VERSITA

Fractional Calculus & Applied Analysis
An International Journal for Theory and Applications
VOLUME 16, NUMBER 4 (2013)
(Print) ISSN 1311-0454
(Electronic) ISSN 1314-2224

RESEARCH PAPER

APPLICATION OF MEASURE OF NONCOMPACTNESS TO A CAUCHY PROBLEM FOR FRACTIONAL DIFFERENTIAL EQUATIONS IN BANACH SPACES

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Abstract

This paper is devoted to study the existence of solutions of a Cauchy type problem for a nonlinear fractional differential equation, via the techniques of measure of noncompactness. The investigation is based on a new fixed point result which is a generalization of the well known Darbo's fixed point theorem. The main result is less restrictive than those given in the literature. Some illustrative examples are given.

MSC 2010: Primary 47H10; Secondary 34A08

Key Words and Phrases: fixed point, measure of noncompactness, fractional differential equation

1. Introduction

The object of this paper is to discuss the existence of solutions to the Cauchy problem for the fractional differential equation

$$\begin{cases} {}^cD^q u(t) = f(t, u(t)), \quad t \in I = [0, T], \\ u(0) = 0, \end{cases} \tag{1.1}$$

where the symbol ${}^cD^q := {}^cD^q_{0+}$ denotes the Caputo fractional derivative of order $q \in (0, 1)$ with the lower limit zero, E is a Banach space equipped with the norm $\|\cdot\|$ and $f : I \times E \to E$ satisfies some certain conditions, specified later. The problem of existence and/or uniqueness of solutions of

pp. 962–977 , DOI: 10.2478/s13540-013-0059-y

Eq. (1.1) has been studied by several authors in recent times, mostly under the usual conditions that $f(t, u)$ is continuous and Lipschitz with respect to u, or dominated by an affine function with respect to u. Our contribution in this paper is to obtain an existence result for Eq. (1.1) in an abstract Banach space setting, under more general and less restrictive conditions than those given in the literature, i.e., $f(t, u)$ satisfies a Carathéodory type condition and dominated by a control function which is radial with respect to u.

The motivation for studying fractional differential equations comes from the fact that the theory of fractional differential equations has essentially been attracted by the enormous numbers of interesting and novel applications arising in physics, chemistry, biology, engineering, finance and other areas which have been developed in the last few decades. To focus on some applications, we refer the reader to the more recent results, e.g., works of Kilbas et al. [27], Podlubny [33] and Caponetto et al. [14] (control theory), Metzler et al. [32] (relaxation in filled polymer networks), Podlubny et al. [34] (heat propagation), Shaw et al. [35] (modeling of viscoelastic materials), Chern [18] (modeling of the behaviour of viscoelastic and viscoplastic materials under external influences), Bai and Feng [8] and Cuesta and Finat Codes [19] (image processing) and Gaul et al. [26] (description of mechanical systems subject to damping). Also, it is worth pointing out that a completely different and very novel applicable field is the area of mathematical psychology where fractional-order systems may be used to model the behaviour of human beings ([6], [15], [36]), more precisely, using fractional operators, a model of memory-dependent phenomena is prepared which is based on human reaction and the external influences depending on the backgrounds that has been made in the past. To see more about applications, we also refer the reader to ([17], [21]-[25]).

Existence of solutions for fractional differential equations has been investigated by many authors in various types. Recently, the first and third authors and Jalilian [2] proved the solvability of a large class of nonlinear fractional integro-differential equations by establishing some fractional integral inequalities and using the nonlinear alternative of Leray-Schauder type. Benchohra et al. [12] investigated a class of boundary value problems for fractional differential equations involving nonlinear integral conditions using the technique associated with measures of weak noncompactness. Li et al. [30] studied on the existence of mild solutions for fractional semilinear differential equations with nonlocal conditions. Wang et al. [38] applied a new variant fixed point theorem to investigate some fractional differential equations in Banach spaces. Ahmad and Nieto [3] focused on a new class of anti-periodic boundary value problems of fractional differential equations with nonlinear term depending on lower order fractional derivative and obtained some existence and uniqueness results using some of the well known fixed point theorems. Very recently, Liang et al. [31] studied the solvability for a coupled system of nonlinear fractional differential equations in a Banach space using the measures of noncompactness and the well-known fixed point theorem of Mönch type (see also [4], [5], [9], [11], [16], [27]-[29], [37]).

