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RESEARCH PAPER

POSITIVE SOLUTIONS FOR A SYSTEM OF NONLOCAL FRACTIONAL BOUNDARY VALUE PROBLEMS

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Abstract

We investigate the existence of positive solutions for a system of nonlinear Riemann-Liouville fractional differential equations, subject to multipoint boundary conditions. Existence results for systems of nonlinear Hammerstein integral equations are also presented. Some nontrivial examples are included.

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1. Introduction

Fractional differential equations describe many phenomena in various fields of engineering and scientific disciplines such as physics, biophysics, chemistry, biology (such as blood flow phenomena), economics, control theory, signal and image processing, aerodynamics, viscoelasticity, electromagnetics, and so on (see [8], [10], [20], [23], [24], [25]). For some recent developments on the topic, see [1]-[7], [9], [11]-[12], [19], [22] and the references therein.

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In this paper, we consider the system of nonlinear ordinary fractional differential equations

$$(S)\qquad \begin{cases} D_{0+}^{\alpha}u(t)+\lambda f(t,u(t),v(t))=0, & t\in(0,1), \quad n-1<\alpha\le n,\\ D_{0+}^{\beta}v(t)+\mu g(t,u(t),v(t))=0, & t\in(0,1), \quad m-1<\beta\le m,\end{cases}$$

with the multi-point boundary conditions

$$(BC)\qquad \begin{cases} u(0)=u'(0)=\cdots=u^{(n-2)}(0)=0, \quad u(1)=\displaystyle\sum_{i=1}^{p}a_iu(\xi_i),\\ v(0)=v'(0)=\cdots=v^{(m-2)}(0)=0, \quad v(1)=\displaystyle\sum_{i=1}^{q}b_iv(\eta_i),\end{cases}$$

where $n,\ m\in\mathbb{N}$, $n,\ m\ge 2$, $p,\ q\in\mathbb{N}$, $a_i\in\mathbb{R}$ for all $i=1,\dots,p$, $b_i\in\mathbb{R}$ for all $i=1,\dots,q$,

$$0<\xi_1<\cdots<\xi_p<1, \quad 0<\eta_1<\cdots<\eta_q<1,$$

D_{0+}^{α} and D_{0+}^{β} denote the Riemann-Liouville derivatives of orders α and β, respectively.

We shall give sufficient conditions on λ, μ, f and g such that positive solutions of $(S)-(BC)$ exist. By a positive solution of problem $(S)-(BC)$ we mean a pair of functions $(u,v)\in C([0,1])\times C([0,1])$ satisfying (S) and (BC) with $u(t)\ge 0$, $v(t)\ge 0$ for all $t\in[0,1]$ and $\sup_{t\in[0,1]}u(t)+\sup_{t\in[0,1]}v(t)>0$. The system (S) with $\alpha=n$, $\beta=m$, $f(t,u,v)=a(t)f_1(u,v)$, $g(t,u,v)=b(t)g_1(u,v)$ and the boundary conditions (BC) has been studied in [14], (see also [15] for a generalization of the problem investigated in [14]).

In Section **2**, we present the necessary definitions and properties from the fractional calculus theory and then we give some auxiliary results which investigate a nonlocal boundary value problem for fractional differential equations. In Section **3**, we prove two existence theorems for the positive solutions with respect to a cone for our problem $(S)-(BC)$, which are based on the Guo-Krasnosel'skii fixed point theorem. Existence results for general systems of nonlinear Hammerstein integral equations are presented in Section **4**. Finally, some examples are given in Section **5** to illustrate our main results.

2. Preliminaries and auxiliary results

We present here the definitions, some lemmas from the theory of fractional calculus and some auxiliary results that will be used to prove our main theorems.

DEFINITION **2.1**. The (left-sided) fractional integral of order $\alpha > 0$ of a function $f : (0, \infty) \to \mathbb{R}$ is given by

$$(I_{0+}^{\alpha} f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds, \ \ t > 0,$$

provided the right-hand side is pointwise defined on $(0, \infty)$, where $\Gamma(\alpha)$ is the Euler gamma function defined by $\Gamma(\alpha) = \displaystyle\int_0^{\infty} t^{\alpha-1} e^{-t}\, dt$, $\alpha > 0$.

DEFINITION **2.2**. The Riemann-Liouville fractional derivative of order $\alpha \geq 0$ for a function $f : (0, \infty) \to \mathbb{R}$ is given by

$$(D_{0+}^{\alpha} f)(t) = \left(\frac{d}{dt}\right)^n \left(I_{0+}^{n-\alpha} f\right)(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{f(s)}{(t-s)^{\alpha-n+1}}\, ds,$$

$t > 0$, where $n = [\![\alpha]\!] + 1$, provided that the right-hand side is pointwise defined on $(0, \infty)$.

The notation $[\![\alpha]\!]$ stands for the largest integer not greater than α. We also denote the Riemann-Liouville fractional derivative of f by $D_{0+}^{\alpha} f(t)$. If $\alpha = m \in \mathbb{N}$ then $D_{0+}^{m} f(t) = f^{(m)}(t)$ for $t > 0$, and if $\alpha = 0$ then $D_{0+}^{0} f(t) = f(t)$ for $t > 0$.

LEMMA **2.1**. ([20]) *a) If $\alpha > 0$, $\beta > 0$ and $f \in L^p(0,1)$, $(1 \leq p \leq \infty)$, then the relation $(I_{0+}^{\alpha} I_{0+}^{\beta} f)(t) = (I_{0+}^{\alpha+\beta} f)(t)$ is satisfied at almost every point $t \in (0,1)$. If $\alpha + \beta > 1$, then the above relation holds at any point of $[0,1]$.*

b) If $\alpha > 0$ and $f \in L^p(0,1)$, $(1 \leq p \leq \infty)$, then the relation $(D_{0+}^{\alpha} I_{0+}^{\alpha} f)(t) = f(t)$ holds almost everywhere on $(0,1)$.

c) If $\alpha > \beta > 0$ and $f \in L^p(0,1)$, $(1 \leq p \leq \infty)$, then the relation $(D_{0+}^{\beta} I_{0+}^{\alpha} f)(t) = (I_{0+}^{\alpha-\beta} f)(t)$ holds almost everywhere on $(0,1)$.

LEMMA **2.2**. ([20]) *Let $\alpha > 0$ and $n = [\![\alpha]\!] + 1$ for $\alpha \notin \mathbb{N}$ and $n = \alpha$ for $\alpha \in \mathbb{N}$; that is, n is the smallest integer greater than or equal to α. Then, the solutions of the fractional differential equation $D_{0+}^{\alpha} u(t) = 0$, $0 < t < 1$, are*

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}, \ \ 0 < t < 1,$$

where $c_1, c_2, \ldots, c_n$ are arbitrary real constants.

LEMMA **2.3**. *Let $\alpha > 0$, n be the smallest integer greater than or equal to α $(n-1 < \alpha \leq n)$ and $y \in L^1(0,1)$. The solutions of the fractional* *equation* $D_{0+}^{\alpha}u(t)+y(t)=0,\ \ 0<t<1$, *are*

$$u(t)=-\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)\,ds+c_1t^{\alpha-1}+\cdots+c_nt^{\alpha-n},\ \ 0<t<1,$$

where $c_1,\ c_2,...,c_n$ *are arbitrary real constants.*

P r o o f. By Lemma **2.1** b), the equation $D_{0+}^{\alpha}u(t)+y(t)=0$ can be written as

$$D_{0+}^{\alpha}u(t)+D_{0+}^{\alpha}(I_{0+}^{\alpha}y)(t)=0 \ \text{ or } \ D_{0+}^{\alpha}(u+I_{0+}^{\alpha}y)(t)=0.$$

By using Lemma **2.2**, the solutions for the above equation are

$$\begin{aligned}&u(t)+I_{0+}^{\alpha}y(t)=c_1t^{\alpha-1}+\cdots+c_nt^{\alpha-n} \ \Leftrightarrow\\ &u(t)=-I_{0+}^{\alpha}y(t)+c_1t^{\alpha-1}+\cdots+c_nt^{\alpha-n}\\ &=-\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)\,ds+c_1t^{\alpha-1}+\cdots+c_nt^{\alpha-n},\ \ 0<t<1,\end{aligned}$$

where $c_1,\ c_2,...,c_n$ are arbitrary real constants. □

We consider now the fractional differential equation

$$D_{0+}^{\alpha}u(t)+y(t)=0,\ \ 0<t<1,\ \ n-1<\alpha\le n, \tag{2.1}$$

with the multi-point boundary conditions

$$u(0)=u'(0)=\cdots=u^{(n-2)}(0)=0,\ \ u(1)=\sum_{i=1}^{p}a_iu(\xi_i), \tag{2.2}$$

where $n\in\mathbb{N}$, $n\ge 2$, $p\in\mathbb{N}$, $0<\xi_1<\cdots<\xi_p<1$, $a_i\in\mathbb{R}$ for all $i=1,\dots,p$.

LEMMA **2.4**. *If* $0<\xi_1<\cdots<\xi_p<1$, $a_i\in\mathbb{R}$ *for all* $i=1,\dots,p$, $\Delta_1=1-\sum_{i=1}^{p}a_i\xi_i^{\alpha-1}\ne 0$ *and* $y\in C([0,1])$, *then the solution of problem (2.1)-(2.2) is given by*

$$\begin{aligned}u(t)=&-\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)\,ds+\frac{t^{\alpha-1}}{\Delta_1\Gamma(\alpha)}\left[\int_0^1(1-s)^{\alpha-1}y(s)\,ds\right.\\ &\left.-\sum_{i=1}^{p}a_i\int_0^{\xi_i}(\xi_i-s)^{\alpha-1}y(s)\,ds\right],\ \ 0\le t\le 1.\end{aligned} \tag{2.3}$$

P r o o f. By Lemma **2.3**, the solutions of equation (2.1) are

$$u(t)=-\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)\,ds+c_1t^{\alpha-1}+\cdots+c_nt^{\alpha-n},$$

with $c_1, ..., c_n \in \mathbb{R}$. By using the conditions $u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0$, we obtain $c_2 = \cdots = c_n = 0$. Then, we conclude

$$u(t) = c_1 t^{\alpha-1} - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\, ds.$$

Now, by condition $u(1) = \sum_{i=1}^{p} a_i u(\xi_i)$, we deduce

$$\begin{aligned} & c_1 - \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} y(s)\, ds \\ & = \sum_{i=1}^{p} a_i \left[c_1 \xi_i^{\alpha-1} - \frac{1}{\Gamma(\alpha)} \int_0^{\xi_i} (\xi_i - s)^{\alpha-1} y(s)\, ds \right], \end{aligned}$$

from which we find

$$c_1 = \frac{1}{\Delta_1} \left(\frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} y(s)\, ds - \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{p} a_i \int_0^{\xi_i} (\xi_i - s)^{\alpha-1} y(s) ds \right).$$

Therefore, we obtain the expression (2.3) for the solution of problem (2.1)-(2.2). □

