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RESEARCH PAPER

UNIQUE POSITIVE SOLUTION FOR A FRACTIONAL BOUNDARY VALUE PROBLEM

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Abstract

In this work we consider the unique positive solution for the following fractional boundary value problem

$$\begin{cases} D_{0+}^{\alpha} u(t) = -f(t, u(t)), \ \ t \in [0,1], \\ u(0) = u'(0) = u'(1) = 0. \end{cases}$$

Here $\alpha \in (2,3]$ is a real number, D_{0+}^{α} is the standard Riemann-Liouville fractional derivative of order α. By using the method of upper and lower solutions and monotone iterative technique, we also obtain that there exists a sequence of iterations uniformly converges to the unique solution.

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1. Introduction

In this paper we investigate the unique positive solution for the fractional boundary value problem with Riemann-liouville derivative

$$\begin{cases} D_{0+}^{\alpha} u(t) = -f(t, u(t)), \ \ t \in [0,1], \\ u(0) = u'(0) = u'(1) = 0, \end{cases} \tag{1.1}$$

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where $\alpha \in (2,3]$ is a real number, D^{α}_{0+} is the standard Riemann-Liouville fractional derivative of order α, and $f : [0,1] \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous.

Fractional calculus and fractional differential equations have been of great interest recently. Because of fractional order derivative memory properties, we can apply it more efficiently in physical memory and genetic properties, and it can better simulate natural physical process and power system process. Thus fractional calculus is used wider and wider in control theory, biological engineering, electrochemical processes, semiconductor physics, mechanical engineering, condensed matter physics, see the relevant books [8,9,11,13,14]. Recently, there are many papers dealing with the existence of solutions (or positive solutions) of nonlinear initial or boundary value problems for fractional differential equations by virtue of techniques of nonlinear analysis (fixed-point theorems, Leray-Schauder theory, lower and upper solution method, etc.), for example, see [1, 2, 5, 6, 10, 15, 16, 17] and the references therein.

In [2], M. El-Shahed concerned with the existence and nonexistence of positive solutions for the nonlinear fractional boundary value problem (1.1) replaced $f(t, u(t))$ by $\lambda a(t) f(u(t))$. The main tool in [2] is Krasnoselskii's fixed point theorem on cone.

In [1, 15, 16], the authors adopted the monotone iterative method to study the existence and uniqueness of solutions for some initial and boundary value problems of fractional differential equations involving Riemann-Liouville sequential fractional derivative.

In [17], by the lower and upper solution method and a fixed point theorem, Zhao et al. established some existence criteria for singular and nonsingular fractional differential equation boundary value problem (1.1). We note that their method involving the subsolution and supersolution is identical with Liang and Zhang's in [10].

Motivated by the works mentioned above, in this paper, we consider the existence and uniqueness of positive solutions for (1.1). Moreover, we can prove that the unique positive solution can be uniformly approximated by an iterative sequence beginning with any function u which is continuous, nonnegative and not identically vanishing on $[0, 1]$.

2. Preliminary results

For the convenience of the reader, we only give the definition of the Riemann-Liouville fractional derivative. The other materials about fractional calculus can be found in the books [8, 9, 11, 13, 14].

Definition **2.1**. The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $f \in AC^n$ is given by

$$D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_0^t \frac{f(s)}{(t-s)^{\alpha-n+1}} \mathrm{d}s,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of number α, provided that the right side is pointwise defined on $(0, +\infty)$.