In the present paper, we investigate the existence of solutions for the Cauchy problem (1.1) using a generalization of Darbo's fixed point theorem obtained by the first author et al. [1] via the Hausdorff measure of noncompactness. We support the obtained result by an example to illustrate the result.

The rest of the paper is organized as follows. In the next section, we give some preliminaries about fractional calculus and the Hausdorff measure of noncompactness. In Section **3**, the existence result and illustrative examples are given.

2. Preliminaries

Suppose that $(E, \|\cdot\|)$ is a Banach space and $\mathbb{R}^+$ and $\mathbb{R}_+$ are $(0,\infty)$ and $[0,\infty)$, respectively. Denote by $C(I,E)$ the space of all continuous functions on $I=[0,T]$ with values in E equipped with the norm $|||h||| = \sup_{s\in I}\|h(s)\|$. Also suppose that $L^1(I,E)$ is the space of Bochner integrable functions $h : I \to E$ with the norm $\|h\|_{L^1(I,E)} = \int_0^T \|u(s)\|ds$.

Let us to recall some essential definitions and auxiliary facts in fractional calculus which will be needed (c.f. [27]).

Definition **2.1**. The *fractional order integral* of the function $y \in L^1([a,b],\mathbb{R})$ of order $q \in \mathbb{R}^+$ is defined by

$$I_a^q y(t) = \frac{1}{\Gamma(q)}\int_a^t (t-s)^{q-1} y(s)ds,$$

where $\Gamma(\cdot)$ is the Gamma function.

Definition **2.2**. For a function y given on the interval $[a,b]$, the *q-th Riemann-Liouville fractional order derivative* of y is defined by

$$D_{a+}^q y(t) = \frac{1}{\Gamma(1-q)}\frac{d}{dt}\int_a^t (t-s)^{-q} y(s)ds,$$

where $q \in (0,1)$.

DEFINITION **2.3**. For a function y given on the interval $[a,b]$, the *Caputo fractional derivative* of y of order $q \in (0,1)$ is defined by

$$^{c}D^{q}_{a+}y(t) = D^{q}_{a+}(y(t) - y(a))$$

which is also expressed by

$$^{c}D^{q}_{a+}y(t) = \frac{1}{\Gamma(1-q)} \int_a^t (t-s)^{-q} y'(s)ds \quad \text{for } \ y \in C^1([a,b]).$$

REMARK **2.1**. Note that for an abstract function y which takes values in E, the integrals which appear in the previous definitions are taken in Bochner's sense.

We recall the Hausdorff measure of noncompactness γ_E defined on a bounded subset M of Banach space E is given by

$$\gamma_E(M) = \inf\{\epsilon > 0 : \text{there exists a finite } \epsilon\text{-net for } M \text{ in } E\}. \tag{2.1}$$

By a finite ϵ-net for M in E we mean, as usual, a set $\{e_1, e_2, ..., e_n\} \subseteq E$ such that $\bigcup_{i=1}^{n} B_\epsilon(E; e_i)$ as finite union of open balls covering M.

To apply the Hausdorff measure of noncompactness γ_E in order to obtain our result, we recall below some of the basic properties of the Hausdorff measure of noncompactness γ_E as a lemma which has been presented by Banaś and Goebel [10].

LEMMA **2.1** ([10]). *Let E be a real Banach space and $\Omega, \Omega_1, \Omega_2 \subseteq E$ be bounded. Then the following properties are satisfied:*

- *(regularity) $\gamma_E(\Omega) = 0$ if and only if Ω is precompact;*
- *(invariance under closure and convex hull) $\gamma_E(\Omega) = \gamma_E(\overline{\Omega}) = \gamma_E(\mathrm{conv}\Omega)$;*
- *(monotonicity) $\Omega_1 \subseteq \Omega_2$ implies $\gamma_E(\Omega_1) \leq \gamma_E(\Omega_2)$;*
- *(semi-additivity) $\gamma_E(\Omega_1 \cup \Omega_2) = \max\{\gamma_E(\Omega_1), \gamma_E(\Omega_2)\}$;*
- *(semi-homogeneity) $\gamma_E(r\Omega) = |r|\gamma_E(\Omega)$ for any $r \in \mathbb{R}$;*
- *(algebraic semi-additivity) $\gamma_E(\Omega_1 + \Omega_2) \leq \gamma_E(\Omega_1) + \gamma_E(\Omega_2)$;*
- *if the mapping $T : D(T) \subseteq E \to F$ is Lipschitz continuous with constant k, then $\gamma_F(TB) \leq k\gamma_E(B)$ for any bounded subset $B \subseteq D(T)$, where F is a Banach space;*
- *If $\{B_n\}_{n=1}^{\infty}$ is a decreasing sequence of bounded closed nonempty subsets of E and $\lim_{n\to\infty} \gamma_E(B_n) = 0$, then $\bigcap_{n=1}^{\infty} B_n$ is nonempty and compact in E.*