LEMMA **2.5**. *Under the assumptions of Lemma* **2.4**, *the Green function for the boundary value problem (2.1)-(2.2) is given by*

$$G_1(t,s) = g_1(t,s) + \frac{t^{\alpha-1}}{\Delta_1} \sum_{i=1}^{p} a_i g_1(\xi_i, s), \quad (t,s) \in [0,1] \times [0,1], \tag{2.4}$$

where

$$g_1(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1}(1-s)^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \le s \le t \le 1, \\ t^{\alpha-1}(1-s)^{\alpha-1}, & 0 \le t \le s \le 1. \end{cases} \tag{2.5}$$

P r o o f. By Lemma **2.4** and relation (2.3), we conclude

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(\alpha)} \left\{ \int_0^t [t^{\alpha-1}(1-s)^{\alpha-1} - (t-s)^{\alpha-1}] y(s)\, ds \right. \\ &+ \int_t^1 t^{\alpha-1}(1-s)^{\alpha-1} y(s)\, ds - \int_0^1 t^{\alpha-1}(1-s)^{\alpha-1} y(s)\, ds \\ &+ \left. \frac{t^{\alpha-1}}{\Delta_1} \left[\int_0^1 (1-s)^{\alpha-1} y(s)\, ds - \sum_{i=1}^{p} a_i \int_0^{\xi_i} (\xi_i - s)^{\alpha-1} y(s)\, ds \right] \right\} \\ &= \frac{1}{\Gamma(\alpha)} \left\{ \int_0^t [t^{\alpha-1}(1-s)^{\alpha-1} - (t-s)^{\alpha-1}] y(s)\, ds \right. \end{aligned}$$

$$+\int_t^1 t^{\alpha-1}(1-s)^{\alpha-1}y(s)\,ds - \frac{1}{\Delta_1}\left(1-\sum_{i=1}^{p} a_i\xi_i^{\alpha-1}\right)$$
$$\times\int_0^1 t^{\alpha-1}(1-s)^{\alpha-1}y(s)\,ds + \frac{t^{\alpha-1}}{\Delta_1}\left[\int_0^1 (1-s)^{\alpha-1}y(s)\,ds\right.$$
$$\left.\left.-\sum_{i=1}^{p} a_i\int_0^{\xi_i}(\xi_i-s)^{\alpha-1}y(s)\,ds\right]\right\} = \frac{1}{\Gamma(\alpha)}\left\{\int_0^t \left[t^{\alpha-1}(1-s)^{\alpha-1}\right.\right.$$
$$\left.-(t-s)^{\alpha-1}\right]y(s)\,ds + \int_t^1 t^{\alpha-1}(1-s)^{\alpha-1}y(s)\,ds$$
$$\left.+\frac{t^{\alpha-1}}{\Delta_1}\left[\sum_{i=1}^{p} a_i\xi_i^{\alpha-1}\int_0^1 (1-s)^{\alpha-1}y(s)\,ds - \sum_{i=1}^{p} a_i\int_0^{\xi_i}(\xi_i-s)^{\alpha-1}y(s)\,ds\right]\right\}$$
$$= \frac{1}{\Gamma(\alpha)}\left\{\int_0^t \left[t^{\alpha-1}(1-s)^{\alpha-1}-(t-s)^{\alpha-1}\right]y(s)\,ds\right.$$
$$+\int_t^1 t^{\alpha-1}(1-s)^{\alpha-1}y(s)\,ds + \frac{t^{\alpha-1}}{\Delta_1}\left[\sum_{i=1}^{p} a_i\int_0^{\xi_i}\left[\xi_i^{\alpha-1}(1-s)^{\alpha-1}\right.\right.$$
$$\left.\left.\left.-\ (\xi_i-s)^{\alpha-1}\right]y(s)\,ds + \sum_{i=1}^{p} a_i\int_{\xi_i}^1 \xi_i^{\alpha-1}(1-s)^{\alpha-1}y(s)\,ds\right]\right\}$$
$$= \int_0^1\left[g_1(t,s) + \frac{t^{\alpha-1}}{\Delta_1}\sum_{i=1}^{p} a_i g_1(\xi_i,s)\right]y(s)\,ds = \int_0^1 G_1(t,s)y(s)\,ds,$$

where g_1 is given by (2.5). Hence $u(t) = \int_0^1 G_1(t,s)y(s)\,ds$, where G_1 is given in (2.4). □

LEMMA **2.6**. *The function g_1 given by (2.5) has the properties:*

a) $g_1 : [0,1]\times[0,1] \to \mathbb{R}_+$ is a continuous function and $g_1(t,s) \geq 0$ for all $(t,s) \in [0,1]\times[0,1]$.

b) $g_1(t,s) \leq g_1(\theta_1(s),s)$, for all $(t,s) \in [0,1]\times[0,1]$.

c) For any $c \in (0,1/2)$,
$$\min_{t\in[c,1-c]} g_1(t,s) \geq \gamma_1 g_1(\theta_1(s),s),\ \text{for all } s\in[0,1],$$
where $\gamma_1 = c^{\alpha-1}$, $\theta_1(s) = s$ if $1<\alpha\leq 2$, $(n=2)$,

$$\text{and } \theta_1(s) = \begin{cases} \dfrac{s}{1-(1-s)^{\frac{\alpha-1}{\alpha-2}}}, & s\in(0,1], \\ \dfrac{\alpha-2}{\alpha-1}, & s=0, \end{cases} \quad \text{if } n-1<\alpha\leq n,\ n\geq 3.$$

The proof of Lemma **2.6** is similar to that of Lemma 3.3 from [18] (see also [16]). For $n \geq 3$, we define θ_1 at $s = 0$ as $\dfrac{\alpha-2}{\alpha-1}$ so that θ_1 is a continuous function.

LEMMA **2.7**. *If $a_i \geq 0$ for all $i = 1, \ldots, p$, $\Delta_1 = 1 - \sum_{i=1}^{p} a_i \xi_i^{\alpha-1} > 0$, $0 < \xi_1 < \cdots < \xi_p < 1$, then the Green function G_1 of the problem (2.1)-(2.2) is continuous on $[0,1] \times [0,1]$ and satisfies $G_1(t,s) \geq 0$ for all $(t,s) \in [0,1] \times [0,1]$. Moreover, if $y \in C([0,1])$ satisfies $y(t) \geq 0$ for all $t \in [0,1]$, then the unique solution u of problem (2.1)-(2.2) satisfies $u(t) \geq 0$ for all $t \in [0,1]$.*

P r o o f. By using the assumptions of this lemma, we have $G_1(t,s) \geq 0$ for all $(t,s) \in [0,1] \times [0,1]$, and so $u(t) \geq 0$ for all $t \in [0,1]$. □

LEMMA **2.8**. *Assume that $a_i \geq 0$ for all $i = 1, \ldots, p$, $\Delta_1 = 1 - \sum_{i=1}^{p} a_i \xi_i^{\alpha-1} > 0$, $0 < \xi_1 < \cdots < \xi_p < 1$. Then the Green function G_1 of the problem (2.1)-(2.2) satisfies the inequalities:*

a) $G_1(t,s) \leq J_1(s)$, $\forall\,(t,s) \in [0,1] \times [0,1]$, where

$$J_1(s) = g_1(\theta_1(s), s) + \frac{1}{\Delta_1} \sum_{i=1}^{p} a_i g_1(\xi_i, s), \quad s \in [0,1].$$

b) For every $c \in (0, 1/2)$, we have

$$\min_{t \in [c, 1-c]} G_1(t,s) \geq \gamma_1 J_1(s) \geq \gamma_1 G_1(t', s), \quad \forall\, t',\, s \in [0,1].$$

P r o o f. The first inequality a) is evident. For part b), for $c \in (0, 1/2)$ and $t \in [c, 1-c]$, t', $s \in [0,1]$, we deduce

$$\begin{aligned} G_1(t,s) &\geq c^{\alpha-1} g_1(\theta_1(s), s) + \frac{c^{\alpha-1}}{\Delta_1} \sum_{i=1}^{p} a_i g_1(\xi_i, s) \\ &= c^{\alpha-1} \left(g_1(\theta_1(s), s) + \frac{1}{\Delta_1} \sum_{i=1}^{p} a_i g_1(\xi_i, s) \right) = \gamma_1 J_1(s) \geq \gamma_1 G_1(t', s). \end{aligned}$$

Therefore, we obtain the inequalities b) of this lemma. □

LEMMA **2.9**. *Assume that* $a_i \geq 0$ *for all* $i = 1, \ldots, p$, $\Delta_1 = 1 - \sum_{i=1}^{p} a_i \xi_i^{\alpha-1} > 0$, $0 < \xi_1 < \cdots < \xi_p < 1$, $c \in (0, 1/2)$ *and* $y \in C([0,1])$, $y(t) \geq 0$ *for all* $t \in [0,1]$. *Then the solution* $u(t)$, $t \in [0,1]$ *of problem (2.1)-(2.2) satisfies the inequality* $\min_{t\in[c,1-c]} u(t) \geq \gamma_1 \max_{t'\in[0,1]} u(t')$.

P r o o f. For $c \in (0, 1/2)$, $t \in [c, 1-c]$, $t' \in [0,1]$, we have

$$u(t) \geq \gamma_1 \int_0^1 J_1(s)y(s)\,ds \geq \gamma_1 \int_0^1 G_1(t', s)y(s)\,ds = \gamma_1 u(t').$$

Then, we deduce the conclusion of this lemma. □

We can also formulate similar results as Lemmas **2.4-2.9** above, for the fractional differential equation

$$D_{0+}^{\beta} v(t) + h(t) = 0, \;\; 0 < t < 1, \;\; m - 1 < \beta \leq m, \tag{2.6}$$

with the multi-point boundary conditions

$$v(0) = v'(0) = \cdots = v^{(m-2)}(0) = 0, \;\; v(1) = \sum_{i=1}^{q} b_i v(\eta_i), \tag{2.7}$$

where $m \in \mathbb{N}$, $m \geq 2$, $q \in \mathbb{N}$, $0 < \eta_1 < \cdots < \eta_q < 1$, $b_i \geq 0$ for all $i = 1, ..., q$ and $h \in C([0,1])$. We denote by Δ_2, γ_2, g_2, θ_2, G_2 and J_2 the corresponding constants and functions for the problem (2.6)-(2.7) defined in a similar manner as Δ_1, γ_1, g_1, θ_1, G_1 and J_1, respectively.

We present now the Guo-Krasnosel'skii fixed point theorem (see [13]) that we will use in the proofs of our main results.

THEOREM **2.1**. *Let* X *be a Banach space and let* $C \subset X$ *be a cone in* X. *Assume* Ω_1 *and* Ω_2 *are bounded open subsets of* X *with* $0 \in \Omega_1 \subset \overline{\Omega_1} \subset \Omega_2$ *and let* $A : C \cap (\overline{\Omega_2} \setminus \Omega_1) \to C$ *be a completely continuous operator such that, either*

i) $\|Au\| \leq \|u\|$, $u \in C \cap \partial\Omega_1$, *and* $\|Au\| \geq \|u\|$, $u \in C \cap \partial\Omega_2$, *or*

ii) $\|Au\| \geq \|u\|$, $u \in C \cap \partial\Omega_1$, *and* $\|Au\| \leq \|u\|$, $u \in C \cap \partial\Omega_2$.