LEMMA **2.1**. (see [2]) *Let D_{0+}^{α} and f be as in (1.1). Then (1.1) is equivalent to the integral equation*

$$u(t) = \int_0^1 G(t,s) f(s, u(s)) \mathrm{d}s, \quad \text{for} \ \ t \in [0,1], \tag{2.1}$$

where

$$G(t,s) := \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1}(1-s)^{\alpha-2} - (t-s)^{\alpha-1}, & 0 \le s \le t \le 1, \\ t^{\alpha-1}(1-s)^{\alpha-2}, & 0 \le t \le s \le 1. \end{cases} \tag{2.2}$$

LEMMA **2.2**. (see [2]) *The function $G(t,s) \in C([0,1] \times [0,1], [0, +\infty))$ satisfies the following inequalities*

$$t^{\alpha-1} s(1-s)^{\alpha-2} \le \Gamma(\alpha) G(t,s) \le s(1-s)^{\alpha-2}, \quad \forall t, s \in [0,1]. \tag{2.3}$$

LEMMA **2.3**. *Let*

$$w_0(t) := \int_0^1 G(t,s) \mathrm{d}s = \frac{1}{\Gamma(\alpha)} \left(\frac{t^{\alpha-1}}{\alpha-1} - \frac{t^{\alpha}}{\alpha}\right), \quad \forall t \in [0,1].$$

Then for any continuous function w on $[0,1]$ with $w(t) \ge 0$ and $w(t) \not\equiv 0$, $\forall t \in [0,1]$, there exist two positive numbers $a_w < b_w$ such that

$$a_w w_0(t) \le \int_0^1 G(t,s) w(s) \mathrm{d}s \le b_w w_0(t), \quad \forall t \in [0,1]. \tag{2.4}$$

P r o o f. For the reason that w is a continuous and nonnegative function on $[0,1]$, we can set $b_w := \max_{t\in[0,1]} w(t) > 0$, it is clear that the right inequality of (2.4) holds true.

In order to the left inequality of (2.4), we need to consider the limit

$$\lim_{t \to 0^+} \frac{\int_0^1 G(t,s) w(s) \mathrm{d}s}{\int_0^1 G(t,s) \mathrm{d}s}$$

$$
\begin{aligned}
&= \lim_{t\to 0^+} \frac{\int_0^1 t^{\alpha-1}(1-s)^{\alpha-2}w(s)\mathrm{d}s - \int_0^t (t-s)^{\alpha-1}w(s)\mathrm{d}s}{(\alpha-1)^{-1}t^{\alpha-1}-\alpha^{-1}t^{\alpha}} \\
&= \lim_{t\to 0^+} \frac{\int_0^1 (1-s)^{\alpha-2}w(s)\mathrm{d}s}{(\alpha-1)^{-1}-\alpha^{-1}t} - \lim_{t\to 0^+} \frac{\int_0^t (t-s)^{\alpha-1}w(s)\mathrm{d}s}{(\alpha-1)^{-1}t^{\alpha-1}-\alpha^{-1}t^{\alpha}} \qquad (2.5) \\
&= (\alpha-1)\int_0^1 (1-s)^{\alpha-2}w(s)\mathrm{d}s > 0.
\end{aligned}
$$

In what follows, we will show (2.5) is true. Indeed, we have by $-b_w \le w(t) \le b_w, \forall t \in [0,1]$,

$$
\frac{-b_w\alpha^{-1}t^{\alpha}}{(\alpha-1)^{-1}t^{\alpha-1}-\alpha^{-1}t^{\alpha}} \le \frac{\int_0^t (t-s)^{\alpha-1}w(s)\mathrm{d}s}{(\alpha-1)^{-1}t^{\alpha-1}-\alpha^{-1}t^{\alpha}} \le \frac{b_w\alpha^{-1}t^{\alpha}}{(\alpha-1)^{-1}t^{\alpha-1}-\alpha^{-1}t^{\alpha}}.
$$

Consequently,

$$
\lim_{t\to 0^+} \frac{-b_w\alpha^{-1}t}{(\alpha-1)^{-1}-\alpha^{-1}t} = \lim_{t\to 0^+} \frac{b_w\alpha^{-1}t}{(\alpha-1)^{-1}-\alpha^{-1}t} = 0
$$

implies that

$$
\lim_{t\to 0^+} \frac{\int_0^t (t-s)^{\alpha-1}w(s)\mathrm{d}s}{(\alpha-1)^{-1}t^{\alpha-1}-\alpha^{-1}t^{\alpha}} = 0.
$$

