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RESEARCH PAPER

A NEW DIFFERENCE SCHEME FOR TIME FRACTIONAL HEAT EQUATIONS BASED ON THE CRANK-NICHOLSON METHOD

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Abstract

In this paper, we consider the numerical solution of a time-fractional heat equation, which is obtained from the standard diffusion equation by replacing the first-order time derivative with the Caputo derivative of order α, where $0 < \alpha < 1$. The main purpose of this work is to extend the idea on the Crank-Nicholson method to the time-fractional heat equations. By the method of the Fourier analysis, we prove that the proposed method is stable and the numerical solution converges to the exact one with the order $O(\tau^{2-\alpha} + h^2)$, conditionally. Numerical experiments are carried out to support the theoretical claims.

MSC 2010: 65M12, 65M06

Key Words and Phrases: Crank-Nicholson difference schemes, initial value problems for time-fractional heat equations, Caputo derivative, stability, convergence, method of Fourier analysis

1. Introduction

In the last decades differential equations involving fractional derivatives and integrals have been studied by many researchers. Due to their ability to model more adequately some phenomena, fractional partial differential equations (FPDEs) have been used in numerous areas such as viscoelasticity [3, 15], finance [28, 30], hydrology and porous media [4, 9], engineering [16, 26], control systems [18], etc.

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Several different methods have been investigated for solving FPDEs. The variational iteration method [23, 24], the Adomian decomposition method [22, 25, 29], the Laplace transform and Fourier transform methods [2, 27] have been adopted to obtain analytical solutions for FPDEs.

On the other hand, numerical methods based on a finite difference approximation to fractional derivative, for solving FDPEs [8, 11, 17, 20, 21] have been proposed. A practical numerical method for solving multi-dimensional fractional partial differential equations, using a variation on the classical alternating-directions implicit (ADI) Euler method is presented in [19]. In [6, 7] implicit and explicit finite difference methods have been examined for numerical solution of fractional partial differential equations and a Fourier analysis has been used to analyze the stability and convergence of the methods.

There are two fundamental types of fractional order analogues of the heat equations: time fractional heat equations and space fractional heat equations. Some difference schemes for the space-fractional heat equations are presented in [1, 31, 32]. It is easy to apply the Crank-Nicholson difference schemes for the space fractional heat equations. But, for the time fractional heat equations, we need to do some manipulations. This can be seen in [10, 14, 33, 34], the equation of the form $\dfrac{\partial^\alpha u(t,x)}{\partial t^\alpha} = \dfrac{\partial^2 u(t,x)}{\partial x^2}$ is firstly converted to the form $\dfrac{\partial u(t,x)}{\partial t} = \dfrac{\partial^{1-\alpha}}{\partial t^{1-\alpha}}\left(\dfrac{\partial^2 u(t,x)}{\partial x^2}\right)$ and then the approximations are obtained. In [12, 13], the Crank-Nicholson method was applied directly to obtain a numerical solution for the time fractional advection dispersion equations with *Riemann-Liouville* derivative.

In this work, we derive a numerical approximation based on the Crank-Nicholson method for fractional derivatives. Then, we propose a method, with the order of $O(\tau^{2-\alpha} + h^2)$, for solving *time*-fractional heat equations involving *Caputo* derivatives. We consider the following time fractional heat equation:

$$\begin{cases} \dfrac{\partial^\alpha u(t,x)}{\partial t^\alpha} = \dfrac{\partial^2 u(t,x)}{\partial x^2} + f(t,x), \quad (0 < x < 1,\, 0 < t < 1), \\ u(0,x) = r(x), \quad 0 \le x \le 1, \\ u(t,0) = 0,\, u(t,1) = 0, \quad 0 \le t \le 1. \end{cases} \tag{1.1}$$

Here, the term $\dfrac{\partial^\alpha u(t,x)}{\partial t^\alpha}$ denotes the α-order Caputo derivative ([5]), defined by the formula (see e.g. [26], [15], etc.):

$$\frac{\partial^\alpha u(t,x)}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{u_t(\tau,x)}{(t-\tau)^\alpha} d\tau, \qquad \text{where } \ 0 < \alpha < 1, \tag{1.2}$$

and $\Gamma(.)$ is the Gamma function.

2. Discretization of the problem

In this section, we introduce the basic ideas for the numerical solution of the time fractional heat equation (1.1) by the Crank-Nicholson difference scheme.

For some positive integers M and N, the grid sizes in space and time for the finite difference algorithm are defined by $h = 1/M$ and $\tau = 1/N$, respectively. The grid points in the space interval $[0, 1]$ are the numbers $x_j = jh$, $j = 0, 1, 2, ..., M$, and the grid points in the time interval $[0, 1]$ are labeled $t_k = k\tau, k = 0, 1, 2, ..., N$. The values of the functions u and f at the grid points are denoted $u_j^k = u(t_k, x_j)$ and $f_j^k = f(t_k, x_j)$, respectively. Let $u(t, x), u_t(t, x)$ and $u_{tt}(t, x)$ are continuous on $[0, 1]$.

