1)$$

and

$$\boldsymbol{s}(\xi_1, \xi_2, \dots, \xi_n) = \begin{vmatrix} x_1(\xi_1, \xi_2, \dots, \xi_n) \\ x_2(\xi_1, \xi_2, \dots, \xi_n) \\ \vdots \\ x_n(\xi_1, \xi_2, \dots, \xi_n) \\ H(\xi_1, \xi_2, \dots, \xi_n) \end{vmatrix} \quad (2)$$

where $\mathbf{s}(\xi_1, \xi_2, \dots, \xi_n)$ denotes a point on the surface with the n -dimensional rectangular or curvilinear coordinates $x_1, x_2, \dots, x_n$ depending on the n parameters $\xi_1, \xi_2, \dots, \xi_n$. The $(n+1)$ st coordinate H is the quantity, like the electron density in the ionosphere, to be represented by the B-spline surface. The points

$$\begin{aligned} \mathbf{p}_{i_1 i_2 \dots i_n} &= [x_{1i_1}, x_{2i_2}, \dots, x_{ni_n}, H_{i_1 i_2 \dots i_n}]' \\ \text{with } i_1 &\in \{0, \dots, m_1\}, i_2 \in \{0, \dots, m_2\}, \dots, \\ & i_n \in \{0, \dots, m_n\} \end{aligned} \quad (3)$$

are the unknown control points. The B-spline surface approximately follows these points.

The B-spline basis functions are efficiently computed by a recursion formula due to Cox (1972) and de Boor (1972) for the half-open

interval for $\xi_1 \in [\xi_{1i}, \xi_{1,i+1})$ where ξ_{1i} denotes a knot. The number m_1 in (1) depends on the number of knots chosen for ξ_1 and on the degree q_1 . The same holds true for the parameters $\xi_2, \dots, \xi_n$. Generally, the knots are not equally spaced which leads to nonuniform B-splines in contrast to uniform ones which are equally spaced. We will work with knot vectors which have the property of endpoint interpolation. The points $\mathbf{s}(\xi_1 = \text{const}, \dots, \xi_n = \text{const})$ with ξ_1 variable and $\xi_2, \dots, \xi_n$ fixed define an isoparametric curve as a function of ξ_1 on the B-spline surface. Accordingly, $\mathbf{s}(\xi_1 = \text{const}, \xi_2, \xi_3 = \text{const}, \dots, \xi_n = \text{const}), \dots, \mathbf{s}(\xi_1 = \text{const}, \xi_2 = \text{const}, \dots, \xi_n)$ are isoparametric curves depending on $\xi_2, \dots, \xi_n$. The isoparametric curve for ξ_1 shall point along the x_1 -axis, the one for ξ_2 along the x_2 -axis and so on and for ξ_n along the x_n -axis.

Introducing weights for the control points leads to a nonuniform rational B-spline (NURBS) surface. However, a NURBS surface can be represented by the nonrational B-spline surface (1) if homogeneous coordinates are introduced. Thus, the B-spline surface includes the representation of a NURBS surface so that we will work with B-spline surfaces.

A B-spline surface shall be applied for a local representation of the electron density of the ionosphere. The density is denoted by N_e and depends on a three-dimensional position and on the time. The position is given in a geographical geocentric coordinate system. The longitude is linearly transformed into the interval $[0, 1]$ to obtain x , the transformation of the latitude gives y , the height above the Earth's surface leads to z , thus $x_1 = x$, $x_2 = y$, $x_3 = z$ in (2). The fourth coordinate follows with $x_4 = t$ from the transformation of the time t and the fifth one with $H = N_e$ from the electron density N_e . A four-dimensional B-spline surface is therefore introduced to represent the density $N_e(x, y, z, t)$. For a graphical depiction of the surface, N_e may be computed with respect to x and y for fixed values of z and t .

3. Estimation by lofting method

Let the n -dimensional rectangular or curvilinear coordinates $x_1, x_2, \dots, x_n$ together with H of $e_1 \times e_2 \times \dots \times e_n$ points $s(\xi_{1a_1}, \xi_{2a_2}, \dots, \xi_{na_n})$ be given, where ξ_{1a_1} with $a_1 \in \{1, \dots, e_1\}$, ξ_{2a_2} with $a_2 \in \{1, \dots, e_2\}$, $\dots$, ξ_{na_n} with $a_n \in \{1, \dots, e_n\}$ denote the location parameters. Let the points $s(\xi_{1a_1}, \xi_{2a_2}, \dots, \xi_{na_n})$ be arranged in an n -dimensional grid. It means that the quantity H is given, for instance, by measurements at the points in the grid. An n -dimensional B-spline surface shall be fitted to the given points so that the control points $p_{i_1 i_2 \dots i_n}$ with $i_1 \in \{0, \dots, m_1\}$, $i_2 \in \{0, \dots, m_2\}$, $\dots$, $i_n \in \{0, \dots, m_n\}$ have to be estimated for $e_1 > m_1 + 1$, $e_2 > m_2 + 1$, $\dots$, $e_n > m_n + 1$.

If the given points $s(\xi_{1a_1}, \xi_{2a_2}, \dots, \xi_{na_n})$ are equally spaced, the location parameters $\xi_{1a_1}, \xi_{2a_2}, \dots, \xi_{na_n}$ may also be equally spaced. If not, the location parameters can be determined by the chord lengths of the given points, cf. Koch (2010a). It means

that for computing ξ_{1a_1} with $a_1 \in \{1, \dots, e_1\}$ the distances between the points at the surface defined by H in the direction of the x_1 axis of the grid are determined for fixed values of $x_2, \dots, x_n$. This is repeated for all values of $x_2, \dots, x_n$ and the mean of these values gives ξ_{1a_1} . Correspondingly, $\xi_{2a_2}, \dots, \xi_{na_n}$ are computed. Eq. (1) then gives a linear relation between the unknown control points $p_{i_1 i_2 \dots i_n}$ and the given points $s(\xi_{1a_1}, \xi_{2a_2}, \dots, \xi_{na_n})$ so that the observation equations for estimating $p_{i_1 i_2 \dots i_n}$ in a linear model are obtained

$$\sum_{i_1=0}^{m_1} \sum_{i_2=0}^{m_2} \dots \sum_{i_n=0}^{m_n} N_{i_1 q_1}(\xi_{1a_1}) N_{i_2 q_2}(\xi_{2a_2}) \dots N_{i_n q_n}(\xi_{na_n}) \cdot p_{i_1 i_2 \dots i_n} = s(\xi_{1a_1}, \xi_{2a_2}, \dots, \xi_{na_n}) + e(\xi_{1a_1}, \xi_{2a_2}, \dots, \xi_{na_n}),$$

$$a_1 \in \{1, \dots, e_1\}, a_2 \in \{1, \dots, e_2\}, \dots, a_n \in \{1, \dots, e_n\} \quad (4)$$

where $e(\xi_{1a_1}, \xi_{2a_2}, \dots, \xi_{na_n})$ denotes the vector of errors of $s(\xi_{1a_1}, \xi_{2a_2}, \dots, \xi_{na_n})$. Eq. (4) results in $e_1 \times e_2 \times \dots \times e_n$ linear equations for determining $(m_1 + 1) \times (m_2 + 1) \times \dots \times (m_n + 1)$ unknown control points.