As a result of this, (2.5) holds, as claimed. By (2.5), we can choose $a_w := \inf_{t\in(0,1)} \frac{\int_0^1 G(t,s)w(s)\mathrm{d}s}{\int_0^1 G(t,s)\mathrm{d}s}$, then $a_w > 0$ and the left inequality of (2.4) is satisfied. This completes the proof. □

Remark **2.1**. In [10], the authors assume that their nonlinearity is a positive function on $[0,1]$. By this, the relevant inequalities with (2.4) can be obtained easily. For example, in Lemma 2.4 of [10], the authors can only take the minimum value $m := \min_{(t,u)\in[0,1]\times[0,M']} f(t,u(t)) > 0$, then the left inequality (2.4) can be proved immediately. However, our nonlinearity is a nonnegative function on $[0,1]$, which the minimum value may be 0. So the method in [10] is invalided. In our paper, we utilize the method of the limit at 0 to prove our inequalities (2.4), of course, we also weaken the condition on f in [10].

Let

$$
E := C[0,1], \|u\| := \max_{t\in[0,1]} |u(t)|, P := \{u \in E : u(t) \ge 0, t \in [0,1]\}.
$$

Then $(E, \|\cdot\|)$ becomes a real Banach space and P is a cone on E. Now, note that u solves (1.1) if and only if u is a fixed point of the operator

$$(Au)(t) := \int_0^1 G(t,s)f(s,u(s))\mathrm{d}s, \text{ for } u \in E. \tag{2.6}$$

Clearly, the continuity of f and G implies $A : E \to E$ is a completely continuous operator.

LEMMA **2.4**. (Krein-Rutman, see [12]) *Let P be a reproducing cone in a real Banach space E and let $L : E \to E$ be a compact linear operator with $L(P) \subset P$. $r(L)$ is the spectral radius of L. If $r(L) > 0$, then there exists $\varphi \in P\backslash\{0\}$ such that $L\varphi = r(L)\varphi$.*

Let

$$(Lu)(t) := \int_0^1 G(t,s)u(s)\mathrm{d}s, \text{ for } u \in E. \tag{2.7}$$

Then $L : E \to E$ is a completely continuous linear operator, satisfying $L(P) \subset P$. That is, L is a positive, completely continuous, linear operator. In the following, we shall utilize Gelfand's theorem to prove that the spectral radius $r(L) > 0$.

LEMMA **2.5**. $r(L) > 0$.

P r o o f. Clearly, for the Green function G, it is easy to obtain the following inequality

$$G(t,s) \geq t^{\alpha-1}G(\tau,s), \text{ for all } t,s,\tau \in [0,1]. \tag{2.8}$$

By direct computation, we have from the left inequality of (2.3)

$$\|L\| = \max_{t\in[0,1]} \int_0^1 G(t,s)\mathrm{d}s \geq \frac{1}{\Gamma(\alpha)} \max_{t\in[0,1]} t^{\alpha-1} \int_0^1 s(1-s)^{\alpha-2}\mathrm{d}s = \frac{\mathbf{B}(2,\alpha-1)}{\Gamma(\alpha)},$$

and

$$\begin{aligned}
\|L^2\| &= \max_{t\in[0,1]} \int_0^1 \int_0^1 G(t,s)G(s,\tau)\mathrm{d}s\mathrm{d}\tau \\
&\geq \frac{1}{\Gamma^2(\alpha)} \max_{t\in[0,1]} t^{\alpha-1} \int_0^1 \int_0^1 s(1-s)^{\alpha-2}s^{\alpha-1}\tau(1-\tau)^{\alpha-2}\mathrm{d}s\mathrm{d}\tau \\
&= \frac{\mathbf{B}(\alpha+1,\alpha-1)\mathbf{B}(2,\alpha-1)}{\Gamma^2(\alpha)}.
\end{aligned}$$