Then A *has a fixed point in* $C \cap (\overline{\Omega_2} \setminus \Omega_1)$.

3. Main results

In this section, we shall give sufficient conditions on λ, μ, f and g such that positive solutions with respect to a cone for our problem $(S) - (BC)$ exist.

We present the assumptions that we shall use in the sequel.

$(H1)$ $0 < \xi_1 < \cdots < \xi_p < 1$, $a_i \geq 0$ for all $i = 1, \dots, p$, $0 < \eta_1 < \cdots < \eta_q < 1$, $b_i \geq 0$ for all $i = 1, \dots, q$, $\Delta_1 = 1 - \sum_{i=1}^{p} a_i \xi_i^{\alpha-1} > 0$, $\Delta_2 = 1 - \sum_{i=1}^{q} b_i \eta_i^{\beta-1} > 0$.

$(H2)$ The functions $f,\ g : [0,1] \times [0,\infty) \times [0,\infty) \to [0,\infty)$ are continuous.

For $c \in (0, 1/2)$, we introduce the following extreme limits:

$$
\begin{aligned}
f_0^s &= \limsup_{u+v \to 0^+} \max_{t \in [0,1]} \frac{f(t,u,v)}{u+v}, \quad g_0^s = \limsup_{u+v \to 0^+} \max_{t \in [0,1]} \frac{g(t,u,v)}{u+v},\\
f_0^i &= \liminf_{u+v \to 0^+} \min_{t \in [c,1-c]} \frac{f(t,u,v)}{u+v}, \quad g_0^i = \liminf_{u+v \to 0^+} \min_{t \in [c,1-c]} \frac{g(t,u,v)}{u+v},\\
f_\infty^s &= \limsup_{u+v \to \infty} \max_{t \in [0,1]} \frac{f(t,u,v)}{u+v}, \quad g_\infty^s = \limsup_{u+v \to \infty} \max_{t \in [0,1]} \frac{g(t,u,v)}{u+v},\\
f_\infty^i &= \liminf_{u+v \to \infty} \min_{t \in [c,1-c]} \frac{f(t,u,v)}{u+v}, \quad g_\infty^i = \liminf_{u+v \to \infty} \min_{t \in [c,1-c]} \frac{g(t,u,v)}{u+v}.
\end{aligned}
$$

By using the Green functions G_1 and G_2 from Section **2** (Lemma **2.5**), our problem $(S) - (BC)$ can be written equivalently as the following nonlinear system of integral equations

$$
\begin{cases}
u(t) = \lambda \displaystyle\int_0^1 G_1(t,s) f(s, u(s), v(s))\, ds, & 0 \leq t \leq 1,\\
v(t) = \mu \displaystyle\int_0^1 G_2(t,s) g(s, u(s), v(s))\, ds, & 0 \leq t \leq 1.
\end{cases}
$$

We consider the Banach space $X = C([0,T])$ with supremum norm $\|\cdot\|$, and the Banach space $Y = X \times X$ with the norm $\|(u,v)\|_Y = \|u\| + \|v\|$. We define the cone $P \subset Y$ by:

$$
P = \{(u,v) \in Y;\ u(t) \geq 0,\ v(t) \geq 0,\ \forall\, t \in [0,1], \text{ and } \inf_{t \in [c,1-c]} (u(t) + v(t)) \geq \gamma \|(u,v)\|_Y\},
$$

where $\gamma = \min\{\gamma_1, \gamma_2\}$ and γ_1, γ_2 are defined in Section **2**.

For λ, $\mu > 0$, we introduce the operators $T_1, T_2 : Y \to X$ and $Q : Y \to Y$ defined by

$$
\begin{aligned}
T_1(u,v)(t) &= \lambda \int_0^1 G_1(t,s) f(s, u(s), v(s))\, ds, \quad 0 \leq t \leq 1,\\
T_2(u,v)(t) &= \mu \int_0^1 G_2(t,s) g(s, u(s), v(s))\, ds, \quad 0 \leq t \leq 1,
\end{aligned}
$$

and $Q(u,v) = (T_1(u,v), T_2(u,v))$, $(u,v) \in Y$. The solutions of our problem $(S) - (BC)$ are the fixed points of the operator Q.

Lemma **3.1**. *If $(H1)-(H2)$ hold and $c \in (0,1/2)$, then $Q : P \to P$ is a completely continuous operator.*

P r o o f. Let $(u,v) \in P$ be an arbitrary element. Because $T_1(u,v)$ and $T_2(u,v)$ satisfy the problem (2.1)-(2.2) for $y(t) = \lambda f(t,u(t),v(t))$, $t \in [0,1]$, and the problem (2.6)-(2.7) for $h(t) = \mu g(t,u(t),v(t))$, $t \in [0,1]$, respectively, then by Lemma **2.9**, we obtain

$$\inf_{t\in[c,1-c]} T_1(u,v)(t) \geq \gamma_1 \max_{t'\in[0,1]} T_1(u,v)(t') = \gamma_1 \|T_1(u,v)\|,$$
$$\inf_{t\in[c,1-c]} T_2(u,v)(t) \geq \gamma_2 \max_{t'\in[0,1]} T_2(u,v)(t') = \gamma_2 \|T_2(u,v)\|.$$

Hence, we conclude

$$\inf_{t\in[c,1-c]} [T_1(u,v)(t) + T_2(u,v)(t)] \geq \inf_{t\in[c,1-c]} T_1(u,v)(t) + \inf_{t\in[c,1-c]} T_2(u,v)(t)$$
$$\geq \gamma_1\|T_1(u,v)\| + \gamma_2\|T_2(u,v)\| \geq \gamma\|(T_1(u,v),T_2(u,v))\|_Y = \gamma\|Q(u,v)\|_Y.$$

By Lemma **2.7** and $(H1)-(H2)$, we obtain $T_1(u,v)(t) \geq 0$, $T_2(u,v)(t) \geq 0$ for all $t \in [0,1]$, and so, we deduce that $Q(u,v) \in P$. Hence, we get $Q(P) \subset P$. By using standard arguments, we can easily show that T_1 and T_2 are completely continuous, and then Q is a completely continuous operator. □

We denote by $A = \int_c^{1-c} J_1(s)\,ds$, $B = \int_0^1 J_1(s)\,ds$, $C = \int_c^{1-c} J_2(s)\,ds$ and $D = \int_0^1 J_2(s)\,ds$, where J_1 and J_2 are defined in Section **2** (Lemma **2.8**).

First, for f_0^s, g_0^s, f_∞^i, $g_\infty^i \in (0,\infty)$ and numbers α_1, $\alpha_2 \geq 0$, $\widetilde{\alpha}_1$, $\widetilde{\alpha}_2 > 0$ such that $\alpha_1+\alpha_2 = 1$ and $\widetilde{\alpha}_1+\widetilde{\alpha}_2 = 1$, we define the numbers L_1, L_2, L_3, L_4, L_2', L_4' by

$$L_1 = \frac{\alpha_1}{\gamma\gamma_1 f_\infty^i A},\ \ L_2 = \frac{\widetilde{\alpha}_1}{f_0^s B},\ \ L_3 = \frac{\alpha_2}{\gamma\gamma_2 g_\infty^i C},\ \ L_4 = \frac{\widetilde{\alpha}_2}{g_0^s D},$$
$$L_2' = \frac{1}{f_0^s B},\ \ L_4' = \frac{1}{g_0^s D}.$$

Theorem **3.1**. *Assume that $(H1)$ and $(H2)$ hold, $c \in (0,1/2)$, α_1, $\alpha_2 \geq 0$, $\widetilde{\alpha}_1$, $\widetilde{\alpha}_2 > 0$ such that $\alpha_1+\alpha_2 = 1$, $\widetilde{\alpha}_1+\widetilde{\alpha}_2 = 1$.*

1) If f_0^s, g_0^s, f_∞^i, $g_\infty^i \in (0,\infty)$, $L_1 < L_2$ and $L_3 < L_4$, then for each $\lambda \in (L_1,L_2)$ and $\mu \in (L_3,L_4)$ there exists a positive solution $(u(t),v(t))$, $t \in [0,1]$ for $(S)-(BC)$.

2) If $f_0^s = 0$, g_0^s, f_∞^i, $g_\infty^i \in (0,\infty)$ and $L_3 < L_4'$, then for each $\lambda \in (L_1,\infty)$ and $\mu \in (L_3,L_4')$ there exists a positive solution $(u(t),v(t))$, $t \in [0,1]$ for $(S)-(BC)$.

3) If $g_0^s = 0$, f_0^s, f_∞^i, $g_\infty^i \in (0,\infty)$ and $L_1 < L_2'$, then for each $\lambda \in (L_1, L_2')$ and $\mu \in (L_3,\infty)$ there exists a positive solution $(u(t),v(t))$, $t \in [0,1]$ for $(S)-(BC)$.

4) If $f_0^s = g_0^s = 0$, f_∞^i, $g_\infty^i \in (0,\infty)$, then for each $\lambda \in (L_1,\infty)$ and $\mu \in (L_3,\infty)$ there exists a positive solution $(u(t),v(t))$, $t \in [0,1]$ for $(S)-(BC)$.

5) If $\{f_0^s,\ g_0^s,\ f_\infty^i \in (0,\infty),\ g_\infty^i = \infty\}$ or $\{f_0^s,\ g_0^s,\ g_\infty^i \in (0,\infty),\ f_\infty^i = \infty\}$ or $\{f_0^s,\ g_0^s \in (0,\infty),\ f_\infty^i = g_\infty^i = \infty\}$, then for each $\lambda \in (0, L_2)$ and $\mu \in (0, L_4)$ there exists a positive solution $(u(t),v(t))$, $t \in [0,1]$ for $(S)-(BC)$.

6) If $\{f_0^s = 0,\ g_0^s,\ f_\infty^i \in (0,\infty),\ g_\infty^i = \infty\}$ or $\{f_0^s = 0,\ f_\infty^i = \infty,\ g_0^s,\ g_\infty^i \in (0,\infty)\}$ or $\{f_0^s = 0,\ g_0^s \in (0,\infty),\ f_\infty^i = g_\infty^i = \infty\}$ then for each $\lambda \in (0,\infty)$ and $\mu \in (0, L_4')$ there exists a positive solution $(u(t),v(t))$, $t \in [0,1]$ for $(S)-(BC)$.

7) If $\{f_0^s,\ f_\infty^i \in (0,\infty),\ g_0^s = 0,\ g_\infty^i = \infty\}$ or $\{f_0^s,\ g_\infty^i \in (0,\infty),\ g_0^s = 0,\ f_\infty^i = \infty\}$ or $\{f_0^s \in (0,\infty),\ g_0^s = 0,\ f_\infty^i = g_\infty^i = \infty\}$ then for each $\lambda \in (0, L_2')$ and $\mu \in (0,\infty)$ there exists a positive solution $(u(t),v(t))$, $t \in [0,1]$ for $(S)-(BC)$.