The lofting method by cross-sectional curve fits is applied for the estimation. Eq. (4) is therefore rewritten by

$$\sum_{i_1=0}^{m_1} N_{i_1 q_1}(\xi_{1a_1}) b_{i_1 a_2 a_3 \dots a_n}^{(1)} = s(\xi_{1a_1}, \xi_{2a_2}, \dots, \xi_{na_n}) + e(\xi_{1a_1}, \xi_{2a_2}, \dots, \xi_{na_n}) \quad (5)$$

with

$$\sum_{i_2=0}^{m_2} N_{i_2 q_2}(\xi_{2a_2}) b_{i_1 i_2 a_3 \dots a_n}^{(2)} = b_{i_1 a_2 a_3 \dots a_n}^{(1)} \quad (6)$$

and so on, with finally

$$\sum_{i_n=0}^{m_n} N_{i_n q_n}(\xi_{na_n}) p_{i_1 i_2 i_3 \dots i_n} = b_{i_1 i_2 i_3 \dots i_n}^{(n-1)} \quad (7)$$

where $b_{i_1 a_2 a_3 \dots a_n}^{(1)}$ denotes the control points of the isoparametric curves $s(\xi_1, \xi_2 = \text{const}, \dots, \xi_n = \text{const})$, $b_{i_1 i_2 a_3 \dots a_n}^{(2)}$ the control points of the isoparametric curves $s(\xi_1 = \text{const}, \xi_2, \xi_3 = \text{const}, \dots, \xi_n = \text{const})$ and so on, finally $p_{i_1 i_2 i_3 \dots i_n}$ the control points of the isoparametric curves $s(\xi_1 = \text{const}, \xi_2 = \text{const}, \dots, \xi_n)$.

The control points $b_{i_1 a_2 a_3 \dots a_n}^{(1)}$ are estimated first by means of the observation equations (5). They read in matrix notation

$$N(\xi_1) B^{(1)} = S + E \quad (8)$$

where the $e_1 \times (m_1 + 1)$ matrix $N(\xi_1)$ of the B-spline basis functions is defined by

$$N(\xi_1) = \begin{bmatrix} N_{0q_1}(\xi_{11}) & \dots & N_{m_1 q_1}(\xi_{11}) \\ \dots & \dots & \dots \\ N_{0q_1}(\xi_{1e_1}) & \dots & N_{m_1 q_1}(\xi_{1e_1}) \end{bmatrix}, \quad (9)$$

the $(m_1 + 1) \times (e_2 \times e_3 \times \dots \times e_n)$ matrix $B^{(1)}$ of control points by

$$B^{(1)} = \begin{bmatrix} b_{011\dots 1}^{(1)} & \dots & b_{0e_2 1\dots 1}^{(1)} & \dots & b_{01e_3\dots 1}^{(1)} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ b_{m_1 11\dots 1}^{(1)} & \dots & b_{m_1 e_2 1\dots 1}^{(1)} & \dots & b_{m_1 1e_3\dots 1}^{(1)} & \dots \\ b_{0e_2 e_3\dots 1}^{(1)} & \dots & b_{011\dots e_n}^{(1)} & \dots & b_{0e_2 1\dots e_n}^{(1)} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ b_{m_1 e_2 e_3\dots 1}^{(1)} & \dots & b_{m_1 11\dots e_n}^{(1)} & \dots & b_{m_1 e_2 1\dots e_n}^{(1)} & \dots \\ & & b_{01e_3\dots e_n}^{(1)} & \dots & b_{0e_2 e_3\dots e_n}^{(1)} & \\ & & \dots & \dots & \dots & \\ b_{m_1 1e_3\dots e_n}^{(1)} & \dots & b_{m_1 e_2 e_3\dots e_n}^{(1)} & & & \end{bmatrix}, \quad (10)$$

and with

$$s(\xi_{1a_1}, \xi_{2a_2}, \xi_{3a_3}, \dots, \xi_{na_n}) = s_{a_1 a_2 a_3 \dots a_n} \quad (11)$$

the $e_1 \times (e_2 \times e_3 \times \dots \times e_n)$ matrix S of given points by

$$S = \begin{bmatrix} s_{111\dots 1} & \dots & s_{1e_2 1\dots 1} & \dots & s_{11e_3\dots 1} & \dots & s_{1e_2 e_3\dots 1} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ s_{e_1 11\dots 1} & \dots & s_{e_1 e_2 1\dots 1} & \dots & s_{e_1 1e_3\dots 1} & \dots & s_{e_1 e_2 e_3\dots 1} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ s_{111\dots e_n} & \dots & s_{1e_2 1\dots e_n} & \dots & s_{11e_3\dots e_n} & \dots & s_{1e_2 e_3\dots e_n} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ s_{e_1 11\dots e_n} & \dots & s_{e_1 e_2 1\dots e_n} & \dots & s_{e_1 1e_3\dots e_n} & \dots & s_{e_1 e_2 e_3\dots e_n} & \dots \end{bmatrix}. \quad (12)$$

Finally, the $e_1 \times (e_2 \times e_3 \times \dots \times e_n)$ matrix E of errors is obtained with replacing s by e in (11) and (12).

Eq. (8) represents the observation equations of a multivariate linear model by which the control points $b_{i_1 a_2 a_3 \dots a_n}^{(1)}$ of the isoparametric curves $s(\xi_1, \xi_2 = \text{const}, \dots, \xi_n = \text{const})$ are estimated e_2 times for each value of ξ_2 up to e_n times for each value of ξ_n . The estimate $\hat{B}^{(1)}$ of $B^{(1)}$ follows by, cf. Koch (1999) p. 241,

$$\hat{B}^{(1)} = (N(\xi_1)' N(\xi_1))^{-1} N(\xi_1)' S. \quad (13)$$

The matrix $N(\xi_1)$ has full column rank so that the matrix $N(\xi_1)' N(\xi_1)$ of normal equations is regular and gives a unique solution if the given points are evenly distributed like on grids.