The mapping $T : C \subseteq E \to E$ is said to be a γ_E-*contraction*, if there exists a positive constant $k < 1$ such that

$$\gamma_E(T(W)) \leq k\gamma_E(W) \tag{2.2}$$

for any bounded closed subset $W \subseteq C$.

THEOREM **2.1**. (Darbo-Sadovskii) *Let C be a nonempty, bounded, closed, and convex subset of a Banach space E and let the continuous mapping $T : C \to C$ be a γ_E-contraction. Then T has at least one fixed point in C.*

From the point of view of historical remarks, we note that G. Darbo [20] initially introduced condition (2.2) for any arbitrary measure of noncompactness μ and presented a similar result if the continuous mapping T is being a μ-contraction. Very recently, the first author et al. [1] extended the Darbo's fixed point theorem using control functions and presented the following result.

THEOREM **2.2** ([1]). *Let C be a nonempty, bounded, closed, and convex subset of a Banach space E and let $T : C \longrightarrow C$ be a continuous function satisfying*

$$\mu(T(W)) \leq \phi(\mu(W)) \tag{2.3}$$

for each $W \subseteq C$, where μ is an arbitrary measure of noncompactness and $\phi : [0,\infty) \to [0,\infty)$ is a monotone increasing (not necessarily continuous) function with $\lim_{n\to\infty} \phi^n(t) = 0$ for all $t \in [0,\infty)$. Then T has at least one fixed point in C.

From now on, without loss of generality and only for convenience, we denote the class of all the functions ϕ which enjoy in conditions of Theorem **2.2** by Φ, and the Hausdorff measure of noncompactness of E and $C(I,E)$ by γ.

3. Cauchy problem in fractional differential equations

We are now in a position to apply Theorem **2.2** as a generalized form of Darbo's fixed point theorem to investigate the existence of solutions to the Cauchy problem (1.1) in a Banach space E using the Hausdorff measure of noncompactness γ which was defined in the previous section. To prove the main result, we need the following assumptions:

(C1): f satisfies Carathéodory type conditions; i.e., $f(.,x)$ is measurable for each fixed x and $f(t,.)$ is continuous for a.e. $t \in I$.

(C2): There exists a function $g \in L^{\frac{1}{q_1}}(I,\mathbb{R}_+)$, $q_1 \in [0,q)$ and a nondecreasing continuous function $\Omega : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|f(t,x)\| \leq g(t)\Omega(\|x\|),$$

for all $x \in E$, and a.e. $t \in I$.

(C3): There exists a function $L \in L^1(I, \mathbb{R}_+)$ and $\phi \in \Phi$ such that for any bounded subset $B \subseteq E$,

$$\gamma(f(t, B)) \leq L(t)\phi(\gamma(B)), \tag{3.1}$$

for a.e. $t \in I$.

(C4): There exists at least one solution $p(t) \in C(I, \mathbb{R}_+)$ to the inequality

$$\frac{\Omega(\|p\|_0)}{\Gamma(q)}\bigg(\int_0^t (t-s)^{q-1} g(s)ds\bigg) \leq p(t), \quad \text{for } t \in I, \tag{3.2}$$

where $\|\cdot\|_0$ is the supremum norm in $C(I, \mathbb{R}_+)$.

The main tools used in our investigation rely on the following lemmas.

Lemma **3.1** ([38]). *A function $u \in C(I, E)$ is a solution of the fractional integral equation*

$$u(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, u(s))ds,$$

if and only if u is a solution of Eq. (1.1).