8) If $\{f_0^s = g_0^s = 0,\ f_\infty^i \in (0,\infty),\ g_\infty^i = \infty\}$ or $\{f_0^s = g_0^s = 0,\ f_\infty^i = \infty,\ g_\infty^i \in (0,\infty)\}$ or $\{f_0^s = g_0^s = 0,\ f_\infty^i = g_\infty^i = \infty\}$ then for each $\lambda \in (0,\infty)$ and $\mu \in (0,\infty)$ there exists a positive solution $(u(t),v(t))$, $t \in [0,1]$ for $(S)-(BC)$.

P r o o f. We consider the above cone $P \subset Y$ and the operators T_1, T_2 and Q. Because the proofs of the above cases are similar, in what follows we will prove one of them, namely the first case in 6). So, we suppose $f_0^s = 0$, g_0^s, $f_\infty^i \in (0,\infty)$ and $g_\infty^i = \infty$. Let $\lambda \in (0,\infty)$ and $\mu \in (0, L_4')$, that is $\mu \in (0, \frac{1}{g_0^s D})$. We choose $\alpha_1' > 0$, $\alpha_1' < \min\{\lambda\gamma\gamma_1 f_\infty^i A, 1\}$ and $\widetilde{\alpha}_2' \in (\mu g_0^s D, 1)$. Let $\alpha_2' = 1 - \alpha_1'$ and $\widetilde{\alpha}_1' = 1 - \widetilde{\alpha}_2'$, and let $\varepsilon > 0$ be a positive number such that $\varepsilon < f_\infty^i$ and

$$\frac{\alpha_1'}{\gamma\gamma_1(f_\infty^i - \varepsilon)A} \le \lambda, \quad \frac{\alpha_2'\varepsilon}{\gamma\gamma_2 C} \le \mu, \quad \frac{\widetilde{\alpha}_1'}{\varepsilon B} \ge \lambda, \quad \frac{\widetilde{\alpha}_2'}{(g_0^s + \varepsilon)D} \ge \mu.$$

By using $(H2)$ and the definitions of f_0^s and g_0^s, we deduce that there exists $R_1 > 0$ such that for all $t \in [0,1]$, u, $v \in \mathbb{R}_+$, with $0 \le u + v \le R_1$, we have $f(t,u,v) \le \varepsilon(u+v)$ and $g(t,u,v) \le (g_0^s + \varepsilon)(u+v)$. We define the set $\Omega_1 = \{(u,v) \in Y,\ \|(u,v)\|_Y < R_1\}$. Now let $(u,v) \in P \cap \partial\Omega_1$, that is $(u,v) \in P$ with $\|(u,v)\|_Y = R_1$ or equivalently $\|u\| + \|v\| = R_1$. Then $u(t) + v(t) \le R_1$ for all $t \in [0,1]$, and by Lemma **2.8**, we obtain

$$T_1(u,v)(t) \le \lambda \int_0^1 J_1(s) f(s,u(s),v(s))\,ds \le \lambda \int_0^1 J_1(s)\varepsilon(u(s)+v(s))\,ds$$
$$\le \lambda\varepsilon \int_0^1 J_1(s)(\|u\| + \|v\|)\,ds = \lambda\varepsilon B\|(u,v)\|_Y \le \widetilde{\alpha}_1'\|(u,v)\|_Y,\ \ \forall\, t \in [0,1].$$

Therefore, $\|T_1(u,v)\| \le \widetilde{\alpha}_1'\|(u,v)\|_Y$. In a similar manner, we conclude

$$\begin{aligned}
&T_2(u,v)(t) \le \mu \int_0^1 J_2(s) g(s,u(s),v(s))\,ds \\
&\le \mu \int_0^1 J_2(s)(g_0^s+\varepsilon)(u(s)+v(s))\,ds \le \mu(g_0^s+\varepsilon)\int_0^1 J_2(s)(\|u\|+\|v\|)\,ds \\
&= \mu(g_0^s+\varepsilon)D\|(u,v)\|_Y \le \widetilde{\alpha}_2'\|(u,v)\|_Y, \ \ \forall\, t\in[0,1].
\end{aligned}$$

Therefore, $\|T_2(u,v)\| \le \widetilde{\alpha}_2'\|(u,v)\|_Y$.

Then, for $(u,v)\in P\cap\partial\Omega_1$, we deduce

$$\|Q(u,v)\|_Y = \|T_1(u,v)\| + \|T_2(u,v)\| \le (\widetilde{\alpha}_1'+\widetilde{\alpha}_2')\|(u,v)\|_Y = \|(u,v)\|_Y.$$

By the definitions of f_∞^i and g_∞^i, there exists $\bar{R}_2>0$ such that $f(t,u,v) \ge (f_\infty^i-\varepsilon)(u+v)$ and $g(t,u,v)\ge \frac{1}{\varepsilon}(u+v)$ for all $u,\,v\ge 0$ with $u+v\ge \bar{R}_2$ and $t\in[c,1-c]$. We consider $R_2=\max\{2R_1,\bar{R}_2/\gamma\}$, and we define $\Omega_2=\{(u,v)\in Y,\ \|(u,v)\|_Y<R_2\}$. Then for $(u,v)\in P$ with $\|(u,v)\|_Y=R_2$, we obtain

$$u(t)+v(t) \ge \inf_{t\in[c,1-c]}(u(t)+v(t)) \ge \gamma\|(u,v)\|_Y = \gamma R_2 \ge \bar{R}_2,$$

for all $t\in[c,1-c]$.

Then, by Lemma **2.8**, we conclude

$$\begin{aligned}
&T_1(u,v)(c) \ge \lambda\gamma_1 \int_0^1 J_1(s) f(s,u(s),v(s))\,ds \\
&\ge \lambda\gamma_1 \int_c^{1-c} J_1(s) f(s,u(s),v(s))ds \ge \lambda\gamma_1\int_c^{1-c} J_1(s)(f_\infty^i-\varepsilon)(u(s)+v(s))ds \\
&\ge \lambda\gamma_1(f_\infty^i-\varepsilon)\int_c^{1-c} J_1(s)\gamma\|(u,v)\|_Y ds \ge \alpha_1'\|(u,v)\|_Y.
\end{aligned}$$

So, $\|T_1(u,v)\| \ge T_1(u,v)(c) \ge \alpha_1'\|(u,v)\|_Y$.

In a similar manner, we deduce

$$\begin{aligned}
&T_2(u,v)(c) \ge \mu\gamma_2 \int_0^1 J_2(s) g(s,u(s),v(s))\,ds \\
&\ge \mu\gamma_2 \int_c^{1-c} J_2(s) g(s,u(s),v(s))\,ds \ge \mu\gamma_2 \int_c^{1-c} J_2(s)\frac{1}{\varepsilon}(u(s)+v(s))\,ds \\
&\ge \tfrac{\mu\gamma_2}{\varepsilon}\int_c^{1-c} J_2(s)\gamma\|(u,v)\|_Y\,ds \ge \alpha_2'\|(u,v)\|_Y.
\end{aligned}$$

So, $\|T_2(u,v)\| \ge T_2(u,v)(c) \ge \alpha_2'\|(u,v)\|_Y$.

Hence, for $(u,v)\in P\cap\partial\Omega_2$, we obtain

$$\|Q(u,v)\|_Y = \|T_1(u,v)\| + \|T_2(u,v)\| \ge (\alpha_1'+\alpha_2')\|(u,v)\|_Y = \|(u,v)\|_Y.$$

By using Lemma **3.1** and Theorem **2.1** i), we conclude that Q has a fixed point $(u,v)\in P\cap(\bar{\Omega}_2\setminus\Omega_1)$ such that $R_1\le\|u\|+\|v\|\le R_2$. $\square$

REMARK **3.1**. We mention that in Theorem **3.1** we have the possibility to choose $\alpha_1 = 0$ or $\alpha_2 = 0$. Therefore, each of the first four cases contains three subcases. For example, in the first case f_0^s, g_0^s, f_∞^i, $g_\infty^i \in (0,\infty)$, we have the following situations:

a) if $\alpha_1, \alpha_2 \in (0,1)$, $\alpha_1 + \alpha_2 = 1$ and $L_1 < L_2$, $L_3 < L_4$, then $\lambda \in (L_1, L_2)$ and $\mu \in (L_3, L_4)$;

b) if $\alpha_1 = 1$, $\alpha_2 = 0$ and $L_1' < L_2$, then $\lambda \in (L_1', L_2)$ and $\mu \in (0, L_4)$, where $L_1' = \frac{1}{\gamma\gamma_1 f_\infty^i A}$;

c) if $\alpha_1 = 0$, $\alpha_2 = 1$ and $L_3' < L_4$, then $\lambda \in (0, L_2)$ and $\mu \in (L_3', L_4)$, where $L_3' = \frac{1}{\gamma\gamma_2 g_\infty^i C}$.

In what follows, for f_0^i, g_0^i, f_∞^s, $g_\infty^s \in (0,\infty)$ and numbers $\alpha_1, \alpha_2 \geq 0$, $\widetilde{\alpha}_1, \widetilde{\alpha}_2 > 0$ such that $\alpha_1 + \alpha_2 = 1$ and $\widetilde{\alpha}_1 + \widetilde{\alpha}_2 = 1$, we define the numbers $\widetilde{L}_1$, $\widetilde{L}_2$, $\widetilde{L}_3$, $\widetilde{L}_4$, $\widetilde{L}_2'$ and $\widetilde{L}_4'$ by

$$\widetilde{L}_1 = \frac{\alpha_1}{\gamma\gamma_1 f_0^i A}, \quad \widetilde{L}_2 = \frac{\widetilde{\alpha}_1}{f_\infty^s B}, \quad \widetilde{L}_3 = \frac{\alpha_2}{\gamma\gamma_2 g_0^i C}, \quad \widetilde{L}_4 = \frac{\widetilde{\alpha}_2}{g_\infty^s D},$$
$$\widetilde{L}_2' = \frac{1}{f_\infty^s B}, \quad \widetilde{L}_4' = \frac{1}{g_\infty^s D}.$$

THEOREM **3.2**. *Assume that $(H1)$ and $(H2)$ hold, $c \in (0,1/2)$, $\alpha_1, \alpha_2 \geq 0$, $\widetilde{\alpha}_1, \widetilde{\alpha}_2 > 0$ such that $\alpha_1 + \alpha_2 = 1$, $\widetilde{\alpha}_1 + \widetilde{\alpha}_2 = 1$.*

1) If f_0^i, g_0^i, f_∞^s, $g_\infty^s \in (0,\infty)$, $\widetilde{L}_1 < \widetilde{L}_2$ and $\widetilde{L}_3 < \widetilde{L}_4$, then for each $\lambda \in (\widetilde{L}_1, \widetilde{L}_2)$ and $\mu \in (\widetilde{L}_3, \widetilde{L}_4)$ there exists a positive solution $(u(t), v(t))$, $t \in [0,1]$ for $(S)-(BC)$.

