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RESEARCH PAPER

A LYAPUNOV-TYPE INEQUALITY FOR A FRACTIONAL BOUNDARY VALUE PROBLEM

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Abstract

In this work we obtain a Lyapunov-type inequality for a fractional differential equation subject to Dirichlet-type boundary conditions. Moreover, we apply this inequality to deduce a criteria for the nonexistence of real zeros of a certain Mittag–Leffler function.

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1. Introduction

In the late XIX century the Russian mathematician, A. M. Lyapunov, proved the following result:

THEOREM **1.1**. (cf. [4]) *If the boundary value problem*

$$\begin{aligned} y''(t) + q(t)y(t) = 0, &\quad a < t < b, \\ y(a) = 0 = y(b), & \end{aligned} \tag{1.1}$$

has a nontrivial solution, where q is a real and continuous function, then

$$\int_a^b |q(s)|ds > \frac{4}{b-a}. \tag{1.2}$$

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The inequality (1.2) proved to be very useful in various problems related with differential equations and, since then, many improvements and generalizations of Theorem **1.1** have appeared in the literature (see [2, 8] and the references therein). To the best of the author's knowledge, none of the mentioned generalizations accomplished to obtain a result in which a fractional derivative is used instead of the (classical) ordinary derivative in equation (1.1). This is one of the issues we are going to address in this work.

The fractional calculus is nowadays an important branch of mathematics, with a positive impact on several applied sciences (the reader is invited to consult, for example, the monograph [3]). Let us introduce the concept of fractional integral and derivative of order $\alpha \geq 0$.

Definition **1.1**. Let $\alpha \geq 0$ and f be a real function defined on $[a, b]$. The Riemann–Liouville fractional integral of order α is defined by $(I_q^0 f)(x) = f(x)$ and

$$({}_aI^\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)ds, \quad \alpha > 0, \quad t \in [a, b].$$

Definition **1.2**. The Riemann–Liouville fractional derivative of order $\alpha \geq 0$ is defined by $({}_aD^0 f)(t) = f(t)$ and $({}_aD^\alpha f)(t) = (D^m {}_aI^{m-\alpha} f)(t)$ for $\alpha > 0$, where m is the smallest integer greater or equal than α.

These differential operators of arbitrary order are nonlocal and this feature has been used to model several real world phenomena, typically by substituting an ordinary derivative by a fractional one in a differential equation (see e.g. [7]).

In this work we will consider the following fractional differential equation

$$({}_aD^\alpha y)(t) + q(t)y(t) = 0, \quad a < t < b, \tag{1.3}$$

together with the boundary conditions

$$y(a) = 0, \quad y(b) = 0, \tag{1.4}$$

where $1 < \alpha \leq 2$ and $q : [a, b] \to \mathbb{R}$ is a continuous function, and try to obtain an inequality similar to (1.2). To do that, we write our BVP as an equivalent integral equation and then, by making use of some properties of its Green function, we are able to obtain such an inequality. After that, we show how this inequality can be used to prove that a Mittag–Leffler-type function has no real zeros on a determined interval.

2. Main result and an application

The Green function for BVP (1.3)–(1.4) was deduced in [1] for $a = 0$ and $b = 1$. Following the same steps as in that work we can easily prove the following result:

LEMMA **2.1**. *y is a solution of the boundary value problem* (1.3)–(1.4) *if, and only if, y satisfies the integral equation*

$$y(t) = \int_a^b G(t,s)q(s)y(s)ds,$$

where

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} \dfrac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}}(b-s)^{\alpha-1} - (t-s)^{\alpha-1}, & a \leq s \leq t \leq b, \\ \dfrac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}}(b-s)^{\alpha-1}, & a \leq t \leq s \leq b. \end{cases}$$

Next, we state three properties of the function G, that will be needed in the sequel. Some were proved in [1] for $a = 0$ and $b = 1$. Nevertheless, since there are some tricky issues within the proof for $a \neq 0$ and $b \neq 1$, we present a proof here for $\{a < b\} \in \mathbb{R}$.

LEMMA **2.2**. *The Green function G defined above satisfies the following conditions:*

(1) $G(t,s) \geq 0$ *for all* $a \leq t, s \leq b$.

(2) $\max_{t\in[a,b]} G(t,s) = G(s,s),\ s \in [a,b]$.

(3) $G(s,s)$ *has a unique maximum, given by*

$$\max_{s\in[a,b]} G(s,s) = G\left(\frac{a+b}{2}, \frac{a+b}{2}\right) = \frac{1}{\Gamma(\alpha)}\left(\frac{b-a}{4}\right)^{\alpha-1}.$$

P r o o f. We start by defining two functions

$$g_1(t,s) = \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}}(b-s)^{\alpha-1} - (t-s)^{\alpha-1},\ a \leq s \leq t \leq b,$$

and

$$g_2(t,s) = \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}}(b-s)^{\alpha-1},\ a \leq t \leq s \leq b.$$

It is clear that $g_2(t,s) \geq 0$. Now, regarding the function g_1, we have that

$$\begin{aligned} g_1(t,s) &= \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}}(b-s)^{\alpha-1} - (t-s)^{\alpha-1} \\ &= \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}}(b-s)^{\alpha-1} \\ &\qquad\qquad - \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}}\left(b - \left(a + \frac{(s-a)(b-a)}{t-a}\right)\right)^{\alpha-1}. \end{aligned}$$