Lemma **3.2** ([10]). *If $W \subseteq C(I, E)$ is bounded and equicontinuous, then the set $\gamma(W(t))$ is continuous on I and*

$$\gamma(W) = \sup_{t \in I} \gamma(W(t)), \qquad \gamma\bigg(\int_0^t W(s)ds\bigg) \leq \int_0^t \gamma(W(s))ds.$$

Lemma **3.3** ([13]). *If $\{u_n\}_{n=1}^{\infty} \subseteq L^1(I, E)$ satisfies $\|u_n(t)\| \leq \kappa(t)$ a.e. on I for all $n \geq 1$ with some $\kappa \in L^1(I, \mathbb{R}_+)$, then the function $\gamma(\{u_n(t)\}_{n=1}^{\infty})$ belongs to $L^1(I, \mathbb{R}_+)$ and*

$$\gamma\bigg(\bigg\{\int_0^t u_n(s)ds : n \geq 1\bigg\}\bigg) \leq 2\int_0^t \gamma(\{u_n(s) : n \geq 1\})ds.$$

Proposition **3.1**. *Suppose that $p(t)$ is a function which satisfies (C4) and put*

$$W_p = \bigg\{u \in C(I, E) : \|u(t)\| \leq p(t), \ \ t \in I\bigg\} \subseteq C(I, E).$$

Then $C_p = \overline{\mathrm{conv}}\mathcal{F}W_p$ is equicontinuous, where $\overline{\mathrm{conv}}$ means the closure of the convex hull in $C(I, E)$ and $\mathcal{F}$ is a operator from $C(I, E)$ into itself given by

$$(\mathcal{F}u)(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s,u(s))ds, \ \ t \in I. \tag{3.3}$$

Moreover, the operator $\mathcal{F}$ is continuous on $C(I,E)$ and also is bounded from W_p into itself.

P r o o f. We recall that both of the concepts boundedness and equicontinuity have hereditary property related to the closure of the convex hull in $C(I,E)$, that is, boundedness and equicontinuity of $\overline{\text{conv}} W \subseteq C(I,E)$ are inherited by boundedness and equicontinuity of $W \subseteq C(I,E)$. So, it suffices to prove that $\mathcal{F}W_p \subseteq C(I,E)$ is equicontinuous. To prove this, let $u \in W_p$ and $0 \leq t_1 < t_2 \leq T$; then we have

$$\begin{aligned}
&\|(\mathcal{F}u)(t_2) - (\mathcal{F}u)(t_1)\| \\
&= \frac{1}{\Gamma(q)} \| \int_0^{t_2} (t_2-s)^{q-1} f(s,u(s))ds - \int_0^{t_1} (t_1-s)^{q-1} f(s,u(s))ds \| \\
&= \frac{1}{\Gamma(q)} \| \int_0^{t_1} (t_2-s)^{q-1} f(s,u(s))ds - \int_0^{t_1} (t_1-s)^{q-1} f(s,u(s))ds \\
&\qquad + \int_{t_1}^{t_2} (t_2-s)^{q-1} f(s,u(s))ds \| \\
&\leq \frac{1}{\Gamma(q)} \Big(\int_0^{t_1} ((t_1-s)^{q-1} - (t_2-s)^{q-1}) \, \|f(s,u(s))\| ds \\
&\qquad + \int_{t_1}^{t_2} (t_2-s)^{q-1} \|f(s,u(s))\| ds \Big) \\
&\leq \frac{\Omega(\|p\|_0)}{\Gamma(q)} \Big(\int_0^{t_1} ((t_1-s)^{q-1} - (t_2-s)^{q-1}) \, |g(s)| ds \\
&\qquad + \int_{t_1}^{t_2} (t_2-s)^{q-1} |g(s)| ds \Big).
\end{aligned}$$

Next, applying the Hölder inequality we derive

$$\begin{aligned}
\|(\mathcal{F}u)(t_2) - (\mathcal{F}u)(t_1)\| \leq \ & \frac{\Omega(\|p\|_0) \|g\|_{L^{\frac{1}{q_1}}(I,\mathbb{R}_+)}}{\Gamma(q)} \\
& \times \Big(\Big[\int_0^{t_1} ((t_1-s)^{q-1} - (t_2-s)^{q-1})^{\frac{1}{1-q_1}} ds \Big]^{1-q_1} \\
& + \Big[\int_{t_1}^{t_2} (t_2-s)^{\frac{q-1}{1-q_1}} ds \Big]^{1-q_1} \Big)
\end{aligned}$$

so

$$\|(\mathcal{F}u)(t_2) - (\mathcal{F}u)(t_1)\| \to 0 \quad \text{as} \quad t_1 \to t_2.$$