Consequently, for $n \in \mathbb{N}_+$, we have

$$\|L^n\| \geq \frac{\mathbf{B}^{n-1}(\alpha+1,\alpha-1)\mathbf{B}(2,\alpha-1)}{\Gamma^n(\alpha)}.$$

Here $\mathbf{B}(\cdot,\cdot)$ denotes the Beta function. Therefore, by Gelfand's theorem, we have

$$r(L) = \lim_{n\to\infty} \sqrt[n]{\|L^n\|} \geq \frac{\mathbf{B}(\alpha+1,\alpha-1)}{\Gamma(\alpha)} > 0.$$

This completes the proof. □

By Lemmas **2.4** and **2.5**, we can get that there exists $\varphi \in P\backslash\{0\}$ such that $L\varphi = r(L)\varphi$, i.e.,

$$\int_0^1 G(t,s)\varphi(s)\mathrm{d}s = r(L)\varphi(t). \tag{2.9}$$

Let

$$(L^*u)(s) := \int_0^1 G(t,s)u(t)\mathrm{d}t, \text{ for } u \in E. \tag{2.10}$$

Then we see $r(L^*) = r(L) > 0$ and there exists $\psi \in P^*\backslash\{0\}$ such that $L^*\psi = r(L)\psi$, i.e.,

$$\int_0^1 G(t,s)\psi(t)\mathrm{d}t = r(L)\psi(s). \tag{2.11}$$

Note that we can normalize ψ such that

$$\int_0^1 \psi(t)\mathrm{d}t = 1. \tag{2.12}$$

Let $\omega := \int_0^1 t^{\alpha-1}\psi(t)\mathrm{d}t > 0$ and $P_0 := \left\{u \in P : \int_0^1 u(t)\psi(t)\mathrm{d}t \geq \omega\|u\|\right\}$. Then we have easily the following lemma.

LEMMA **2.6**. $L(P) \subset P_0$.

P r o o f. By (2.8), for $u \in P$, we obtain

$$\int_0^1 (Lu)(t)\psi(t)\mathrm{d}t = \int_0^1 \left(\int_0^1 G(t,s)u(s)\mathrm{d}s\right)\psi(t)\mathrm{d}t$$
$$\geq \int_0^1 t^{\alpha-1}\psi(t)\mathrm{d}t \int_0^1 G(\tau,s)u(s)\mathrm{d}s = \omega(Lu)(\tau), \ \forall \tau \in [0,1].$$

Therefore, $\int_0^1 (Lu)(t)\psi(t)\mathrm{d}t \geq \omega\|Lu\|$, as claimed. This completes the proof. □

We denote $B_\rho := \{u \in E : \|u\| < \rho\}$ for $\rho > 0$ in the sequel.

LEMMA **2.7**. (see [7]) *Let E be a real Banach space and P a cone of E. If $A : (\overline{B}_R\backslash B_r) \cap P \to P$ is a completely continuous operator with $0 < r < R$. If either (1) $Av \nleq v$ for each $P \cap \partial B_r$ and $Av \ngeq v$ for each $P \cap \partial B_R$ or (2) $Av \ngeq v$ for each $P \cap \partial B_r$ and $Av \nleq v$ for each $P \cap \partial B_R$. Then A has at least one fixed point on $(\overline{B}_R\backslash B_r) \cap P$.*

Lemma **2.8**. (see [3]) *Let E be a partial order Banach space, and $x_0, y_0 \in E$ with $x_0 \leq y_0$, $D = [x_0, y_0]$. Suppose that $A : D \to E$ satisfies the following conditions:*

(i) A is an increasing operator;

(ii) $x_0 \leq Ax_0$, $y_0 \geq Ay_0$, i.e., x_0 and y_0 is a subsolution and a supersolution of A;

(iii) A is a completely continuous operator.

Then A has the smallest fixed point x^* and the largest fixed point y^* in $[x_0, y_0]$, respectively. Moreover, $x^* = \lim_{n\to\infty} A^n x_0$ and $y^* = \lim_{n\to\infty} A^n y_0$.