In the next step, (6) is used as observation equations for estimating the unknown control points $\mathbf{b}_{i_1 i_2 a_3 \dots a_n}^{(2)}$ of the isoparametric curves $s(\xi_1 = \text{const}, \xi_2, \xi_3 = \text{const}, \dots, \xi_n = \text{const})$. Applying matrix notation we find

$$\mathbf{N}(\xi_2) \mathbf{B}^{(2)} = \hat{\mathbf{B}}^{(1)} + \mathbf{E}_{\hat{\mathbf{B}}^{(1)}} \quad (14)$$

where the $e_2 \times (m_2 + 1)$ matrix $\mathbf{N}(\xi_2)$ of the B-spline basis functions is defined by

$$\mathbf{N}(\xi_2) = \begin{bmatrix} N_{0q_2}(\xi_{21}) & \dots & N_{m_2 q_2}(\xi_{21}) \\ \dots & \dots & \dots \\ N_{0q_2}(\xi_{2e_2}) & \dots & N_{m_2 q_2}(\xi_{2e_2}) \end{bmatrix}, \quad (15)$$

the $(m_2 + 1) \times ((m_1 + 1) \times e_3 \times \dots \times e_n)$ matrix $\mathbf{B}^{(2)}$ of control points by

$$\mathbf{B}^{(2)} = \begin{bmatrix} \mathbf{b}_{001\dots 1}^{(2)} & \dots & \mathbf{b}_{m_1 01\dots 1}^{(2)} & \dots & \mathbf{b}_{00e_3\dots 1}^{(2)} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \mathbf{b}_{0m_2 1\dots 1}^{(2)} & \dots & \mathbf{b}_{m_1 m_2 1\dots 1}^{(2)} & \dots & \mathbf{b}_{0m_2 e_3\dots 1}^{(2)} & \dots \\ \mathbf{b}_{m_1 0e_3\dots 1}^{(2)} & \dots & \mathbf{b}_{001\dots e_n}^{(2)} & \dots & \mathbf{b}_{m_1 01\dots e_n}^{(2)} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \mathbf{b}_{m_1 m_2 e_3\dots 1}^{(2)} & \dots & \mathbf{b}_{0m_2 1\dots e_n}^{(2)} & \dots & \mathbf{b}_{m_1 m_2 1\dots e_n}^{(2)} & \dots \\ \dots & \dots & \mathbf{b}_{00e_3\dots e_n}^{(2)} & \dots & \mathbf{b}_{m_1 0e_3\dots e_n}^{(2)} & \dots \\ \mathbf{b}_{0m_2 e_3\dots e_n}^{(2)} & \dots & \mathbf{b}_{m_1 m_2 e_3\dots e_n}^{(2)} & \dots & \dots & \dots \end{bmatrix}, \quad (16)$$

the $e_2 \times ((m_1 + 1) \times e_3 \times \dots \times e_n)$ matrix $\hat{\mathbf{B}}^{(1)}$ of estimated control points by

$$\hat{\mathbf{B}}^{(1)} = \begin{bmatrix} \hat{\mathbf{b}}_{011\dots 1}^{(1)} & \dots & \hat{\mathbf{b}}_{m_1 11\dots 1}^{(1)} & \dots & \hat{\mathbf{b}}_{01e_3\dots 1}^{(1)} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \hat{\mathbf{b}}_{0e_2 1\dots 1}^{(1)} & \dots & \hat{\mathbf{b}}_{m_1 e_2 1\dots 1}^{(1)} & \dots & \hat{\mathbf{b}}_{0e_2 e_3\dots 1}^{(1)} & \dots \\ \hat{\mathbf{b}}_{m_1 1e_3\dots 1}^{(1)} & \dots & \hat{\mathbf{b}}_{011\dots e_n}^{(1)} & \dots & \hat{\mathbf{b}}_{m_1 11\dots e_n}^{(1)} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \hat{\mathbf{b}}_{m_1 e_2 e_3\dots 1}^{(1)} & \dots & \hat{\mathbf{b}}_{0e_2 1\dots e_n}^{(1)} & \dots & \hat{\mathbf{b}}_{m_1 e_2 1\dots e_n}^{(1)} & \dots \\ \dots & \dots & \hat{\mathbf{b}}_{01e_3\dots e_n}^{(1)} & \dots & \hat{\mathbf{b}}_{m_1 1e_3\dots e_n}^{(1)} & \dots \\ \hat{\mathbf{b}}_{0e_2 e_3\dots e_n}^{(1)} & \dots & \hat{\mathbf{b}}_{m_1 e_2 e_3\dots e_n}^{(1)} & \dots & \dots & \dots \end{bmatrix}, \quad (17)$$

and the $e_2 \times ((m_1 + 1) \times e_3 \times \dots \times e_n)$ matrix $\mathbf{E}_{\hat{\mathbf{B}}^{(1)}}$ of errors of $\hat{\mathbf{B}}^{(1)}$ by replacing $\hat{\mathbf{b}}^{(1)}$ in (17) by $\mathbf{e}^{(1)}$.

Eq. (14) represents the observation equations of a multivariate linear model by which the control points $\mathbf{b}_{i_1 i_2 a_3 \dots a_n}^{(2)}$ of the isoparametric curves $s(\xi_1 = \text{const}, \xi_2, \xi_3 = \text{const}, \dots, \xi_n = \text{const})$ are estimated $m_1 + 1$ times for each value of ξ_1 , e_3 times for each

value of ξ_3 up to e_n times for each value of ξ_n . The estimate $\hat{\mathbf{B}}^{(2)}$ of $\mathbf{B}^{(2)}$ follows by

$$\hat{\mathbf{B}}^{(2)} = (\mathbf{N}(\xi_2)' \mathbf{N}(\xi_2))^{-1} \mathbf{N}(\xi_2)' \hat{\mathbf{B}}^{(1)}. \quad (18)$$