This completes the proof of the first part. Turning to the second assertion, the continuity of $\mathcal{F}: C(I,E) \to C(I,E)$, we first note that $\mathcal{F}$ is obviously well-defined since f satisfies both conditions (C1) and (C2). Next, let $\{u_n\}$ be a sequence of functions in $C(I,E)$ which converges to $u \in C(I,E)$. We have to show that $|||\mathcal{F}u_n - \mathcal{F}u||| \to 0$ as $n \to \infty$. Using the Carathéodory continuity of f we easily have $\|f(s,u_n(s)) - f(s,u(s))\| \to 0$ as $n \to \infty$. Next, applying condition (C2) we conclude the inequality

$$\|f(s,u_n(s)) - f(s,u(s))\| \leq 2g(s)\Bigg(\Omega(\|u(s)\|) + \Omega(\|u_n(s)\|)\Bigg).$$

We notice that since the function $s \to g(s)\Omega(\|u(s)\|)$ is Lebesgue integrable over $[0,t]$, the function $s \to (t-s)^{q-1}g(s)\Omega(\|u(s)\|)$ is also. This fact together with the Lebesgue dominated convergence theorem implies that

$$\|(\mathcal{F}u_n)(t) - (\mathcal{F}u)(t)\| \leq \frac{1}{\Gamma(q)}(\int_0^t (t-s)^{q-1}\|f(s,u_n(s)) - f(s,u(s))\|ds) \to 0$$

as $n \to \infty$ for all $t \in I$. This shows that $\mathcal{F}u_n \rightrightarrows \mathcal{F}u$ and so $|||\mathcal{F}u_n - \mathcal{F}u||| \to 0$ on I as $n \to \infty$. Finally, we prove that $\mathcal{F}$ is bounded on W_p. To prove this, let $u \in W_p$ and $t \in I$; then using (C4) we get

$$\begin{aligned}\|(\mathcal{F}u)(t)\| &\leq \frac{1}{\Gamma(q)}\left(\int_0^t (t-s)^{q-1}\|f(s,u(s))\|ds\right)\\ &\leq \frac{\Omega(\|p\|_0)}{\Gamma(q)}\left(\int_0^t (t-s)^{q-1}g(s)ds\right) \leq p(t),\end{aligned}$$

and the conclusion follows. □

The essential ingredient in the proof of the main result is the following lemma presented by Bothe [13].

LEMMA **3.4** ([13]). *If W is bounded, then for each $\epsilon > 0$, there exists a sequence $\{u_n\}_{n=1}^{\infty} \subseteq W$, such that*

$$\gamma(W) \leq 2\gamma(\{u_n\}_{n=1}^{\infty}) + \epsilon.$$

Now, we have the possibility to formulate our existence result.

THEOREM **3.1**. *Assume that conditions $(C1)$–$(C4)$ are satisfied. Then the Cauchy problem (1.1) has at least one solution $u \in C(I,E)$.*

P r o o f. Consider the operator $\mathcal{F}: C(I,E) \to C(I,E)$ as defined in (3.3). Using Lemma **3.1**, it is clear that the fixed point of operator $\mathcal{F}$ is the solution of Eq. (1.1). Suppose W_p and C_p are as defined in Proposition **3.1**.

Obviously, $W_p \subseteq C(I,E)$ is bounded, closed and convex. Since $\mathcal{F}W_p \subseteq W_p$, we easily obtain the following

$$C_p = \overline{\text{conv}}\mathcal{F}W_p \subseteq \overline{\text{conv}}W_p = W_p \implies \mathcal{F}C_p \subseteq \mathcal{F}W_p \subseteq C_p$$

which shows that $\mathcal{F} : C_p \to C_p$ is well-defined and continuous. On the other hand, for any $C \subseteq C_p$, using Lemma **3.4** and the fact that C is bounded we infer for given $\epsilon > 0$, there exists a sequence $\{u_n\}_{n=1}^{\infty} \subseteq C$ so that

$$\begin{aligned}\gamma((\mathcal{F}C)(t)) &= \gamma\left(\left\{\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s,u(s))ds : u \in C\right\}\right)\\ &\le 2\gamma\left(\left\{\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s,\{u_n(s)\}_{n=1}^{\infty})ds\right\}\right) + \epsilon.\end{aligned}$$