3. Main Results

Let $\lambda_1 := (r(L))^{-1}$. Now, we list our hypotheses on f.

(H1) $\liminf_{u\to 0^+} \frac{f(t,u)}{u} > \lambda_1$ uniformly with respect to $t \in [0,1]$.

(H2) $\limsup_{u\to\infty} \frac{f(t,u)}{u} < \lambda_1$ uniformly with respect to $t \in [0,1]$.

(H3) $f(t, \lambda u) \geq \lambda f(t,u)$ for each $\lambda \in (0,1)$, $u \in \mathbb{R}^+$ and $t \in [0,1]$.

(H4) $f(t,u)$ is increasing in u, that is, the inequality $f(t,u_1) \leq f(t,u_2)$ holds for all $u_1, u_2 \in \mathbb{R}^+$ satisfying $u_1 \leq u_2$.

Theorem **3.1**. *If (H1) and (H2) are satisfied, then* (1.1) *has at least one positive solution.*

P r o o f. By (H1), there are $r > 0$ and $\varepsilon_1 > 0$ such that

$$f(t,u) \geq (\lambda_1 + \varepsilon_1)u, \text{ for all } u \in [0,r] \text{ and } t \in [0,1]. \tag{3.1}$$

This leads to

$$(Au)(t) \geq (\lambda_1 + \varepsilon_1) \int_0^1 G(t,s)u(s)\mathrm{d}s \tag{3.2}$$

for all $u \in \overline{B}_r \cap P$. Let $\mathscr{M}_1 := \{u \in \overline{B}_r \cap P : u \geq Au\}$. We claim that $\mathscr{M}_1 = \{0\}$. Indeed, if the claim is false, then there exists $u_1 \in \partial B_r \cap P$ such that $u_1 \geq Au_1$. Combining with (3.2), we obtain

$$u_1(t) \geq (\lambda_1 + \varepsilon_1) \int_0^1 G(t,s)u_1(s)\mathrm{d}s.$$

Multiply by $\psi(t)$ on both sides of the above and integrate over [0,1] and use (2.11) and (2.12) to obtain

$$\int_0^1 u_1(t)\psi(t)\mathrm{d}t \geq (\lambda_1 + \varepsilon_1)\lambda_1^{-1} \int_0^1 u_1(t)\psi(t)\mathrm{d}t$$

and thus,

$$\int_0^1 u_1(t)\psi(t)\mathrm{d}t = 0,$$

whence $u_1(t) \equiv 0$, contradicting $u_1 \in \partial B_r \cap P$. Consequently,

$$u \not\geq Au, \forall u \in \partial B_r \cap P. \tag{3.3}$$

In addition, by (H2), there exist $\varepsilon_2 \in (0, \lambda_1)$ and $b > 0$ such that

$$f(t,u) \le (\lambda_1 - \varepsilon_2)u + b, \text{ for all } u \in \mathbb{R}^+ \text{ and } t \in [0,1]. \tag{3.4}$$

Let $\mathscr{M}_2 := \{u \in P : \ u \le Au\}$. We shall prove that $\mathscr{M}_2$ is bounded in P. Indeed, if $u \in \mathscr{M}_2$, then we have

$$\begin{aligned} u(t) \le (Au)(t) &= \int_0^1 G(t,s)f(s,u(s))\mathrm{d}s \le \int_0^1 G(t,s)((\lambda_1 - \varepsilon_2)u(s) + b)\mathrm{d}s \\ &= (\lambda_1 - \varepsilon_2)(Lu)(t) + bw_0(t), \end{aligned} \tag{3.5}$$

where $w_0 \in P \setminus \{0\}$ being defined by Lemma **2.3**. Notice $r((\lambda_1 - \varepsilon_2)L) < 1$. This implies the inverse operator of $I - (\lambda_1 - \varepsilon_2)L$ exists and equals