Correspondingly, the estimates of the control points of the isoparametric curves continue, until in the n th and final step, (7) is used as observation equations for estimating the unknown control points $\mathbf{p}_{i_1 i_2 i_3 \dots i_n}$ of the isoparametric curves $s(\xi_1 = \text{const}, \xi_2 = \text{const}, \dots, \xi_n)$. The observation equations are given in matrix notation by

$$\mathbf{N}(\xi_n) \hat{\mathbf{P}} = \hat{\mathbf{B}}^{(n-1)} + \mathbf{E}_{\hat{\mathbf{B}}^{(n-1)}}, \quad (19)$$

where the $e_n \times (m_n + 1)$ matrix $\mathbf{N}(\xi_n)$ of B-spline basis functions is given by

$$\mathbf{N}(\xi_n) = \begin{bmatrix} N_{0q_n}(\xi_{n1}) & \dots & N_{m_n q_n}(\xi_{n1}) \\ \dots & \dots & \dots \\ N_{0q_n}(\xi_{ne_n}) & \dots & N_{m_n q_n}(\xi_{ne_n}) \end{bmatrix}, \quad (20)$$

the $(m_n + 1) \times ((m_1 + 1) \times (m_2 + 1) \times \dots \times (m_{n-1} + 1))$ matrix $\hat{\mathbf{P}}$ by

$$\hat{\mathbf{P}} = \begin{bmatrix} p_{000\dots 0} & \dots & p_{m_1 00\dots 0} & \dots & p_{0m_2 0\dots 0} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ p_{000\dots m_n} & \dots & p_{m_1 00\dots m_n} & \dots & p_{0m_2 0\dots m_n} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ p_{m_1 m_2 0\dots 0} & \dots & p_{00m_3\dots 0} & \dots & p_{m_1 0m_3\dots 0} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ p_{m_1 m_2 0\dots m_n} & \dots & p_{00m_3\dots m_n} & \dots & p_{m_1 0m_3\dots m_n} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ p_{0m_2 m_3\dots 0} & \dots & p_{m_1 m_2 m_3\dots 0} & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ p_{0m_2 m_3\dots m_n} & \dots & p_{m_1 m_2 m_3\dots m_n} & \dots & \dots & \dots \end{bmatrix}, \quad (21)$$

the $e_n \times ((m_1 + 1) \times (m_2 + 1) \times \dots \times (m_{n-1} + 1))$ matrix $\hat{\mathbf{B}}^{(n-1)}$ by

$$\hat{\mathbf{B}}^{(n-1)} = \begin{bmatrix} \hat{\mathbf{b}}_{000\dots 1}^{(n-1)} & \dots & \hat{\mathbf{b}}_{m_1 00\dots 1}^{(n-1)} & \dots & \hat{\mathbf{b}}_{0m_2 0\dots 1}^{(n-1)} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \hat{\mathbf{b}}_{000\dots e_n}^{(n-1)} & \dots & \hat{\mathbf{b}}_{m_1 00\dots e_n}^{(n-1)} & \dots & \hat{\mathbf{b}}_{0m_2 0\dots e_n}^{(n-1)} & \dots \\ \hat{\mathbf{b}}_{m_1 m_2 0\dots 1}^{(n-1)} & \dots & \hat{\mathbf{b}}_{00m_3\dots 1}^{(n-1)} & \dots & \hat{\mathbf{b}}_{m_1 0m_3\dots 1}^{(n-1)} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \hat{\mathbf{b}}_{m_1 m_2 0\dots e_n}^{(n-1)} & \dots & \hat{\mathbf{b}}_{00m_3\dots e_n}^{(n-1)} & \dots & \hat{\mathbf{b}}_{m_1 0m_3\dots e_n}^{(n-1)} & \dots \\ \dots & \dots & \hat{\mathbf{b}}_{0m_2 m_3\dots 1}^{(n-1)} & \dots & \hat{\mathbf{b}}_{m_1 m_2 m_3\dots 1}^{(n-1)} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \hat{\mathbf{b}}_{0m_2 m_3\dots e_n}^{(n-1)} & \dots & \hat{\mathbf{b}}_{m_1 m_2 m_3\dots e_n}^{(n-1)} & \dots & \dots & \dots \end{bmatrix}, \quad (22)$$

and the $e_n \times ((m_1 + 1) \times (m_2 + 1) \times \dots \times (m_{n-1} + 1))$ matrix $E_{\hat{B}^{(n-1)}}$ of errors of $\hat{B}^{(n-1)}$ by replacing $\hat{b}^{(n-1)}$ in (22) by $e^{(n-1)}$. Eq. (19) represents the observation equations of a multivariate linear model by which the control points $P_{i_1 i_2 \dots i_n}$ of the isoparametric curves $s(\xi_1 = \text{const}, \xi_2 = \text{const}, \dots, \xi_n)$ are estimated $m_1 + 1$ times for each value of ξ_1 , $m_2 + 1$ times for each value of ξ_2 up to $m_{n-1} + 1$ times for each value of ξ_{n-1} . The estimate $\hat{P}$ of P follows by

$$\hat{P} = (N(\xi_n)' N(\xi_n))^{-1} N(\xi_n)' \hat{B}^{(n-1)}. \quad (23)$$

The matrix $\hat{E}$ of residuals follows from (19) and with the estimate $\hat{P}$ from (23) by

$$\hat{B}^{(n-1)} = N(\xi_n) \hat{P}, \quad (24)$$

and so on, from (14) by

$$\hat{B}^{(1)} = N(\xi_2) \hat{B}^{(2)}, \quad (25)$$

and finally from (8) by

$$\hat{E} = N(\xi_1) \hat{B}^{(1)} - S. \quad (26)$$

The matrix $\hat{E}$ of residuals, i.e. of the estimated errors E of the coordinates of the given points is needed to compute the variance factor of the estimation which leads to the estimated variance of the measurements, cf. Koch (2007) p. 85. The standard deviation of the measurements gives an indication about the quality of the fit of the B-spline surface to the given points. A minimum standard deviation should be searched for by selecting a minimum number of control points and minimum degrees for the B-spline basis functions.

If points $s(\xi_{1w_1}, \xi_{2w_2}, \dots, \xi_{nw_n})$ with given location parameters $\xi_{1w_1}, \xi_{2w_2}, \dots, \xi_{nw_n}$ shall be computed on the estimated B-spline surface, the matrices $N(\xi_{1w_1}), N(\xi_{2w_2}), \dots, N(\xi_{nw_n})$ corresponding to (9), (15) and (20) have to be computed. Eqs. (24) to (26) then give

$$\hat{B}_w^{(n-1)} = N(\xi_{nw_n}) \hat{P}, \quad (27)$$

$$\hat{B}_w^{(1)} = N(\xi_{2w_2}) \hat{B}_w^{(2)}, \quad (28)$$

$$\hat{S}_w = N(\xi_{1w_1}) \hat{B}_w^{(1)}. \quad (29)$$

Any points $\hat{S}_w$ on the estimated surface may be computed by these equations.

The estimates (13) are efficiently computed by one Cholesky factorization of the $(m_1 + 1) \times (m_1 + 1)$ matrix $N(\xi_1)' N(\xi_1)$ of normal equations followed by $e_2 \times e_3 \times \dots \times e_n$ back solutions, cf. Koch (1999) p. 30. Correspondingly, the estimates (18) and (23) are obtained.