Next, Lemma **3.2** and Lemma **3.3** together with condition (C3) imply that

$$\begin{aligned}\gamma((\mathcal{F}C)(t)) &\le \frac{4}{\Gamma(q)}\int_0^t (t-s)^{q-1}\gamma(\{f(s,u_n(s)) : n \in \mathbb{N}\})ds + \epsilon\\ &\le \frac{4\phi(\gamma(\{u_n\}_{n=1}^{\infty}))}{\Gamma(q)}\int_0^t (t-s)^{q-1}L(s)ds + \epsilon.\end{aligned} \tag{3.4}$$

To simplify the notation we let

$$\widetilde{L} = \int_0^T L(s)ds.$$

Clearly, $0 \le \widetilde{L} < \infty$. Linking inequality (3.4) and the recent notation together with the fact that $\phi(\gamma(\{u_n\}_{n=1}^{\infty})) \le \phi(\gamma(C))$ we derive the following

$$\gamma((\mathcal{F}C)) \le \frac{4\widetilde{L}T^{q-1}}{\Gamma(q)}\phi(\gamma(C)) + \epsilon.$$

Now, considering $\epsilon \to 0$ we obtain

$$\gamma((\mathcal{F}C)) \le \frac{4\widetilde{L}T^{q-1}}{\Gamma(q)}\phi(\gamma(C)).$$

By taking $\psi(s) = \frac{4\widetilde{L}T^{q-1}}{\Gamma(q)}\phi(s)$ we easily conclude that $\psi \in \Phi$. Now, the operator $\mathcal{F}$ is a continuous mapping from the bounded, closed and convex set C_p into itself and inequality (2.3) holds. So all conditions of Theorem **2.2** are satisfied and hence $\mathcal{F}$ has a fixed point such $u \in C_p \subseteq C(I,E)$ and this completes the proof. □

Remark **3.1**. Keep in mind that if $\widetilde{L} = 0$, that is, $L = 0$ a.e., then $\mathcal{F}C$ is precompact so $\mathcal{F} : C \to C$ is compact and conclusion of the theorem immediately follows by Schauder's fixed point theorem.

REMARK **3.2**. Note that if we impose the condition $\frac{4\widetilde{L}T^{q-1}}{\Gamma(q)} < 1$, then using the fact that $\phi(s) \leq s$ for all $s \geq 0$ and $\phi \in \Phi$, one can easily obtain the result applying Darbo's fixed point theorem. This condition need not be added to the assumptions in Theorem **2.2**, and so it can be considered as a redundant condition.

Now, we give some examples illustrating our obtained result.

EXAMPLE **3.1**. Consider the following fractional differential equation

$$\begin{cases} {}^cD^q u(t) = f(t, u(t)), \quad t \in I = [0, T], \ 0 < T < (\frac{M}{2})^{\frac{1}{q}}, \\ u(0) = 0, \end{cases} \tag{3.5}$$

where $M = \min_{q\in(0,1)} \Gamma(q+1) \cong 0.8856$ and $f : I \times c_0 \to c_0$ is given by

$$f(t,x) = \frac{1}{t^2+\frac{1}{2}} \left\{ \ln(|x_k|+1) + \frac{t}{k^2} \right\}_{k=1}^{\infty} \quad \text{for } t \in I, \ x = \{x_k\}_k \in c_0, \tag{3.6}$$

and c_0 represents the space of all sequences converging to zero, which is a Banach space with respect to the norm $\|x\|_\infty = \sup_k |x_k|$. We prove the existence of a solution $u \in C(I, c_0)$ for Eq. (3.5). To do this, we have to show that conditions (C1)–(C4) are satisfied. Note that we can easily deduce the function f satisfies the Carathéodory type conditions, so (C1) holds. To justify condition (C2), let $t \in I$ and $x = \{x_k\}_k \in c_0$. Then we have

$$\begin{aligned} \|f(t,x)\|_\infty &= \frac{1}{t^2+\frac{1}{2}} \|(\ln(|x_k|+1) + \frac{t}{k^2})_k\|_\infty \\ &\leq \frac{1}{t^2+\frac{1}{2}} (\sup_k |x_k| + t) \\ &= \frac{1}{t^2+\frac{1}{2}} (\|x\|_\infty + t) \\ &\leq g(t)\,\Omega(\|x\|_\infty), \end{aligned}$$

where $\Omega : \mathbb{R}_+ \to \mathbb{R}_+$ and $g : I \to \mathbb{R}_+$ are given by $\Omega(t) = t + T$ and $g(t) = (t^2 + \frac{1}{2})^{-1}$. This shows that (C2) holds. To prove (3.1), we recall that the Hausdorff measure of noncompactness γ in the space c_0 can be computed by means of the formula

$$\gamma(B) = \lim_{n\to\infty} \sup_{x \in B} \|(I - P_n)x\|_\infty,$$

where B is a bounded subset in c_0 and P_n is the projection onto the linear span of the first n vectors in the standard basis (c.f. [7]).