Observe now that

$$\begin{aligned} a + \frac{(s-a)(b-a)}{t-a} \geq s \Leftrightarrow & \frac{a(t-a)+(s-a)(b-a)}{t-a} \geq s \\ \Leftrightarrow & \; a(t-b) + s(b-t) \geq 0 \\ \Leftrightarrow & \; s \geq a, \end{aligned}$$

and therefore $g_1(t,s) \geq 0$, which concludes the proof of 1. To prove 2. we only need to differentiate g_1 with respect to t for every fixed s and then perform an analysis like we did for g_1 in 1. (we leave the details to the reader). We then conclude that g_1 is a decreasing function. Obviously, g_2 is an increasing function of t.

Finally, let $f(s) = G(s,s) = \frac{(s-a)^{\alpha-1}}{(b-a)^{\alpha-1}}(b-s)^{\alpha-1}$, $s \in [a,b]$. Then, for $s \in (a,b)$ we have

$$f'(s) = (\alpha-1)\frac{[(s-a)(b-s)]^{\alpha-2}}{(b-a)^{\alpha-1}}(-2s+a+b),$$

which implies that $f'(s) = 0$ only at $s = \frac{a+b}{2}$ and $f'(s) > 0$ for $s < \frac{a+b}{2}$ and $f'(s) < 0$ for $s > \frac{a+b}{2}$. The proof is complete. □

It follows our main result, i.e. the Lyapunov inequality for the *fractional case.*

THEOREM **2.1**. *If the following fractional boundary value problem (FBVP)*

$$\begin{aligned} ({}_aD^\alpha y)(t) + q(t)y(t) = 0, \quad a < t < b, \\ y(a) = 0 = y(b), \end{aligned}$$

has a nontrivial solution, where q is a real and continuous function, then

$$\int_a^b |q(s)|ds > \Gamma(\alpha)\left(\frac{4}{b-a}\right)^{\alpha-1}. \tag{2.1}$$

P r o o f. Let $\mathcal{B} = C[a,b]$ be the Banach space endowed with norm $\|y\| = \sup_{t\in[a,b]} |y(t)|$.

It follows from Lemma **2.1** that a solution to the FBVP satisfies the integral equation

$$y(t) = \int_a^b G(t,s)q(s)y(s)ds.$$

Hence,

$$\|y\| \le \max_{t\in[a,b]} \int_a^b |G(t,s)q(s)|ds\|y\|,$$

or, equivalently,

$$1 \le \max_{t\in[a,b]} \int_a^b |G(t,s)q(s)|ds.$$

Appealing now to the properties of the Green function G provided in Lemma **2.2**, we get

$$1 < \frac{1}{\Gamma(\alpha)} \left(\frac{b-a}{4}\right)^{\alpha-1} \int_a^b |q(s)|ds,$$

from which the inequality (2.1) follows. □

Remark **2.1**. Note that if we let $\alpha = 2$ in (2.1), one obtains Lyapunov's classical inequality (1.2).

We will end this work presenting an application of Theorem **2.1**. More specifically, we will show how inequality (2.1) can be used to determine intervals for the real zeros of the Mittag–Leffler function:

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+\alpha)},\ z\in\mathbb{C},\ \Re(\alpha) > 0.$$

Let now $a = 0$ and $b = 1$ for simplicity and consider the following fractional Sturm–Liouville eigenvalue problem (SLEP):

$$\begin{aligned} ({}_0D^\alpha y)(t) + \lambda y(t) = 0, &\quad 0 < t < 1, \\ y(0) = 0 = y(1). & \end{aligned} \tag{2.2}$$

In contrast with the classical case (when $\alpha = 2$) there is not a solid (spectral) theory for this problem when $\alpha < 2$ (we refer the reader to the works [5, 6], where some results about the existence of eigenvalues to fractional Sturm–Liouville problems are presented, although not necessarily for the differential equation (2.2)).

Now, by [3, Corollary 5.1] we know that the eigenvalues $\lambda \in \mathbb{R}$ of the SLEP are the solutions of

$$E_\alpha(-\lambda) = 0, \tag{2.3}$$

and the corresponding eigenfunctions are given by

$$y(t) = t^{\alpha-1} E_\alpha(-\lambda t^\alpha),\ t \in [0, 1].$$

To be more precise, we should mention that [3, Corollary 5.1] provides only the explicit formula for the solution of the differential equation (2.2). It is nevertheless true that $y(t) = t^{\alpha-1} E_\alpha(-\lambda t^\alpha)$ is nontrivial since it is not differentiable at $t = 0$.

By Theorem **2.1**, if $\lambda \in \mathbb{R}$ is an eigenvalue of the SLEP, i.e. λ is a zero of equation (2.3), then $|\lambda| > \Gamma(\alpha) 4^{\alpha-1}$. We have just proved the following

THEOREM **2.2**. *Let $1 < \alpha \leq 2$. Then, the Mittag–Leffler function $E_\alpha(z)$ has no real zeros for $|z| \leq \Gamma(\alpha) 4^{\alpha-1}$.*

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