We assume for an approximate assessment of the numerical complexity of the computations in case of a large number of given points

$$e_1 \approx e_2 \approx \dots \approx e_n \approx m_1 + 1 \approx m_2 + 1 \approx \dots \approx m_n + 1 \approx v. \quad (30)$$

The complexity of the Cholesky factorization is $O(v^3)$ and of one back solution $O(v^2)$. By repeating the back solutions v^{n-1} times, the numerical complexity $O(v^{n+1})$ is obtained for estimating the v^n control points of an n -dimensional surface by the lofting method. Under the assumption of (30), the complexity of estimating the control points of a three-dimensional surface by the lofting method is $O(v^4)$ and of a two-dimensional surface $O(v^3)$, Koch (2009a) and Koch (2010b).

Conditions have to be fulfilled to apply the lofting method. Eq. (2) introduces the coordinates $x_1, x_2, \dots, x_n$ as functions of the parameters $\xi_1, \xi_2, \dots, \xi_n$. Inverting these relations gives the parameters as functions of all coordinates. However, the dependency of the parameters has to be restricted to one coordinate, i.e. $\xi_1(x_1), \xi_2(x_2), \dots, \xi_n(x_n)$, to use the observation equations (4). Furthermore, each isoparametric curve gets an identical knot vector and an identical degree for the B-spline basis functions, and the coordinates of the $e_1 \times e_2 \times \dots \times e_n$ given points have to be ordered in an n -dimensional grid.

4. Simultaneous estimate of control points and proof of equivalence

By generalizing the result of Koch (2010b) for expressing the tensor product by Kronecker products, we conclude that the observation equations

$$(N(\xi_n) \otimes \dots \otimes N(\xi_2) \otimes N(\xi_1)) \text{vec} P = \text{vec} S + \text{vec} E \quad (31)$$

are equivalent to the ones of (4) with the $(m_1 + 1) \times ((m_2 + 1) \times (m_3 + 1) \times \dots \times (m_n + 1))$ matrix P of unknown control points defined by

$$P = \begin{vmatrix} p_{000\dots 0} & \dots & p_{0m_2 0\dots 0} & \dots & p_{00m_3\dots 0} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ p_{m_1 00\dots 0} & \dots & p_{m_1 m_2 0\dots 0} & \dots & p_{m_1 0m_3\dots 0} & \dots \\ p_{0m_2 m_3\dots 0} & \dots & p_{000\dots m_n} & \dots & p_{0m_2 0\dots m_n} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ p_{m_1 m_2 m_3\dots 0} & \dots & p_{m_1 00\dots m_n} & \dots & p_{m_1 m_2 0\dots m_n} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ p_{00m_3\dots m_n} & \dots & p_{0m_2 m_3\dots m_n} & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ p_{m_1 0m_3\dots m_n} & \dots & p_{m_1 m_2 m_3\dots m_n} & \dots & \dots & \dots \end{vmatrix}. \quad (32)$$

To show the equivalence, we set $a_1 = a_2 = \dots = a_n = 1$ and obtain with the definition of the Kronecker product, cf.

Koch (1999) p. 18,

$$\begin{aligned}
 & N_{0q_n}(\xi_{n1}) \dots N_{0q_3}(\xi_{31}) \sum_{i_2=0}^{m_2} N_{i_2q_2}(\xi_{21}) \\
 & \quad \sum_{i_1=0}^{m_1} N_{i_1q_1}(\xi_{11}) \mathbf{p}_{i_1 i_2 0 \dots 0} + \dots \\
 & + N_{0q_n}(\xi_{n1}) \dots N_{m_3q_3}(\xi_{31}) \sum_{i_2=0}^{m_2} N_{i_2q_2}(\xi_{21}) \\
 & \quad \sum_{i_1=0}^{m_1} N_{i_1q_1}(\xi_{11}) \mathbf{p}_{i_1 i_2 m_3 \dots 0} + \dots \\
 & + N_{m_nq_n}(\xi_{n1}) \dots N_{0q_3}(\xi_{31}) \sum_{i_2=0}^{m_2} N_{i_2q_2}(\xi_{21}) \\
 & \quad \sum_{i_1=0}^{m_1} N_{i_1q_1}(\xi_{11}) \mathbf{p}_{i_1 i_2 0 \dots m_n} + \dots \\
 & + N_{m_nq_n}(\xi_{n1}) \dots N_{m_3q_3}(\xi_{31}) \sum_{i_2=0}^{m_2} N_{i_2q_2}(\xi_{21}) \\
 & \quad \sum_{i_1=0}^{m_1} N_{i_1q_1}(\xi_{11}) \mathbf{p}_{i_1 i_2 m_3 \dots m_n} = \\
 & \quad \mathbf{s}_{111\dots 1} + \mathbf{e}_{111\dots 1} \quad (33)
 \end{aligned}$$

which is the first observation equation of (4). By setting $\alpha_1 \in \{1, \dots, e_1\}$, $\alpha_2 \in \{1, \dots, e_2\}$, ..., $\alpha_n \in \{1, \dots, e_n\}$, all observation equations are obtained.

The simultaneous estimate $\text{vec} \hat{\mathbf{P}}$ of $\text{vec} \mathbf{P}$ of the $(m_1 + 1) \times (m_2 + 1) \times \dots \times (m_n + 1)$ unknown control points results from (31) with

$$\begin{aligned}
 \text{vec} \hat{\mathbf{P}} &= [(\mathbf{N}(\xi_n) \otimes \dots \otimes \mathbf{N}(\xi_2) \otimes \mathbf{N}(\xi_1))' \\
 & \quad (\mathbf{N}(\xi_n) \otimes \dots \otimes \mathbf{N}(\xi_2) \otimes \mathbf{N}(\xi_1))]^{-1} \\
 & \quad (\mathbf{N}(\xi_n) \otimes \dots \otimes \mathbf{N}(\xi_2) \otimes \mathbf{N}(\xi_1))' \text{vec} \mathbf{S}. \quad (34)
 \end{aligned}$$

For an assessment of the numerical complexity of solving (34), we assume again like in (30)

$$m_1 + 1 \approx m_2 + 1 \approx \dots \approx m_n + 1 \approx \nu \quad (35)$$

and obtain the complexity of $O(\nu^{3n})$ for simultaneously estimating the ν^n control points of an n -dimensional surface. The complexity is $O(\nu^9)$ for simultaneously estimating the control points of a three-dimensional surface and $O(\nu^6)$ for a two-dimensional surface, Koch (2009a) and Koch (2010b). As mentioned in the previous section, the corresponding values for the lofting method are $O(\nu^4)$ and $O(\nu^3)$. If possible, the simultaneous estimate should therefore be avoided in case of many unknown control points because the lofting method gives identical results not only for two- or three-dimensional surfaces but also for n -dimensional ones.