Next, let $u = \{u_k\}_k \in B \subseteq c_0$ and $t \in I$. Fix $n \in \mathbb{N}$; then we have

$$\ln(|u_k| + 1) \leq \ln(\|(I - P_n)(u_k)_k\|_\infty + 1) \quad \text{for all } \ k > n,$$

which by taking the supremum implies that

$$\begin{aligned}\sup_{u\in B} \|(I-P_n)(\ln(|u_k| + 1))_k\|_\infty &\leq \sup_{u\in B} \ln(\|(I-P_n)(u_k)_k\|_\infty + 1)\\ &= \ln(\sup_{u\in B} \|(I-P_n)(u_k)_k\|_\infty + 1).\end{aligned}$$

Letting $n \to \infty$ we deduce that

$$\begin{aligned}\lim_{n\to\infty} \sup_{u\in B} \|(I-P_n)(\ln(|u_k| + 1))_k\|_\infty &\leq \lim_{n\to\infty} \ln(\sup_{u\in B} \|(I-P_n)(u_k)_k\|_\infty + 1)\\ &= \ln(\lim_{n\to\infty} (\sup_{u\in B} \|(I-P_n)(u_k)_k\|_\infty) + 1),\end{aligned}$$

which yields

$$\begin{aligned}\gamma(f(t,B)) &= \frac{1}{t^2 + \frac{1}{2}} \lim_{n\to\infty} \sup_{u\in B} \|(I - P_n)(\ln(|u_k| + 1) + \frac{t}{k^2})_k\|_\infty\\ &\leq \frac{1}{t^2 + \frac{1}{2}} \lim_{n\to\infty} \sup_{u\in B} \|(I - P_n)(\ln(|u_k| + 1))_k\|_\infty\\ &\leq \frac{1}{t^2 + \frac{1}{2}} \ln(\lim_{n\to\infty} (\sup_{u\in B} \|(I - P_n)(u_k)_k\|_\infty) + 1)\\ &= L(t)\phi(\gamma(B)),\end{aligned}$$

where $L \in L^1(I, \mathbb{R}_+)$ and $\phi \in \Phi$ are defined by $L(t) = (t^2 + \frac{1}{2})^{-1}$ and $\phi(t) = \ln(t+1)$. Hence condition (C3) is satisfied. Now, it remains to show that there is a solution $p(t) \in C(I, \mathbb{R}_+)$ to the equation

$$\frac{\Omega(\|p\|_0)}{\Gamma(q)}\left(\int_0^t \frac{(t-s)^{q-1}}{s^2 + \frac{1}{2}} ds\right) \leq p(t) \quad \text{ for } \ t \in I.$$

To do this, let the function $p(t)$ be the constant function $p(t) = \lambda$ where $\lambda \geq 2T^{q+1}(\Gamma(q + 1) - 2T^q)^{-1}$. This function is a solution for the recent equation because

$$\begin{aligned}\frac{\Omega(\|\lambda\|_0)}{\Gamma(q)}\left(\int_0^t \frac{(t-s)^{q-1}}{s^2 + \frac{1}{2}} ds\right) &\leq \frac{\lambda + T}{\Gamma(q)}\left(2\int_0^t (t-s)^{q-1} ds\right)\\ &\leq \frac{2(\lambda + T)T^q}{\Gamma(q + 1)} \leq \lambda.\end{aligned}$$

Remark **3.3**. Notice that in the previous example, the Darbo's fixed point theorem can not be applied for some orders $q \in (0, 1)$. More precisely, by taking the bounded set $B = \{(e_k^{(i)})_k; i \in \mathbb{N}\} \subseteq c_0$ where

$$e_k^{(i)} = \begin{cases} 1, & k = i, \\ 0, & \text{otherwise}, \end{cases}$$

since

$$\sup_{i\in\mathbb{N}} \|(I - P_n)(\ln(|e_k^{(i)}| + 1) + \frac{t}{k^2})_k\|_\infty = \ln 2 + \frac{t}{(n+1)^2},$$

we have

$$\begin{aligned}\gamma(f(t,B)) &= \frac{1}{t^2 + \frac{1}{2}} \lim_{n\to\infty} \sup_{i\in\mathbb{N}} \|(I - P_n)(\ln(|e_k^{(i)}| + 1) + \frac{t}{k^2})_k\|_\infty \\ &= \frac{2\ln 2}{2t^2 + 1} \geq \frac{2\ln 2}{2T^2 + 1}.\end{aligned}$$