$$(I - (\lambda_1 - \varepsilon_2)L)^{-1} = I + (\lambda_1 - \varepsilon_2)L + (\lambda_1 - \varepsilon_2)^2 L^2 + \ldots + (\lambda_1 - \varepsilon_2)^n L^n + \ldots,$$

from which we obtain $(I - (\lambda_1 - \varepsilon_2)L)^{-1}(P) \subset P$. Applying this to (3.5) gives

$$u \le (I - (\lambda_1 - \varepsilon_2)L)^{-1}u_0$$

for all $u \in \mathscr{M}_2$. This proves the boundedness of $\mathscr{M}_2$, as required. Choosing $R > \sup\{\|u\| : u \in \mathscr{M}_2\}$ and $R > \rho$, we have

$$u \nleq Au, \ \forall u \in \partial B_R \cap P. \tag{3.6}$$

Now Lemma **2.7** implies that A has at least one fixed point on $(B_R \backslash \overline{B}_r) \cap P$. Therefore (1.1) has at least one positive solution, which completes the proof. □

THEOREM **3.2**. *If (H1)-(H3) hold, then* (1.1) *has exactly one positive solution.*

P r o o f. We first prove that (1.1) has at most one positive solution. Indeed, if u_1 and u_2 are two positive solutions of (1.1), then

$$u_i(t) = \int_0^1 G(t,s)f(s,u_i(s))\mathrm{d}s.$$

Lemma **2.3** implies there are positive numbers $b_i \ge a_i$ $(i = 1, 2)$ such that

$$a_1 w_0 \le u_1 \le b_1 w_0 \quad \text{and} \quad a_2 w_0 \le u_2 \le b_2 w_0.$$

Therefore $u_2 \ge \frac{a_2}{b_1} u_1$. Let

$$\mu_0 := \sup\{\mu > 0 : u_2 \ge \mu u_1\}.$$

Thus $\mu_0 > 0$ and $u_2 \ge \mu_0 u_1$. We claim $\mu_0 \ge 1$. Suppose the contrary. Then $\mu_0 < 1$ and

$$u_2(t) \ge \int_0^1 G(t,s)f(s,\mu_0 u_1(s))\mathrm{d}s.$$

We denote

$$g(t) := f(t, \mu_0 u_1(t)) - \mu_0 f(t, u_1(t)).$$

Therefore, (H3) gives $g(t) \in P\backslash\{0\}$. By Lemma **2.3**, there are positive numbers $b_3 \geq a_3$ such that

$$a_3 w_0 \leq \int_0^1 G(t,s)g(s)\mathrm{d}s \leq b_3 w_0.$$

Consequently,

$$u_2(t) \geq \int_0^1 G(t,s)g(s)\mathrm{d}s + \mu_0 u_1(t) \geq \frac{a_3}{b_1}u_1(t) + \mu_0 u_1(t),$$

contradicting the definition of μ_0. As a result, we find $\mu_0 \geq 1$, and thus $u_2 \geq u_1$. Similarly $u_1 \geq u_2$. Therefore $u_1 = u_2$. Thus the problem (1.1) has at most one positive solution. Combining this and Theorem **3.1**, the problem (1.1) has exactly one positive solution. This completes the proof. □

THEOREM **3.3**. *Let (H1) and (H2) hold. The following two conclusions are satisfied:*

(i) Take sufficiently small $\delta > 0$ such that $\delta\|\varphi\| \leq r$, where φ and r are determined by (2.9) *and* (3.1), *respectively. Then we have $A\delta\varphi \geq \delta\varphi$, i.e., $\delta\varphi$ is a subsolution of A.*

(ii) Choose sufficiently large $M > 0$ such that $M \geq (\varepsilon_2 a_w)^{-1}b$, where ε_2, b, a_w are defined by (3.4) *and* (2.4), *respectively. Then we have $AM\varphi \leq M\varphi$, i.e., $M\varphi$ is a supersolution of A.*

P r o o f. From (3.1), we have

$$f(t,\delta\varphi) \geq (\lambda_1 + \varepsilon_1)\delta\varphi, \text{ for all } \delta\varphi \in [0,r] \text{ and } t \in [0,1]. \tag{3.7}$$

Therefore, by (2.9) and (3.7), we see

$$(A\delta\varphi)(t) \geq (\lambda_1 + \varepsilon)\int_0^1 G(t,s)\delta\varphi(s)\mathrm{d}s = \frac{\lambda_1 + \varepsilon}{\lambda_1}\delta\varphi(t) \geq \delta\varphi(t). \tag{3.8}$$

Hence, $\delta\varphi$ is a subsolution of A.