To show it, we expand (34) by the rules of the Kronecker product, cf. Koch (1999) p. 18, and obtain

$$\text{vec} \hat{\mathbf{P}} = (\mathbf{M}(\xi_n) \otimes \dots \otimes \mathbf{M}(\xi_2) \otimes \mathbf{M}(\xi_1)) \text{vec} \mathbf{S} \quad (36)$$

where the $(m_n + 1) \times e_n$ matrix $\mathbf{M}(\xi_n)$ is given by

$$\begin{aligned}
 \mathbf{M}(\xi_n) &= (\mathbf{N}(\xi_n)' \mathbf{N}(\xi_n))^{-1} \mathbf{N}(\xi_n)' \\
 &= \begin{bmatrix} M_{01}(\xi_n) & \dots & M_{0e_n}(\xi_n) \\ \dots & \dots & \dots \\ M_{m_n 1}(\xi_n) & \dots & M_{m_n e_n}(\xi_n) \end{bmatrix}, \quad (37)
 \end{aligned}$$

the $(m_2 + 1) \times e_2$ matrix $\mathbf{M}(\xi_2)$ by

$$\begin{aligned}
 \mathbf{M}(\xi_2) &= (\mathbf{N}(\xi_2)' \mathbf{N}(\xi_2))^{-1} \mathbf{N}(\xi_2)' \\
 &= \begin{bmatrix} M_{01}(\xi_2) & \dots & M_{0e_2}(\xi_2) \\ \dots & \dots & \dots \\ M_{m_2 1}(\xi_2) & \dots & M_{m_2 e_2}(\xi_2) \end{bmatrix}, \quad (38)
 \end{aligned}$$

and the $(m_1 + 1) \times e_1$ matrix $\mathbf{M}(\xi_1)$ by

$$\begin{aligned}
 \mathbf{M}(\xi_1) &= (\mathbf{N}(\xi_1)' \mathbf{N}(\xi_1))^{-1} \mathbf{N}(\xi_1)' \\
 &= \begin{bmatrix} M_{01}(\xi_1) & \dots & M_{0e_1}(\xi_1) \\ \dots & \dots & \dots \\ M_{m_1 1}(\xi_1) & \dots & M_{m_1 e_1}(\xi_1) \end{bmatrix}. \quad (39)
 \end{aligned}$$

The first equation of (34) gives with (13), (18) and (23) the estimate $\hat{\mathbf{P}}_{000\dots 0}$

$$\begin{aligned}
 & M_{01}(\xi_n) \dots M_{01}(\xi_3) M_{01}(\xi_2) (M_{01}(\xi_1) \mathbf{s}_{111\dots 1} + \dots \\
 & \quad + M_{0e_1}(\xi_1) \mathbf{s}_{e_1 11\dots 1}) + \dots \\
 & + M_{01}(\xi_n) \dots M_{01}(\xi_3) M_{0e_2}(\xi_2) (M_{01}(\xi_1) \mathbf{s}_{1e_2 1\dots 1} + \dots \\
 & \quad + M_{0e_1}(\xi_1) \mathbf{s}_{e_1 e_2 1\dots 1}) + \dots \\
 & + M_{01}(\xi_n) \dots M_{0e_3}(\xi_3) M_{01}(\xi_2) (M_{01}(\xi_1) \mathbf{s}_{11e_3\dots 1} + \dots \\
 & \quad + M_{0e_1}(\xi_1) \mathbf{s}_{e_1 1e_3\dots 1}) + \dots \\
 & + M_{01}(\xi_n) \dots M_{0e_3}(\xi_3) M_{0e_2}(\xi_2) (M_{01}(\xi_1) \mathbf{s}_{1e_2 e_3\dots 1} + \dots \\
 & \quad + M_{0e_1}(\xi_1) \mathbf{s}_{e_1 e_2 e_3\dots 1}) + \dots \\
 & + M_{0e_n}(\xi_n) \dots M_{01}(\xi_3) M_{01}(\xi_2) (M_{01}(\xi_1) \mathbf{s}_{111\dots e_n} + \dots \\
 & \quad + M_{0e_1}(\xi_1) \mathbf{s}_{e_1 11\dots e_n}) + \dots \\
 & + M_{0e_n}(\xi_n) \dots M_{01}(\xi_3) M_{0e_2}(\xi_2) (M_{01}(\xi_1) \mathbf{s}_{1e_2 1\dots e_n} + \dots \\
 & \quad + M_{0e_1}(\xi_1) \mathbf{s}_{e_1 e_2 1\dots e_n}) + \dots \\
 & + M_{0e_n}(\xi_n) \dots M_{0e_3}(\xi_3) M_{01}(\xi_2) (M_{01}(\xi_1) \mathbf{s}_{11e_3\dots e_n} + \dots \\
 & \quad + M_{0e_1}(\xi_1) \mathbf{s}_{e_1 1e_3\dots e_n}) + \dots \\
 & + M_{0e_n}(\xi_n) \dots M_{0e_3}(\xi_3) M_{0e_2}(\xi_2) (M_{01}(\xi_1) \mathbf{s}_{1e_2 e_3\dots e_n} + \dots \\
 & \quad + M_{0e_1}(\xi_1) \mathbf{s}_{e_1 e_2 e_3\dots e_n}) =
 \end{aligned}$$

$$\begin{aligned}
& M_{01}(\xi_n) \dots M_{01}(\xi_3)(M_{01}(\xi_2)\hat{b}_{011\dots 1}^{(1)} + \dots \\
& \quad + M_{0e_2}(\xi_2)\hat{b}_{0e_21\dots 1}^{(1)} + \dots \\
& + M_{01}(\xi_n) \dots M_{0e_3}(\xi_3)(M_{01}(\xi_2)\hat{b}_{01e_3\dots 1}^{(1)} + \dots \\
& \quad + M_{0e_2}(\xi_2)\hat{b}_{0e_2e_3\dots 1}^{(1)} + \dots \\
& + M_{0e_n}(\xi_n) \dots M_{01}(\xi_3)(M_{01}(\xi_2)\hat{b}_{011\dots e_n}^{(1)} + \dots \\
& \quad + M_{0e_2}(\xi_2)\hat{b}_{0e_21\dots e_n}^{(1)} + \dots \\
& + M_{0e_n}(\xi_n) \dots M_{0e_3}(\xi_3)(M_{01}(\xi_2)\hat{b}_{01e_3\dots e_n}^{(1)} + \dots \\
& \quad + M_{0e_2}(\xi_2)\hat{b}_{0e_2e_3\dots e_n}^{(1)}) = \\
& \\
& M_{01}(\xi_n) \dots (M_{01}(\xi_3)\hat{b}_{001\dots 1}^{(2)} + \dots \\
& \quad + M_{0e_3}(\xi_3)\hat{b}_{00e_3\dots 1}^{(2)} + \dots \\
& + M_{0e_n}(\xi_n) \dots (M_{01}(\xi_3)\hat{b}_{001\dots e_n}^{(2)} + \dots \\
& \quad + M_{0e_3}(\xi_3)\hat{b}_{00e_3\dots e_n}^{(2)}) =
\end{aligned}$$

$$M_{01}(\xi_n)\hat{b}_{000\dots 1}^{(n-1)} + \dots + M_{0e_n}(\xi_n)\hat{b}_{000\dots e_n}^{(n-1)} = \hat{p}_{000\dots 0} \quad (40)$$