2) If f_0^i, g_0^i, $f_\infty^s \in (0,\infty)$, $g_\infty^s = 0$ and $\widetilde{L}_1 < \widetilde{L}_2'$, then for each $\lambda \in (\widetilde{L}_1, \widetilde{L}_2')$ and $\mu \in (\widetilde{L}_3, \infty)$ there exists a positive solution $(u(t), v(t))$, $t \in [0,1]$ for $(S)-(BC)$.

3) If f_0^i, g_0^i, $g_\infty^s \in (0,\infty)$, $f_\infty^s = 0$ and $\widetilde{L}_3 < \widetilde{L}_4'$, then for each $\lambda \in (\widetilde{L}_1, \infty)$ and $\mu \in (\widetilde{L}_3, \widetilde{L}_4')$ there exists a positive solution $(u(t), v(t))$, $t \in [0,1]$ for $(S)-(BC)$.

4) If f_0^i, $g_0^i \in (0,\infty)$, $f_\infty^s = g_\infty^s = 0$, then for each $\lambda \in (\widetilde{L}_1, \infty)$ and $\mu \in (\widetilde{L}_3, \infty)$ there exists a positive solution $(u(t), v(t))$, $t \in [0,1]$ for $(S)-(BC)$.

5) If $\{f_0^i = \infty$, g_0^i, f_∞^s, $g_\infty^s \in (0,\infty)\}$ or $\{f_0^i$, f_∞^s, $g_\infty^s \in (0,\infty)$, $g_0^i = \infty\}$ or $\{f_0^i = g_0^i = \infty$, f_∞^s, $g_\infty^s \in (0,\infty)\}$, then for each $\lambda \in (0, \widetilde{L}_2)$ and $\mu \in (0, \widetilde{L}_4)$ there exists a positive solution $(u(t), v(t))$, $t \in [0,1]$ for $(S)-(BC)$.

6) If $\{f_0^i = \infty$, g_0^i, $f_\infty^s \in (0,\infty)$, $g_\infty^s = 0\}$ or $\{f_0^i$, $f_\infty^s \in (0,\infty)$, $g_0^i = \infty$, $g_\infty^s = 0\}$ or $\{f_0^i = g_0^i = \infty$, $f_\infty^s \in (0,\infty)$, $g_\infty^s = 0\}$, then for each $\lambda \in (0, \widetilde{L}_2')$ and $\mu \in (0,\infty)$ there exists a positive solution $(u(t), v(t))$, $t \in [0,1]$ for $(S)-(BC)$.

7) If $\{f_0^i=\infty,\ g_0^i,\ g_\infty^s\in(0,\infty),\ f_\infty^s=0\}$ *or* $\{f_0^i,\ g_\infty^s\in(0,\infty),\ g_0^i=\infty,\ f_\infty^s=0\}$ *or* $\{f_0^i=g_0^i=\infty,\ f_\infty^s=0,\ g_\infty^s\in(0,\infty)\}$*, then for each* $\lambda\in(0,\infty)$ *and* $\mu\in(0,\widetilde{L}_4')$ *there exists a positive solution* $(u(t),v(t)),\ \ t\in[0,1]$ *for* $(S)-(BC)$.

8) If $\{f_0^i=\infty,\ g_0^i\in(0,\infty),\ f_\infty^s=g_\infty^s=0\}$ *or* $\{f_0^i\in(0,\infty),\ g_0^i=\infty,\ f_\infty^s=g_\infty^s=0\}$ *or* $\{f_0^i=g_0^i=\infty,\ f_\infty^s=g_\infty^s=0\}$*, then for each* $\lambda\in(0,\infty)$ *and* $\mu\in(0,\infty)$ *there exists a positive solution* $(u(t),v(t)),\ \ t\in[0,1]$ *for* $(S)-(BC)$.

P r o o f. We consider the above cone $P\subset Y$ and the operators T_1, T_2 and Q. Because the proofs of the above cases are similar, in what follows we will prove one of them, namely the case 2). So, we suppose $f_0^i,\ g_0^i,\ f_\infty^s\in(0,\infty)$, $g_\infty^s=0$ and $\widetilde{L}_1<\widetilde{L}_2'$. Let $\lambda\in(\widetilde{L}_1,\widetilde{L}_2')$ and $\mu\in(\widetilde{L}_3,\infty)$; that is, $\lambda\in(\frac{\alpha_1}{\gamma\gamma_1 f_0^i A},\frac{1}{f_\infty^s B})$ and $\mu\in(\frac{\alpha_2}{\gamma\gamma_2 g_0^i C},\infty)$. We choose $\widetilde{\alpha}_1'\in(\lambda f_\infty^s B,1)$. Let $\widetilde{\alpha}_2'=1-\widetilde{\alpha}_1'$ and let $\varepsilon>0$ be a positive number such that $\varepsilon<\min\{f_0^i,g_0^i\}$ and

$$\frac{\alpha_1}{\gamma\gamma_1(f_0^i-\varepsilon)A}\le\lambda,\quad \frac{\alpha_2}{\gamma\gamma_2(g_0^i-\varepsilon)C}\le\mu,\quad \frac{\widetilde{\alpha}_1'}{(f_\infty^s+\varepsilon)B}\ge\lambda,\quad \frac{\widetilde{\alpha}_2'}{\varepsilon D}\ge\mu.$$

By using (H2) and the definitions of f_0^i and g_0^i, we deduce that there exists $R_3>0$ such that $f(t,u,v)\ge(f_0^i-\varepsilon)(u+v)$, $g(t,u,v)\ge(g_0^i-\varepsilon)(u+v)$ for all $u,\ v\ge0$ with $0\le u+v\le R_3$ and $t\in[c,1-c]$. We denote by $\Omega_3=\{(u,v)\in Y,\ \ \|(u,v)\|_Y<R_3\}$. Let $(u,v)\in P$ with $\|(u,v)\|_Y=R_3$, that is $\|u\|+\|v\|=R_3$. Because $u(t)+v(t)\le\|u\|+\|v\|=R_3$ for all $t\in[0,1]$, then by using Lemma **2.8**, we obtain

$$\begin{aligned}
&T_1(u,v)(c)\ge\lambda\gamma_1\int_c^{1-c}J_1(s)f(s,u(s),v(s))\,ds\\
&\ge\lambda\gamma_1\int_c^{1-c}J_1(s)(f_0^i-\varepsilon)(u(s)+v(s))\,ds\\
&\ge\lambda\gamma\gamma_1(f_0^i-\varepsilon)\int_c^{1-c}J_1(s)(\|u\|+\|v\|)\,ds\ge\alpha_1\|(u,v)\|_Y.
\end{aligned}$$

Therefore, $\|T_1(u,v)\|\ge T_1(u,v)(c)\ge\alpha_1\|(u,v)\|_Y$. In a similar manner, we conclude

$$\begin{aligned}
&T_2(u,v)(c)\ge\mu\gamma_2\int_c^{1-c}J_2(s)g(s,u(s),v(s))\,ds\\
&\ge\mu\gamma_2\int_c^{1-c}J_2(s)(g_0^i-\varepsilon)(u(s)+v(s))\,ds\\
&\ge\mu\gamma\gamma_2(g_0^i-\varepsilon)\int_c^{1-c}J_2(s)(\|u\|+\|v\|)\,ds\ge\alpha_2\|(u,v)\|_Y.
\end{aligned}$$

So, $\|T_2(u,v)\| \geq T_2(u,v)(c) \geq \alpha_2\|(u,v)\|_Y$.

Thus, for an arbitrary element $(u,v) \in P \cap \partial\Omega_3$, we deduce $\|Q(u,v)\|_Y \geq (\alpha_1+\alpha_2)\|(u,v)\|_Y = \|(u,v)\|_Y$.

Now, we define the functions f^*, $g^*: [0,1]\times\mathbb{R}_+ \to \mathbb{R}_+$, $f^*(t,x) = \max_{0\leq u+v\leq x} f(t,u,v)$, $g^*(t,x) = \max_{0\leq u+v\leq x} g(t,u,v)$, $t\in[0,1]$, $x\in\mathbb{R}_+$. Then $f(t,u,v) \leq f^*(t,x)$, $g(t,u,v) \leq g^*(t,x)$ for all $t\in[0,1]$, $u\geq 0$, $v\geq 0$ and $u+v\leq x$. The functions $f^*(t,\cdot)$, $g^*(t,\cdot)$ are nondecreasing for every $t\in[0,1]$, and they satisfy the conditions

$$\limsup_{x\to\infty}\max_{t\in[0,1]}\frac{f^*(t,x)}{x} \leq f_\infty^s, \quad \lim_{x\to\infty}\max_{t\in[0,1]}\frac{g^*(t,x)}{x} = 0.$$

Therefore, for $\varepsilon > 0$, there exists $\bar{R}_4 > 0$ such that for all $x \geq \bar{R}_4$ and $t\in[0,1]$, we have

$$\begin{aligned}\frac{f^*(t,x)}{x} &\leq \limsup_{x\to\infty}\max_{t\in[0,1]}\frac{f^*(t,x)}{x} + \varepsilon \leq f_\infty^s + \varepsilon,\\ \frac{g^*(t,x)}{x} &\leq \lim_{x\to\infty}\max_{t\in[0,1]}\frac{g^*(t,x)}{x} + \varepsilon = \varepsilon,\end{aligned}$$

and so $f^*(t,x) \leq (f_\infty^s+\varepsilon)x$ and $g^*(t,x) \leq \varepsilon x$.

We consider $R_4 = \max\{2R_3, \bar{R}_4\}$ and we denote by $\Omega_4 = \{(u,v)\in Y,\ \|(u,v)\|_Y < R_4\}$. Let $(u,v)\in P\cap\partial\Omega_4$. By the definitions of f^* and g^*, we conclude that for all $t\in[0,1]$

$$f(t,u(t),v(t)) \leq f^*(t,\|(u,v)\|_Y), \quad g(t,u(t),v(t)) \leq g^*(t,\|(u,v)\|_Y).$$

Then for all $t\in[0,1]$, we obtain

$$\begin{aligned}&T_1(u,v)(t) \leq \lambda\int_0^1 J_1(s)f(s,u(s),v(s))\,ds \leq \lambda\int_0^1 J_1(s)f^*(s,\|(u,v)\|_Y)\,ds\\ &\leq \lambda(f_\infty^s+\varepsilon)\int_0^1 J_1(s)\|(u,v)\|_Y\,ds \leq \widetilde{\alpha}_1'\|(u,v)\|_Y,\end{aligned}$$

and so, $\|T_1(u,v)\| \leq \widetilde{\alpha}_1'\|(u,v)\|_Y$.

In a similar manner, we deduce

$$\begin{aligned}&T_2(u,v)(t) \leq \mu\int_0^1 J_2(s)g(s,u(s),v(s))\,ds \leq \mu\int_0^1 J_2(s)g^*(s,\|(u,v)\|_Y)\,ds\\ &\leq \mu\varepsilon\int_0^1 J_2(s)\|(u,v)\|_Y\,ds \leq \widetilde{\alpha}_2'\|(u,v)\|_Y,\end{aligned}$$

and so, $\|T_2(u,v)\| \leq \widetilde{\alpha}_2'\|(u,v)\|_Y$.