On the other hand, from (2.9), (3.4) and (2.4) (choosing $w(t) = \varphi(t) \in P\backslash\{0\}$), we get

$$\begin{aligned}(AM\varphi)(t) &\leq \int_0^1 G(t,s)((\lambda_1 - \varepsilon_2)M\varphi(s) + b)\mathrm{d}s \\ &= \frac{\lambda_1 - \varepsilon_2}{\lambda_1}M\varphi(t) + b\int_0^1 G(t,s)\mathrm{d}s \\ &\leq \frac{\lambda_1 - \varepsilon_2}{\lambda_1}M\varphi(t) + \frac{b}{a_w}\int_0^1 G(t,s)\varphi(s)\mathrm{d}s \\ &= \frac{\lambda_1 - \varepsilon_2}{\lambda_1}M\varphi(t) + \frac{b}{\lambda_1 a_w}\varphi(t) \leq M\varphi(t).\end{aligned}$$

Therefore, $M\varphi$ is a supersolution of A. This completes the proof. □

THEOREM **3.4**. *Let (H1)-(H4) hold and u^* be the unique positive solution of* (1.1). *Then for all $u_0 \in [\delta\varphi, M\varphi]$ with $f(t, u_0(t)) \not\equiv 0$, the sequence*

$$u_n(t) = \int_0^1 G(t,s) f(s, u_{n-1}(s)) \mathrm{d}s, \quad n = 1, 2, \ldots,$$

uniformly converges to u^*.

P r o o f. (H4) implies that A is an increasing operator. It follows from Lemma **2.8** and Theorem **3.3** that A has the smallest fixed point u_{**} and the largest fixed point u^{**} in $[\delta\varphi, M\varphi]$, respectively. By this, we first show $u^* \in [\delta\varphi, M\varphi]$. Indeed, for all $n \in \mathbb{N}_+$, we have

$$\delta\varphi \leq A^n \delta\varphi \leq A^n M\varphi \leq M\varphi. \tag{3.9}$$

Let $n \to \infty$ in (3.9), we see $\delta\varphi \leq u_{**} \leq u^* \leq u^{**} \leq M\varphi$. For all $\delta\varphi \leq u_0 \leq M\varphi$ and $n \in \mathbb{N}_+$, we have

$$A^n \delta\varphi \leq A^n u_0 = u_n \leq A^n M\varphi. \tag{3.10}$$

By Lemma **2.8** and (1.1) has only a positive solution, $\lim_{n\to\infty} A^n \delta\varphi = \lim_{n\to\infty} A^n M\varphi = u^*$, and thus $\lim_{n\to\infty} u_n \to u^*$. This completes the proof. □

REMARK **3.1**. (1) In Example 4.1 of [10], Liang and Zhang impose their nonlinearity $f(t, u) = t + u^\mu, t \in [0, 1]$, $\mu \in (0, 1)$. Clearly, (H1)-(H4) hold true, i.e., our results can also apply to the example.

(2) In [4], Guo considered the existence and uniqueness of positive solutions for boundary value problems of polynomial type

$$\frac{\mathrm{d}^2 u}{\mathrm{d}t^2} + \sum_{i=1}^n c_i(t) u^{a_i} = 0, \quad 0 \leq t \leq 1, \tag{3.11}$$

subject to the following two boundary conditions respectively

$$u(0) = u(1) = 0 \quad \text{or} \quad u(0) = u'(1) = 0,$$

where $c_i \in C[0, 1]$ are nonnegative functions and there exists $i_0 \in \{1, 2, ..., n\}$ such that $c_{i_0} > 0$ for all $t \in [0, 1]$, $a_i \in (0, 1)$.