By writing down in addition the equations for $\hat{p}_{000\dots 1}, \dots, \hat{p}_{000\dots m_n}, \dots, \hat{p}_{m_100\dots 0}, \dots, \hat{p}_{m_100\dots m_n}, \dots, \hat{p}_{0m_20\dots 0}, \dots, \hat{p}_{0m_20\dots m_n}$ up to $\hat{p}_{m_1m_2m_3\dots m_n}$, one recognizes that $\hat{P}$ is obtained in agreement with (23) of the lofting method so that both estimations are equivalent. The residuals follow with (31) from

$$\text{vec} \hat{E} = (N(\xi_n) \otimes \dots \otimes N(\xi_2) \otimes N(\xi_1)) \text{vec} \hat{P} - \text{vec} S. \quad (41)$$

The observation equations (4) and (31) are equivalent and the estimates $\hat{P}$ of the control points agree. The matrix $\hat{E}$ of residuals from the simultaneous estimation must therefore be identical with the one obtained by the lofting method from (24) to (26). The same holds true for computing the points $\text{vec} \hat{S}_w$ with the location parameters $\xi_{1w_1}, \xi_{2w_2}, \dots, \xi_{nw_n}$ on the estimated B-spline surface. They follow from (41) by

$$\text{vec} \hat{S}_w = (N(\xi_{nw_n}) \otimes \dots \otimes N(\xi_{2w_2}) \otimes N(\xi_{1w_1})) \text{vec} \hat{P} \quad (42)$$

and agree with the points from (27) to (29) by the lofting method.

5. Extending B-spline surfaces from simultaneous estimates to higher dimensions by the lofting method

Let a k -dimensional B-spline surface be depending on the k parameters $\xi_1(x_1), \xi_2(x_2), \dots, \xi_k(x_k)$. Let the conditions mentioned at the end of Section 3 be fulfilled so that the lofting method can be applied to fit a k -dimensional B-spline surface to the quantities H .

We will again assume that the conditions for applying the lofting method are fulfilled. The quantities H , however, shall not be given at the $e_1 \times e_2 \times \dots \times e_k$ points but at different positions or given by functions of different positions. Thus, the observation equations (4) cannot be formed so that the simultaneous estimation of the control points is needed.

The lofting method gives a unique solution as mentioned in connection with (13). The solution of the simultaneous estimation is also unique if its matrix of normal equations is regular. It then agrees because of the uniqueness with the solution of the lofting method. The solution can therefore be assumed as being computed by the lofting method. If we want to extend the representation of the quantity H from the k -dimensional surface to the $(k+1)$ -dimensional one, we only have to repeat the simultaneous estimation m_{k+1} times and apply the lofting method. To assess the numerical complexity of this method, the approximations (30) and (35) are assumed again. The numerical complexity of simultaneously estimating v^k control points is $O(v^{3k})$. By repeating the estimates v times, the complexity $O(v^{3k+1})$ is obtained. The complexity does not increase by applying the lofting method because it is only $O(v^{k+1})$. The numerical complexity of increasing a k -dimensional surface obtained by the simultaneous estimation to a $(k+1)$ -dimensional surface using the lofting method is therefore $O(v^{3k+1})$. Simultaneously estimating v^{k+1} control points for a $(k+1)$ -dimensional surface gives a complexity of $O(v^{3k+3})$. Computational time is therefore saved by the lofting method. This holds also true when increasing a k -dimensional surface to $k+2, k+3, \dots, n$ dimensions.

The method of increasing dimensions will be applied to locally improving IRI. As mentioned in the introduction, Schmidt (2007) and Zeilhofer et al. (2009) estimated three-dimensional B-spline surfaces to represent the electron densities of points in identical spatial grids at different time epochs. They used STEC data and prior information to avoid a singular matrix of normal equations in a simultaneous adjustment. This kind of data does not allow to apply the lofting method. Instead of a three-dimensional B-spline surface at different time epochs, a four-dimensional surface is more appropriate. As explained above, it can be obtained by the lofting method. This requires a numerical complexity of $O(v^{10})$ in comparison to $O(v^{12})$ for a complete simultaneous estimation. The simultaneous estimation first and then the lofting method will therefore be applied for future analysis of STEC data to locally improve IRI.

6. Example

The identity of the control points of a four-dimensional B-spline surface computed by the lofting method and the simultaneous estimation shall be confirmed by a simple example. It will show the potential of a four-dimensional B-spline surface to represent data, but it is not intended to set up a model for IRI. The electron density N_e with dimension [electrons/ m^3] of the ionosphere from

IRI, version 2007, Bilitza and Reinisch (2008), is calculated for 10 points of longitudes 270° , 275° to 315° , for 10 points of latitudes -25° , -20° to 20° , for 10 points of heights 100 km, 150 km to 550 km and for 12 points of times 0 h, 2 h to 22 h UT for June 21, 2007. The data include the equatorial anomaly with maximum variations of N_e between the longitudes 280° to 300° , at the height around 300 km and at the time around 18 h. As an example of large variations of N_e , the height $z=300$ km and time $t=18$ h are chosen. The global variations of N_e at that time and height are shown in Figure 1. The 10×10 points with given densities N_e from IRI lie within the white box and are depicted in black in Figure 2 with respect to the x, y plane, where x is expressed by longitude and y by latitude. The 10×10 points in black with height $z=350$ km and and time $t=20$ h are shown in Figure 3 as a second example with large variations of N_e .

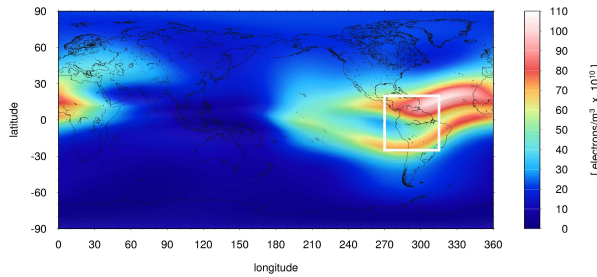


Figure 1. Global distribution of the electron density N_e calculated from IRI for June 21, 2007 for height $z=300$ km at time $t=18$ h UT. The points with given densities N_e within the white box are shown as black points in Figure 2.