Therefore, for $(u,v)\in P\cap\partial\Omega_4$, it follows that $\|Q(u,v)\|_Y \leq (\widetilde{\alpha}_1'+\widetilde{\alpha}_2')\|(u,v)\|_Y = \|(u,v)\|_Y$. By using Lemma **3.1** and Theorem **2.1** ii), we conclude that Q has a fixed point $(u,v)\in P\cap(\bar{\Omega}_4\setminus\Omega_3)$ such that $R_3 \leq \|(u,v)\|_Y \leq R_4$. □

4. Systems of Hammerstein integral equations

We consider the following system of nonlinear Hammerstein integral equations

$$(\widetilde{S}) \qquad \begin{cases} u(t) = \widetilde{\lambda} \displaystyle\int_0^1 K_1(t,s)\widetilde{f}(s,u(s),v(s))\,ds, & 0 \le t \le 1, \\ v(t) = \widetilde{\mu} \displaystyle\int_0^1 K_2(t,s)\widetilde{g}(s,u(s),v(s))\,ds, & 0 \le t \le 1, \end{cases}$$

under the assumptions:

$(\widetilde{H1})$ The functions $K_1,\ K_2 : [0,1] \times [0,1] \to \mathbb{R}$ are continuous and there exist continuous functions $I_1,\ I_2 : [0,1] \to \mathbb{R}$ and $c \in (0,1/2)$ such that

a) $0 \le K_i(t,s) \le I_i(s),\ \ \forall\, t,\ s \in [0,1],\ \ i = 1,2;$

b) there exist $\delta_i > 0,\ i = 1,2$ such that $\displaystyle\min_{t\in[c,1-c]} K_i(t,s) \ge \delta_i I_i(s),\ \ \forall\, s \in [0,1],\ \ i = 1,2;$

c) there exist $t_1,\ t_2 \in [c, 1-c]$ such that $I_1(t_1) \ne 0,\ I_2(t_2) \ne 0$.

$(\widetilde{H2})$ The functions $\widetilde{f}, \widetilde{g} : [0,1] \times [0,\infty) \times [0,\infty) \to [0,\infty)$ are continuous.

For $c \in (0,1/2)$, we introduce the following extreme limits:

$$\widetilde{f}_0^s = \limsup_{u+v\to 0^+} \max_{t\in[0,1]} \frac{\widetilde{f}(t,u,v)}{u+v}, \ \ \widetilde{g}_0^s = \limsup_{u+v\to 0^+} \max_{t\in[0,1]} \frac{\widetilde{g}(t,u,v)}{u+v},$$
$$\widetilde{f}_0^i = \liminf_{u+v\to 0^+} \min_{t\in[c,1-c]} \frac{\widetilde{f}(t,u,v)}{u+v}, \ \ \widetilde{g}_0^i = \liminf_{u+v\to 0^+} \min_{t\in[c,1-c]} \frac{\widetilde{g}(t,u,v)}{u+v},$$
$$\widetilde{f}_\infty^s = \limsup_{u+v\to \infty} \max_{t\in[0,1]} \frac{\widetilde{f}(t,u,v)}{u+v}, \ \ \widetilde{g}_\infty^s = \limsup_{u+v\to \infty} \max_{t\in[0,1]} \frac{\widetilde{g}(t,u,v)}{u+v},$$
$$\widetilde{f}_\infty^i = \liminf_{u+v\to \infty} \min_{t\in[c,1-c]} \frac{\widetilde{f}(t,u,v)}{u+v}, \ \ \widetilde{g}_\infty^i = \liminf_{u+v\to \infty} \min_{t\in[c,1-c]} \frac{\widetilde{g}(t,u,v)}{u+v}.$$

In this section, we shall give sufficient conditions on $\widetilde{\lambda}$, $\widetilde{\mu}$, $\widetilde{f}$ and $\widetilde{g}$ such that positive solutions of $(\widetilde{S})$ exist. By a positive solution of system $(\widetilde{S})$ we mean a pair of functions $(u,v) \in C([0,1]) \times C([0,1])$ satisfying $(\widetilde{S})$ with $u(t) \ge 0,\ v(t) \ge 0$ for all $t \in [0,1]$ and $\displaystyle\sup_{t\in[0,1]} u(t) + \sup_{t\in[0,1]} v(t) > 0$.

We denote by $\widetilde{A} = \displaystyle\int_c^{1-c} I_1(s)\,ds$, $\widetilde{B} = \displaystyle\int_0^1 I_1(s)\,ds$, $\widetilde{C} = \displaystyle\int_c^{1-c} I_2(s)\,ds$ and $\widetilde{D} = \displaystyle\int_0^1 I_2(s)\,ds$. By $(\widetilde{H1})$, we have $\widetilde{A},\ \widetilde{B},\ \widetilde{C},\ \widetilde{D} > 0$.

First, for $\widetilde{f}_0^s,\ \widetilde{g}_0^s,\ \widetilde{f}_\infty^i,\ \widetilde{g}_\infty^i \in (0,\infty)$ and numbers $\alpha_1,\ \alpha_2 \ge 0,\ \widetilde{\alpha}_1,\ \widetilde{\alpha}_2 > 0$ such that $\alpha_1 + \alpha_2 = 1$ and $\widetilde{\alpha}_1 + \widetilde{\alpha}_2 = 1$, we define the numbers $M_1,\ M_2,\ M_3,$

M_4, M_2', M_4' by

$$M_1 = \frac{\alpha_1}{\delta\delta_1 \widetilde{f}^i_\infty \widetilde{A}}, \ \ M_2 = \frac{\widetilde{\alpha}_1}{\widetilde{f}^s_0 \widetilde{B}}, \ \ M_3 = \frac{\alpha_2}{\delta\delta_2 \widetilde{g}^i_\infty \widetilde{C}}, \ \ M_4 = \frac{\widetilde{\alpha}_2}{\widetilde{g}^s_0 \widetilde{D}},$$
$$M_2' = \frac{1}{\widetilde{f}^s_0 \widetilde{B}}, \ \ M_4' = \frac{1}{\widetilde{g}^s_0 \widetilde{D}},$$

where $\delta = \min\{\delta_1, \delta_2\}$.

THEOREM **4.1**. *Assume that* $(\widetilde{H}1)$ *and* $(\widetilde{H}2)$ *hold,* $\alpha_1,\ \alpha_2 \geq 0$, $\widetilde{\alpha}_1,\ \widetilde{\alpha}_2 > 0$ *such that* $\alpha_1 + \alpha_2 = 1$, $\widetilde{\alpha}_1 + \widetilde{\alpha}_2 = 1$.

1) If $\widetilde{f}^s_0$, $\widetilde{g}^s_0$, $\widetilde{f}^i_\infty$, $\widetilde{g}^i_\infty \in (0,\infty)$, $M_1 < M_2$ *and* $M_3 < M_4$, *then for each* $\widetilde{\lambda} \in (M_1, M_2)$ *and* $\widetilde{\mu} \in (M_3, M_4)$ *there exists a positive solution* $(u(t), v(t)),\ \ t \in [0,1]$ *for* $(\widetilde{S})$.

2) If $\widetilde{f}^s_0 = 0$, $\widetilde{g}^s_0$, $\widetilde{f}^i_\infty$, $\widetilde{g}^i_\infty \in (0,\infty)$ *and* $M_3 < M_4'$, *then for each* $\widetilde{\lambda} \in (M_1, \infty)$ *and* $\widetilde{\mu} \in (M_3, M_4')$ *there exists a positive solution* $(u(t), v(t)),\ \ t \in [0,1]$ *for* $(\widetilde{S})$.

3) If $\widetilde{g}^s_0 = 0$, $\widetilde{f}^s_0$, $\widetilde{f}^i_\infty$, $\widetilde{g}^i_\infty \in (0,\infty)$ *and* $M_1 < M_2'$, *then for each* $\widetilde{\lambda} \in (M_1, M_2')$ *and* $\widetilde{\mu} \in (M_3, \infty)$ *there exists a positive solution* $(u(t), v(t)),\ \ t \in [0,1]$ *for* $(\widetilde{S})$.

4) If $\widetilde{f}^s_0 = \widetilde{g}^s_0 = 0$, $\widetilde{f}^i_\infty$, $\widetilde{g}^i_\infty \in (0,\infty)$, *then for each* $\widetilde{\lambda} \in (M_1, \infty)$ *and* $\widetilde{\mu} \in (M_3, \infty)$ *there exists a positive solution* $(u(t), v(t)),\ \ t \in [0,1]$ *for* $(\widetilde{S})$.

5) If $\{\widetilde{f}^s_0,\ \widetilde{g}^s_0,\ \widetilde{f}^i_\infty \in (0,\infty),\ \widetilde{g}^i_\infty = \infty\}$ *or* $\{\widetilde{f}^s_0,\ \widetilde{g}^s_0,\ \widetilde{g}^i_\infty \in (0,\infty),\ \widetilde{f}^i_\infty = \infty\}$ *or* $\{\widetilde{f}^s_0,\ \widetilde{g}^s_0 \in (0,\infty),\ \widetilde{f}^i_\infty = \widetilde{g}^i_\infty = \infty\}$, *then for each* $\widetilde{\lambda} \in (0, M_2)$ *and* $\widetilde{\mu} \in (0, M_4)$ *there exists a positive solution* $(u(t), v(t)),\ \ t \in [0,1]$ *for* $(\widetilde{S})$.

6) If $\{\widetilde{f}^s_0 = 0,\ \widetilde{g}^s_0,\ \widetilde{f}^i_\infty \in (0,\infty),\ \widetilde{g}^i_\infty = \infty\}$ *or* $\{\widetilde{f}^s_0 = 0,\ \widetilde{f}^i_\infty = \infty,\ \widetilde{g}^s_0,\ \widetilde{g}^i_\infty \in (0,\infty)\}$ *or* $\{\widetilde{f}^s_0 = 0,\ \widetilde{g}^s_0 \in (0,\infty),\ \widetilde{f}^i_\infty = \widetilde{g}^i_\infty = \infty\}$ *then for each* $\widetilde{\lambda} \in (0,\infty)$ *and* $\widetilde{\mu} \in (0, M_4')$ *there exists a positive solution* $(u(t), v(t)),\ \ t \in [0,1]$ *for* $(\widetilde{S})$.

7) If $\{\widetilde{f}^s_0,\ \widetilde{f}^i_\infty \in (0,\infty),\ \widetilde{g}^s_0 = 0,\ \widetilde{g}^i_\infty = \infty\}$ *or* $\{\widetilde{f}^s_0,\ \widetilde{g}^i_\infty \in (0,\infty),\ \widetilde{g}^s_0 = 0,\ \widetilde{f}^i_\infty = \infty\}$ *or* $\{\widetilde{f}^s_0 \in (0,\infty),\ \widetilde{g}^s_0 = 0,\ \widetilde{f}^i_\infty = \widetilde{g}^i_\infty = \infty\}$ *then for each* $\widetilde{\lambda} \in (0, M_2')$ *and* $\widetilde{\mu} \in (0,\infty)$ *there exists a positive solution* $(u(t), v(t)),\ \ t \in [0,1]$ *for* $(\widetilde{S})$.