This shows that $\gamma(f(t,B))$ is strictly greater than

$$\gamma(B) = \lim_{n\to\infty} \sup_{i\in\mathbb{N}} \|(I - P_n)(u_k^{(i)})_k\|_\infty = 1,$$

where $0.9908 \cong (2\ln \frac{M}{2})(\ln(\frac{2\ln 2 - 1}{2}))^{-1} < q < 1$. This proves the claim.

Remark **3.4**. Note that in Example **3.1** if we take the mapping f defined by $f(0,x) = \{0\}_k$ for $x \in c_0$ and by (3.6) otherwise, then f is discontinuous at $t = 0$ but we easily obtain the same result in a similar way. Indeed, as the measure of noncompactness of a finite set is zero, then $\gamma(f(0,B)) = \gamma(\{\{0\}_k\}) = 0$ which shows that condition (C3) holds. The rest of the proof is similar.

Remark **3.5**. We recall that the Hausdorff measure of noncompactness of the unit ball in finite dimensional space $E = \mathbb{R}^n$ is zero. Comparing this fact to our assumptions implies that condition (C3) is redundant in this case, and we can immediately obtain the following result in the vector case.

Corollary **3.1**. *Suppose that conditions (C1), (C2) and (C4) are satisfied. Then the following system of nonlinear fractional differential equations has at least one solution* $\mathbf{u} \in C(I, \mathbb{R}^n)$

$$\begin{cases} {}^cD^q \mathbf{u}(t) = f(t, \mathbf{u}(t)), & t \in I = [0,T], \\ \mathbf{u}(0) = \mathbf{0}, \end{cases}$$

where $\mathbf{0} \in \mathbb{R}^n$, $0 < q < 1$ *and* $\mathbf{u}$ *is a real-valued vector function.*

EXAMPLE **3.2**. Consider the following system of fractional differential equation

$$\begin{cases} {}^cD^q\mathbf{u}(t) = f(t, \mathbf{u}(t)), & t \in I = [0,1], \\ \mathbf{u}(0) = (0,0,0), \end{cases} \tag{3.7}$$

where $\mathbf{u} = (u_1, u_2, u_3) \in C(I, \mathbb{R}^3)$ and $f : I \times \mathbb{R}^3 \to \mathbb{R}^3$ is given by

$$f(t, \mathbf{w}) = \begin{cases} (t\sin(x+y+z), t\cos(x+y+z), 3), & t \in [0, \frac{1}{2}], \\ (t^2, 2\sin(x+y+z), 2\cos(x+y+z)), & t \in (\frac{1}{2}, 1], \end{cases}$$

for all $\mathbf{w} = (x, y, z) \in \mathbb{R}^3$.

Clearly, for every fixed $\mathbf{w} \in \mathbb{R}^3$, $f(t, \mathbf{w})$ is discontinuous at $t = \frac{1}{2}$ but satisfies Carathéodory type conditions and $\|f(t, \mathbf{w})\| \leq \sqrt{t^2 + 9}$ for $t \in I$ and $\mathbf{w} \in \mathbb{R}^3$. Next, take $g(t) = \sqrt{t^2 + 9}$ and $\Omega(s) = 1$ for $t \in I$ and $s \in \mathbb{R}_+$ in condition (C2). Also, the function $p(t) \in C(I, \mathbb{R}_+)$ defined as

$$p(t) = \frac{1}{\Gamma(q)} \left(\int_0^t (t-s)^{q-1} \sqrt{s^2 + 9}\, ds \right) \quad \text{for } t \in I,$$

is a solution of inequality (3.2), so conditions (C1), (C2) and (C4) are satisfied. By Corollary **3.1** system (3.7) has a solution in $C(I, \mathbb{R}^3)$.

Acknowledgements

The third author has been supported in part by the Government of Spain and FEDER (Grant MTM2010-16499).

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Received: April 10, 2013

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Please cite to this paper as published in:

Fract. Calc. Appl. Anal., Vol. **16**, No 4 (2013), pp. 962–977;
DOI: 10.2478/s13540-013-0059-y