In our paper, we can weaken the condition about c_i and only need $c_i \in C[0, 1]$ are nonnegative functions with $c_i \not\equiv 0$, for all $i = 1, 2, \ldots, n$ and $t \in [0, 1]$. The polynomial $\sum_{i=1}^n c_i(t) u^{a_i}$ must satisfy the conditions (H1)-(H4).

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References

[1] C. Bai, Impulsive periodic boundary value problems for fractional differential equation involving Riemann-Liouville sequential fractional derivative. *J. Math. Anal. Appl.* **384**, No 2 (2011), 211–231.

[2] M. El-Shahed, Positive solutions for boundary value problems of nonlinear fractional differential equation. *Abs. Appl. Anal.* Volume 2007, Article ID 10368.

[3] D. Guo, V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*. Academic Press, Orlando (1988).

[4] D. Guo, Positive solutions of Hammerstein integral equations of polynomial type with applications. *Chinese Ann. Math. Ser. A.* **4**, No 5 (1983), 645–656.

[5] J. Graef, L. Kong, B. Yang, Positive solutions for a semipositone fractional boundary value problem with a forcing term. *Fract. Calc. Appl. Anal.* **15**, No 1 (2012), 8–24; DOI: 10.2478/s13540-012-0002-7; http://link.springer.com/article/10.2478/s13540-012-0002-7.

[6] J. Graef, L. Kong, Q. Kong, M. Wang, Uniqueness of positive solutions of fractional boundary value problems with nonhomogeneous integral boundary conditions. *Fract. Calc. Appl. Anal.* **15**, No 3 (2012), 509–528; DOI: 10.2478/s13540-012-0036-x; http://link.springer.com/article/10.2478/s13540-012-0036-x.

[7] M. Krasnoselski, P. Zabreiko, *Geometrical Methods of Nonlinear Analysis*. Springer (1984).

[8] A. Kilbas, H. Srivastava, J. Trujillo, *Theory and Applications of Fractional Differential Equations*. Volume 204 of North-Holland Mathematics Studies, Elsevier Science, Amsterdam (2006).

[9] V. Lakshmikantham, S. Leela, J. Vasundhara Devi, *Theory of Fractional Dynamic Systems*. Cambridge Academic Publishers, Cambridge (2009).

[10] S. Liang, J. Zhang, Positive solutions for boundary value problems of nonlinear fractional differential equation. *Nonlinear Anal.* **71**, No 11 (2009), 5545–5550.

[11] K. Miller, B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*. Wiley, New York (1993).

[12] R. Nussbaum, Eigenvectors of nonlinear positive operators and the linear Krein-Rutman theorem. In: *Fixed Point Theory*, Lecture Notes in Mathematics **886** (1981), Springer, 309–330.

[13] I. Podlubny, *Fractional Differential Equations*. Mathematics in Science and Engineering **198**, Academic Press, San Diego - CA (1999).

[14] S. Samko, A. Kilbas, O. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*. Gordon and Breach, Yverdon (1993).

[15] Z. Wei, Q. Li, J. Che, Initial value problems for fractional differential equations involving Riemann-Liouville sequential fractional derivative. *J. Math. Anal. Appl.* **367**, No 1 (2010), 260–272.

[16] Z. Wei, W. Dong, J. Che, Periodic boundary value problems for fractional differential equations involving a Riemann-Liouville fractional derivative. *Nonlinear Anal.* **73**, No 10 (2010), 3232–3238.

[17] Y. Zhao, S. Sun, Z. Han, Q. Li, The existence of multiple positive solutions for boundary value problems of nonlinear fractional differential equations. *Commun. Nonlinear Sci. Numer. Simul.* **16**, No 4 (2011), 2086–2097.

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