8) If $\{\widetilde{f}^s_0 = \widetilde{g}^s_0 = 0,\ \widetilde{f}^i_\infty \in (0,\infty),\ \widetilde{g}^i_\infty = \infty\}$ *or* $\{\widetilde{f}^s_0 = \widetilde{g}^s_0 = 0,\ \widetilde{f}^i_\infty = \infty,\ \widetilde{g}^i_\infty \in (0,\infty)\}$ *or* $\{\widetilde{f}^s_0 = \widetilde{g}^s_0 = 0,\ \widetilde{f}^i_\infty = \widetilde{g}^i_\infty = \infty\}$ *then for each* $\widetilde{\lambda} \in (0,\infty)$ *and* $\widetilde{\mu} \in (0,\infty)$ *there exists a positive solution* $(u(t), v(t)),\ \ t \in [0,1]$ *for* $(\widetilde{S})$.

In what follows, for $\widetilde{f}^i_0$, $\widetilde{g}^i_0$, $\widetilde{f}^s_\infty$, $\widetilde{g}^s_\infty \in (0,\infty)$ and numbers $\alpha_1,\ \alpha_2 \geq 0$, $\widetilde{\alpha}_1,\ \widetilde{\alpha}_2 > 0$ such that $\alpha_1 + \alpha_2 = 1$ and $\widetilde{\alpha}_1 + \widetilde{\alpha}_2 = 1$, we define the numbers

$\widetilde{M_1}$, $\widetilde{M_2}$, $\widetilde{M_3}$, $\widetilde{M_4}$, $\widetilde{M_2'}$ and $\widetilde{M_4'}$ by

$$\widetilde{M_1}=\frac{\alpha_1}{\delta\delta_1\widetilde{f}_0^i\widetilde{A}},\ \ \widetilde{M_2}=\frac{\widetilde{\alpha}_1}{\widetilde{f}_\infty^s\widetilde{B}},\ \ \widetilde{M_3}=\frac{\alpha_2}{\delta\delta_2\widetilde{g}_0^i\widetilde{C}},\ \ \widetilde{M_4}=\frac{\widetilde{\alpha}_2}{\widetilde{g}_\infty^s\widetilde{D}},$$
$$\widetilde{M_2'}=\frac{1}{\widetilde{f}_\infty^s\widetilde{B}},\ \ \widetilde{M_4'}=\frac{1}{\widetilde{g}_\infty^s\widetilde{D}}.$$

THEOREM **4.2**. *Assume that $(\widetilde{H1})$ and $(\widetilde{H2})$ hold, $\alpha_1,\ \alpha_2\geq 0$, $\widetilde{\alpha}_1,\ \widetilde{\alpha}_2>0$ such that $\alpha_1+\alpha_2=1$, $\widetilde{\alpha}_1+\widetilde{\alpha}_2=1$.*

1) If $\widetilde{f}_0^i,\ \widetilde{g}_0^i,\ \widetilde{f}_\infty^s,\ \widetilde{g}_\infty^s\in(0,\infty)$, $\widetilde{M_1}<\widetilde{M_2}$ and $\widetilde{M_3}<\widetilde{M_4}$, then for each $\widetilde{\lambda}\in(\widetilde{M_1},\widetilde{M_2})$ and $\widetilde{\mu}\in(\widetilde{M_3},\widetilde{M_4})$ there exists a positive solution $(u(t),v(t)),\ \ t\in[0,1]$ for $(\widetilde{S})$.

2) If $\widetilde{f}_0^i,\ \widetilde{g}_0^i,\ \widetilde{f}_\infty^s\in(0,\infty)$, $\widetilde{g}_\infty^s=0$ and $\widetilde{M_1}<\widetilde{M_2'}$, then for each $\widetilde{\lambda}\in(\widetilde{M_1},\widetilde{M_2'})$ and $\widetilde{\mu}\in(\widetilde{M_3},\infty)$ there exists a positive solution $(u(t),v(t)),\ \ t\in[0,1]$ for $(\widetilde{S})$.

3) If $\widetilde{f}_0^i,\ \widetilde{g}_0^i,\ \widetilde{g}_\infty^s\in(0,\infty)$, $\widetilde{f}_\infty^s=0$ and $\widetilde{M_3}<\widetilde{M_4'}$, then for each $\widetilde{\lambda}\in(\widetilde{M_1},\infty)$ and $\widetilde{\mu}\in(\widetilde{M_3},\widetilde{M_4'})$ there exists a positive solution $(u(t),v(t)),\ \ t\in[0,1]$ for $(\widetilde{S})$.

4) If $\widetilde{f}_0^i,\ \widetilde{g}_0^i\in(0,\infty)$, $\widetilde{f}_\infty^s=\widetilde{g}_\infty^s=0$, then for each $\widetilde{\lambda}\in(\widetilde{M_1},\infty)$ and $\widetilde{\mu}\in(\widetilde{M_3},\infty)$ there exists a positive solution $(u(t),v(t)),\ \ t\in[0,1]$ for $(\widetilde{S})$.

5) If $\{\widetilde{f}_0^i=\infty,\ \widetilde{g}_0^i,\ \widetilde{f}_\infty^s,\ \widetilde{g}_\infty^s\in(0,\infty)\}$ or $\{\widetilde{f}_0^i,\ \widetilde{f}_\infty^s,\ \widetilde{g}_\infty^s\in(0,\infty),\ \widetilde{g}_0^i=\infty\}$ or $\{\widetilde{f}_0^i=\widetilde{g}_0^i=\infty,\ \widetilde{f}_\infty^s,\ \widetilde{g}_\infty^s\in(0,\infty)\}$, then for each $\widetilde{\lambda}\in(0,\widetilde{M_2})$ and $\widetilde{\mu}\in(0,\widetilde{M_4})$ there exists a positive solution $(u(t),v(t)),\ \ t\in[0,1]$ for $(\widetilde{S})$.

6) If $\{\widetilde{f}_0^i=\infty,\ \widetilde{g}_0^i,\ \widetilde{f}_\infty^s\in(0,\infty),\ \widetilde{g}_\infty^s=0\}$ or $\{\widetilde{f}_0^i,\ \widetilde{f}_\infty^s\in(0,\infty),\ \widetilde{g}_0^i=\infty,\ \widetilde{g}_\infty^s=0\}$ or $\{\widetilde{f}_0^i=\widetilde{g}_0^i=\infty,\ \widetilde{f}_\infty^s\in(0,\infty),\ \widetilde{g}_\infty^s=0\}$, then for each $\widetilde{\lambda}\in(0,\widetilde{M_2'})$ and $\widetilde{\mu}\in(0,\infty)$ there exists a positive solution $(u(t),v(t)),\ \ t\in[0,1]$ for $(\widetilde{S})$.

7) If $\{\widetilde{f}_0^i=\infty,\ \widetilde{g}_0^i,\ \widetilde{g}_\infty^s\in(0,\infty),\ \widetilde{f}_\infty^s=0\}$ or $\{\widetilde{f}_0^i,\ \widetilde{g}_\infty^s\in(0,\infty),\ \widetilde{g}_0^i=\infty,\ \widetilde{f}_\infty^s=0\}$ or $\{\widetilde{f}_0^i=\widetilde{g}_0^i=\infty,\ \widetilde{f}_\infty^s=0,\ \widetilde{g}_\infty^s\in(0,\infty)\}$, then for each $\widetilde{\lambda}\in(0,\infty)$ and $\widetilde{\mu}\in(0,\widetilde{M_4'})$ there exists a positive solution $(u(t),v(t)),\ \ t\in[0,1]$ for $(\widetilde{S})$.

8) If $\{\widetilde{f}_0^i=\infty,\ \widetilde{g}_0^i\in(0,\infty),\ \widetilde{f}_\infty^s=\widetilde{g}_\infty^s=0\}$ or $\{\widetilde{f}_0^i\in(0,\infty),\ \widetilde{g}_0^i=\infty,\ \widetilde{f}_\infty^s=\widetilde{g}_\infty^s=0\}$ or $\{\widetilde{f}_0^i=\widetilde{g}_0^i=\infty,\ \widetilde{f}_\infty^s=\widetilde{g}_\infty^s=0\}$, then for each $\widetilde{\lambda}\in(0,\infty)$ and $\widetilde{\mu}\in(0,\infty)$ there exists a positive solution $(u(t),v(t)),\ \ t\in[0,1]$ for $(\widetilde{S})$.

The proofs of Theorems **4.1** and **4.2** are similar to those of Theorems **3.1** and **3.2**, respectively. For other papers in which there are studied Hammerstein integrals systems, we refer the reader to [17], [21], [26].

5. Examples

Let $\alpha = 3/2$, $(n = 2)$, $\beta = 7/3$, $(m = 3)$, $p = 3$, $q = 2$, $\xi_1 = 1/4$, $\xi_2 = 1/2$, $\xi_3 = 3/4$, $a_1 = 1/2$, $a_2 = 1/3$, $a_3 = 1/4$, $\eta_1 = 1/3$, $\eta_2 = 2/3$, $b_1 = 1$, $b_2 = 1/2$.

We consider the system of fractional differential equations

$$(S_0) \qquad \begin{cases} D_{0+}^{3/2}u(t) + \lambda f(t, u(t), v(t)) = 0, \ \ t \in (0,1), \\ D_{0+}^{7/3}v(t) + \mu g(t, u(t), v(t)) = 0, \ \ t \in (0,1), \end{cases}$$

with the multi-point boundary conditions

$$(BC_0) \qquad \begin{cases} u(0) = 0, \ \ u(1) = \frac{1}{2}u\left(\frac{1}{4}\right) + \frac{1}{3}u\left(\frac{1}{2}\right) + \frac{1}{4}u\left(\frac{3}{4}\right), \\ v(0) = v'(0) = 0, \ \ v(1) = v\left(\frac{1}{3}\right) + \frac{1}{2}v\left(\frac{2}{3}\right). \end{cases}$$

Then we obtain $\Delta_1 = 1 - \sum_{i=1}^{3} a_i \xi_i^{\alpha-1} = (18 - 4\sqrt{2} - 3\sqrt{3})/(24) \approx 0.2978 > 0$, $\Delta_2 = 1 - \sum_{i=1}^{2} b_i \eta_i^{\beta-1} = (3\sqrt[3]{3} - 1 - \sqrt[3]{2})/(3\sqrt[3]{3}) \approx 0.4777 > 0$. We also deduce

$$g_1(t,s) = \frac{2}{\sqrt{\pi}} \begin{cases} t^{1/2}(1-s)^{1/2} - (t-s)^{1/2}, \ \ 0 \le s \le t \le 1, \\ t^{1/2}(1-s)^{1/2}, \ \ 0 \le t \le s \le 1, \end{cases}$$

$$g_2(t,s) = \frac{1}{\Gamma(7/3)} \begin{cases} t^{4/3}(1-s)^{4/3} - (t-s)^{4/3}, \ \ 0 \le s \le t \le 1, \\ t^{4/3}(1-s)^{4/3}, \ \ 0 \le t \le s \le 1, \end{cases}$$