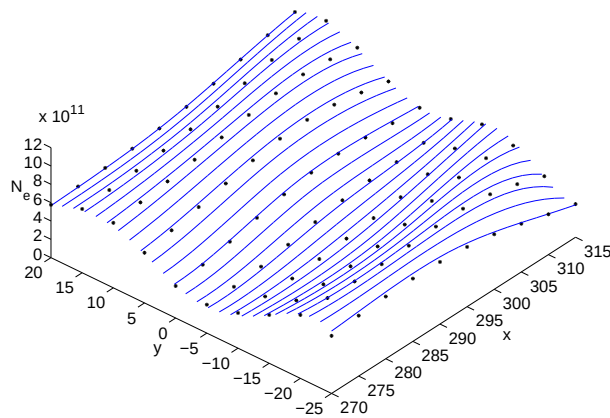


Figure 2. Electron densities N_e of given points in black and isoparametric curves from points computed on the surface in blue for height $z=300$ km and time $t=18$ h.

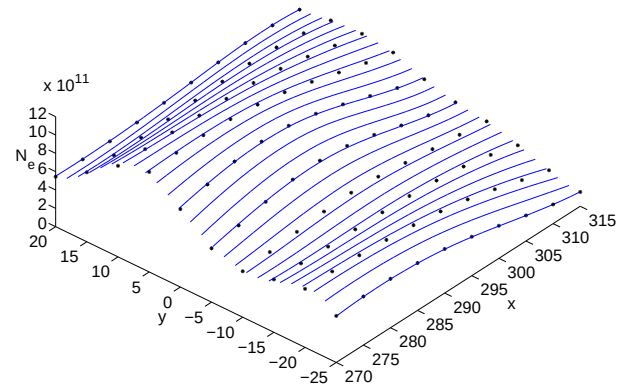


Figure 3. Electron densities N_e of given points in black and isoparametric curves from points computed on the surface in blue for height $z=350$ km and time $t=20$ h.

A four-dimensional B-spline surface is fitted to the $10 \times 10 \times 10 \times 12$ given points of the spatial and temporal grid first by the lofting method with (13), (18), (23) and then by the simultaneous estimation with (34). The points are equally spaced, the location parameters are therefore also chosen to be equally spaced. The quality of the fit is judged by the square root of the variance factor of the electron density, i.e. by the standard deviation with which the electron densities of the given points are approximated by the B-spline surface. It is computed with (24) to (26) by the residuals of the lofting method and with (41) by the residuals of the simultaneous estimation. A small standard deviation with few unknown control points is desirable for a good fit to avoid oscillations of the B-spline surface between the given points. It has been found with $m_1=5$, $m_2=7$, $m_3=8$ and $m_4=10$, i.e. with 2 800 unknown control points, and with the orders $q_1 = \dots = q_4 = 3$ in (4). The maximum value N_e of the $10 \times 10 \times 10 \times 12$ given points is 1.11×10^{12} , the minimum 5.27×10^8 . The standard deviation of the fit is 1.43×10^{10} which is large in comparison to the maximum and minimum value of N_e . It is caused by the large spacing between the positions of the given points along the longitude, latitude, height and time. A denser grid of points reduces the standard deviation of the fit. Such a grid has to be chosen for modeling IRI.

To get an impression of the shape of the four-dimensional surface, a spatial and temporal grid of $26 \times 26 \times 19 \times 23$ points is computed by (27) to (29) of the lofting method and by (42) of the simultaneous estimation on the fitted surface. It is represented with respect to the x, y plane by isoparametric curves for the parameter ξ_1 obtained by connecting the points on the surface with equal values of y by straight lines. The grid in the x, y plane for the computed points differs from the grid of the given points. The isoparametric curves are depicted in blue in Figure 2 for the height $z=300$ km and time $t=18$ h and in Figure 3 for the height $z=350$ km and time $t=20$ h, i.e.

for the values of z and t for which the given points are shown in Figure 2 and Figure 3. The results indicate that the B-spline surface is smooth and does not oscillate between the points with given densities. Figure 4 depicts the isoparametric curves for the height $z=325$ km and time $t=17$ h in red as well as $t=21$ h in blue, which means at a height and a time for which no points with electron densities are given. Figure 4 shows how the density at height $z=325$ km is changing between $t=17$ h and $t=21$ h. The surface in red lies close to the surface of Figure 2 and the surface in blue close to the surface of Figure 3. It should be mentioned that points on the four-dimensional B-spline surface may be computed at any position and time.

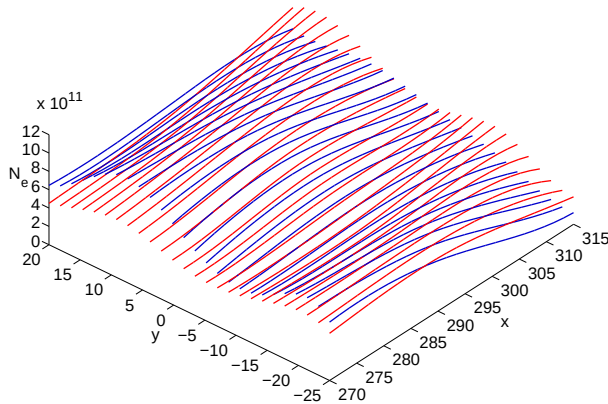


Figure 4. Isoparametric curves from points computed on the surface for height $z=325$ km, time $t=17$ h in red and $t=21$ h in blue.

Using double precision for the computations, the electron densities of the control points and of the points on the surface obtained by the lofting method and the simultaneous estimation agree with at least 12 significant digits. The standard deviation of the electron densities agrees with 15 digits. This confirms the analytical proof that the lofting method and the simultaneous estimation give identical results.

7. Conclusions

Estimating the unknown control points of an n -dimensional B-spline surface by the lofting method using cross-sectional curve fits is much faster than the simultaneous estimation. It is shown by an analytical proof and confirmed by an example of a four-dimensional surface that the results of both methods are identical. The simultaneous estimation should therefore be avoided if it is possible. If not, the results of simultaneously estimated B-spline surfaces of lower dimensions can be extended to higher dimensions by the lofting method, thus getting more efficient algorithms. An application is the simultaneous estimate of the control points of three-dimensional B-spline surfaces for a local

representation of the electron density of the ionosphere at different time epochs. The representation can be extended to a four-dimensional surface by the lofting method. This method takes less computational time than solely using the simultaneous estimation. It will be used for future analysis of STEC data to locally improve IRI.

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