$\theta_1(s) = s$, $\theta_2(s) = \dfrac{1}{4 - 6s + 4s^2 - s^3}$ for all $s \in [0,1]$. For the functions J_1 and J_2, we obtain:

$$J_1(s) = \begin{cases} \frac{2}{\sqrt{\pi}} s^{1/2}(1-s)^{1/2} + \frac{1}{\Delta_1\sqrt{\pi}}\left[\frac{1}{2}\left((1-s)^{1/2} - (1-4s)^{1/2}\right)\right. \\ \quad \left. + \frac{\sqrt{2}}{3}\left((1-s)^{1/2} - (1-2s)^{1/2}\right) + \frac{1}{4}\left(\sqrt{3}(1-s)^{1/2} - (3-4s)^{1/2}\right)\right], \\ \qquad 0 \le s < \frac{1}{4}, \\ \frac{2}{\sqrt{\pi}} s^{1/2}(1-s)^{1/2} + \frac{1}{\Delta_1\sqrt{\pi}}\left[\frac{1}{2}(1-s)^{1/2} + \frac{\sqrt{2}}{3}\left((1-s)^{1/2} - (1-2s)^{1/2}\right)\right. \\ \quad \left. + \frac{1}{4}\left(\sqrt{3}(1-s)^{1/2} - (3-4s)^{1/2}\right)\right], \ \ \frac{1}{4} \le s < \frac{1}{2}, \\ \frac{2}{\sqrt{\pi}} s^{1/2}(1-s)^{1/2} + \frac{1}{\Delta_1\sqrt{\pi}}\left[\frac{1}{2}(1-s)^{1/2} + \frac{\sqrt{2}}{3}(1-s)^{1/2}\right. \\ \quad \left. + \frac{1}{4}\left(\sqrt{3}(1-s)^{1/2} - (3-4s)^{1/2}\right)\right], \ \ \frac{1}{2} \le s < \frac{3}{4}, \\ \frac{2}{\sqrt{\pi}} s^{1/2}(1-s)^{1/2} + \frac{1}{\Delta_1\sqrt{\pi}}\left[\frac{1}{2}(1-s)^{1/2} + \frac{\sqrt{2}}{3}(1-s)^{1/2}\right. \\ \quad \left. + \frac{\sqrt{3}}{4}(1-s)^{1/2}\right], \ \ \frac{3}{4} \le s \le 1, \end{cases}$$

and

$$
J_2(s) = \begin{cases}
\frac{1}{\Gamma(7/3)} \left\{ \dfrac{(1-s)^{4/3}s}{(4-6s+4s^2-s^3)^{1/3}} + \dfrac{1}{\Delta_2} \left[\dfrac{1}{3\sqrt[3]{3}} \left((1-s)^{4/3} - (1-3s)^{4/3} \right) \right. \right. \\
\quad \left. \left. + \dfrac{1}{6\sqrt[3]{3}} \left(2\sqrt[3]{2}(1-s)^{4/3} - (2-3s)^{4/3} \right) \right] \right\}, \quad 0 \le s < \frac{1}{3}, \\
\frac{1}{\Gamma(7/3)} \left\{ \dfrac{(1-s)^{4/3}s}{(4-6s+4s^2-s^3)^{1/3}} + \dfrac{1}{\Delta_2} \left[\dfrac{1}{3\sqrt[3]{3}} (1-s)^{4/3} \right. \right. \\
\quad \left. \left. + \dfrac{1}{6\sqrt[3]{3}} \left(2\sqrt[3]{2}(1-s)^{4/3} - (2-3s)^{4/3} \right) \right] \right\}, \quad \frac{1}{3} \le s < \frac{2}{3}, \\
\frac{1}{\Gamma(7/3)} \left\{ \dfrac{(1-s)^{4/3}s}{(4-6s+4s^2-s^3)^{1/3}} + \dfrac{1}{\Delta_2} \left[\dfrac{1}{3\sqrt[3]{3}} (1-s)^{4/3} \right. \right. \\
\quad \left. \left. + \dfrac{\sqrt[3]{2}}{3\sqrt[3]{3}} (1-s)^{4/3} \right] \right\}, \quad \frac{2}{3} \le s \le 1.
\end{cases}
$$

For $c = \frac{1}{4}$, we deduce $\gamma_1 = \frac{1}{2}$, $\gamma = \gamma_2 = \frac{1}{4\sqrt[3]{4}}$. After some computations, we conclude $A = \int_{1/4}^{3/4} J_1(s)\,ds \approx 0.87174239$, $B = \int_0^1 J_1(s)\,ds \approx 1.35119248$, $C = \int_{1/4}^{3/4} J_2(s)\,ds \approx 0.19552670$, $D = \int_0^1 J_2(s)\,ds \approx 0.27533468$.

Example **5.1**. We consider the functions

$$
f(t,u,v) = \frac{\sqrt{1-t}\,[p_1(u+v)+1](u+v)(q_1+\sin v)}{u+v+1},
$$
$$
g(t,u,v) = \frac{\sqrt[3]{1-t}\,[p_2(u+v)+1](u+v)(q_2+\cos u)}{u+v+1},
$$

for $t \in [0,1]$, u, $v \ge 0$, where p_1, $p_2 > 0$ and q_1, $q_2 > 1$.

We have $f_0^s = q_1$, $g_0^s = q_2 + 1$, $f_\infty^i = p_1(q_1-1)/2$, $g_\infty^i = p_2(q_2-1)/\sqrt[3]{4}$. For α_1, $\alpha_2 > 0$ with $\alpha_1 + \alpha_2 = 1$, we consider $\widetilde{\alpha}_1 = \alpha_1$ and $\widetilde{\alpha}_2 = \alpha_2$. Then, we obtain

$$
L_1 = \frac{16\sqrt[3]{4}\alpha_1}{p_1(q_1-1)A}, \quad L_2 = \frac{\alpha_1}{q_1 B}, \quad L_3 = \frac{64\alpha_2}{p_2(q_2-1)C}, \quad L_4 = \frac{\alpha_2}{(q_2+1)D}.
$$

The conditions $L_1 < L_2$ and $L_3 < L_4$ become

$$
\frac{p_1(q_1-1)}{q_1} > \frac{16\sqrt[3]{4}B}{A}, \quad \frac{p_2(q_2-1)}{q_2+1} > \frac{64D}{C}.
$$

For example, if $\frac{p_1(q_1-1)}{q_1} \geq 40$ and $\frac{p_2(q_2-1)}{q_2+1} \geq 91$, then the above conditions are satisfied. Therefore, by Theorem **3.1** 1), for each $\lambda \in (L_1, L_2)$ and $\mu \in (L_3, L_4)$, there exists a positive solution $(u(t), v(t))$, $t \in [0,1]$ for the problem $(S_0) - (BC_0)$.

EXAMPLE **5.2**. We consider the functions

$$f(t,u,v) = \frac{a_0 t(u+v)^2(q_1 + \sin v)}{u+v+1}, \quad g(t,u,v) = \frac{b_0 t^2(u+v)^2(q_2 + \cos u)}{u+v+1},$$

for $t \in [0,1]$, u, $v \geq 0$, where a_0, $b_0 > 0$, q_1, $q_2 > 1$.

We have $f_0^s = g_0^s = 0$, $f_\infty^i = a_0(q_1-1)/4$, $g_\infty^i = b_0(q_2-1)/(16)$. Then, for α_1, $\alpha_2 \geq 0$ with $\alpha_1 + \alpha_2 = 1$, we conclude $L_1 = \frac{32\sqrt[3]{4}\alpha_1}{a_0(q_1-1)A}$, $L_3 = \frac{256\sqrt[3]{16}\alpha_2}{b_0(q_2-1)C}$. Then, by Theorem **3.1** 4), we deduce that for each $\lambda \in (L_1, \infty)$ and $\mu \in (L_3, \infty)$, there exists a positive solution $(u(t), v(t))$, $t \in [0,1]$ for the problem $(S_0) - (BC_0)$.

EXAMPLE **5.3**. We consider the functions

$$f(t,u,v) = \frac{t}{1+t^2}[p_1(u^2+v^2) + q_1(u+v)],$$
$$g(t,u,v) = \frac{t^2}{1+t}[p_2(u^2+v^2) + q_2(u+v)],$$

for $t \in [0,1]$, u, $v \geq 0$, where p_1, p_2, q_1, $q_2 > 0$.

We have $f_0^s = q_1/2$, $g_0^s = q_2/2$, $f_\infty^i = g_\infty^i = \infty$. Then, for $\widetilde{\alpha}_1$, $\widetilde{\alpha}_2 > 0$ with $\widetilde{\alpha}_1 + \widetilde{\alpha}_2 = 1$, we obtain $L_2 = \frac{2\widetilde{\alpha}_1}{q_1 B}$, $L_4 = \frac{2\widetilde{\alpha}_2}{q_2 D}$. Then, by Theorem **3.1** 5), we deduce that for each $\lambda \in (0, L_2)$ and $\mu \in (0, L_4)$, there exists a positive solution $(u(t), v(t))$, $t \in [0,1]$ for the problem $(S_0) - (BC_0)$.

EXAMPLE **5.4**. We consider the functions

$$f(t,u,v) = p_1 t^a (u^2+v^2), \;\; g(t,u,v) = p_2(1-t)^b(e^{u+v}-1), \;\; t \in [0,1], \;\; u,\, v \geq 0,$$

where a, b, p_1, $p_2 > 0$.

We have $f_0^s = 0$, $g_0^s = p_2$, $f_\infty^i = g_\infty^i = \infty$, $L_4' = \frac{1}{p_2 D}$. Then, by Theorem **3.1** 6), we conclude that for each $\lambda \in (0, \infty)$ and $\mu \in (0, L_4')$, there exists a positive solution $(u(t), v(t))$, $t \in [0,1]$ for the problem $(S_0) - (BC_0)$.

EXAMPLE **5.5**. We consider the functions

$$f(t,u,v) = a(u+v)^{p_1}, \;\; g(t,u,v) = b(u+v)^{p_2}, \;\; t \in [0,1], \;\; u,\, v \geq 0,$$

where a, $b > 0$ and p_1, $p_2 > 1$.

We have $f_0^s = g_0^s = 0$, $f_\infty^i = g_\infty^i = \infty$. Then, by Theorem **3.1** 8), for each $\lambda \in (0, \infty)$ and $\mu \in (0, \infty)$, there exists a positive solution $(u(t), v(t))$, $t \in [0, 1]$ for the problem $(S_0) - (BC_0)$.

Example **5.6**. We consider the functions

$$f(t, u, v) = a(u + v)^{q_1}, \quad g(t, u, v) = b(u + v)^{q_2}, \quad t \in [0, 1], \quad u, v \geq 0,$$

where a, $b > 0$ and q_1, $q_2 \in (0, 1)$.

We have $f_0^i = g_0^i = \infty$, $f_\infty^s = g_\infty^s = 0$. Then, by Theorem **3.2** 8), for each $\lambda \in (0, \infty)$ and $\mu \in (0, \infty)$, there exists a positive solution $(u(t), v(t))$, $t \in [0, 1]$ for the problem $(S_0) - (BC_0)$.

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