As in the classical Crank-Nicholson difference scheme, a discrete approximation to the fractional derivative $\dfrac{\partial^\alpha u(t,x)}{\partial t^\alpha}$ at $(t_{k+\frac{1}{2}}, x_j)$ can be obtained by the following quadrature formula:

$$\frac{\partial^\alpha u(t_{k+\frac{1}{2}}, x_j)}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int\limits_0^{t_{k+1/2}} u_t(s, x_j)(t_{k+1/2} - s)^{-\alpha} ds$$

$$= \frac{1}{\Gamma(1-\alpha)} \left[\int\limits_0^{t_k} u_t(s, x_j) \left((k + \frac{1}{2})\tau - s \right)^{-\alpha} ds \right.$$

$$\left. + \int\limits_{t_k}^{t_{k+1/2}} \left[\frac{u_j^{k+1} - u_j^k}{\tau} + O(\tau) \right] \left((k + \frac{1}{2})\tau - s \right)^{-\alpha} ds \right]$$

$$= \frac{1}{\Gamma(1-\alpha)} \sum_{m=1}^{k} \int\limits_{(m-1)\tau}^{m\tau} \left[\frac{u_j^m - u_j^{m-1}}{\tau} \right.$$

$$\left. + \left(s - t_{m-\frac{1}{2}} \right) u_{tt}(c_m, x_j) \right] \left((k + \frac{1}{2})\tau - s \right)^{-\alpha} ds$$

$$+ \frac{1}{\Gamma(1-\alpha)} \int\limits_{k\tau}^{(k+1/2)\tau} \left[\frac{u_j^{k+1} - u_j^k}{\tau} + O(\tau) \right] \left((k + \frac{1}{2})\tau - s \right)^{-\alpha} ds.$$

Then,

$$\frac{\partial^\alpha u(t_{k+\frac{1}{2}}, x_j)}{\partial t^\alpha}$$

$$= \frac{1}{\Gamma(1-\alpha)}\sum_{m=1}^{k}\left(\frac{u_j^m-u_j^{m-1}}{\tau}\right)\int\limits_{(m-1)\tau}^{m\tau}\left((k+\frac{1}{2})\tau-s\right)^{-\alpha}ds$$

$$+\frac{1}{\Gamma(1-\alpha)}\sum_{m=1}^{k}\int\limits_{(m-1)\tau}^{m\tau}\left(s-t_{m-\frac{1}{2}}\right)u_{tt}(c_m,x_j)\left((k+\frac{1}{2})\tau-s\right)^{-\alpha}ds$$

$$+\frac{1}{\Gamma(1-\alpha)}\left[\frac{u_j^{k+1}-u_j^k}{\tau}+O(\tau)\right]\int\limits_{k\tau}^{(k+1/2)\tau}\left((k+\frac{1}{2})\tau-s\right)^{-\alpha}ds$$

$$= \frac{1}{\Gamma(1-\alpha)}\frac{1}{\tau^\alpha}\frac{1}{1-\alpha}\sum_{m=1}^{k}\left[u_j^m-u_j^{m-1}\right]\left[(k-m+3/2)^{1-\alpha}\right.$$
$$\left.-(k-m+1/2)^{1-\alpha}\right]+\frac{1}{\Gamma(1-\alpha)}\frac{1}{\tau^\alpha}\frac{1}{1-\alpha}(u_j^{k+1}-u_j^k)\frac{1}{2^{1-\alpha}}$$
$$+\quad\frac{1}{\Gamma(1-\alpha)}\sum_{m=1}^{k}\left[\int\limits_{(m-1)\tau}^{m\tau}\left(s-t_{m-\frac{1}{2}}\right)u_{tt}(c_m,x_j)\left((k+\frac{1}{2})\tau-s\right)^{-\alpha}ds\right]$$
$$+\frac{1}{\Gamma(1-\alpha)}\frac{1}{1-\alpha}\frac{1}{2^{1-\alpha}}O(\tau^{2-\alpha}).$$

So, we obtain,

$$\frac{\partial^\alpha u(t_{k+\frac{1}{2}},xj)}{\partial t^\alpha}$$
$$= \frac{1}{\Gamma(1-\alpha)}\frac{1}{\tau^\alpha}\frac{1}{1-\alpha}\left\{(u_j^{k+1}-u_j^k)\frac{1}{2^{1-\alpha}}\right.$$
$$\left.+\sum_{m=1}^{k}\left[u_j^m-u_j^{m-1}\right]\left[(k-m+\frac{3}{2})^{1-\alpha}-(k-m+\frac{1}{2})^{1-\alpha}\right]\right\}$$
$$+R_1+R_2,$$

where

$$R_1=\frac{1}{\Gamma(1-\alpha)}\sum_{m=1}^{k}\left[\int\limits_{(m-1)\tau}^{m\tau}\left(s-t_{m-\frac{1}{2}}\right)u_{tt}(c_m,x_j)\left((k+\frac{1}{2})\tau-s\right)^{-\alpha}ds\right]$$

and

$$R_2=\frac{1}{\Gamma(2-\alpha)}\frac{1}{2^{1-\alpha}}O(\tau^{2-\alpha}).$$

Here,

$$R_1 = \frac{1}{\Gamma(1-\alpha)} \sum_{m=1}^{k} \left[\int\limits_{(m-1)\tau}^{m\tau} \left(s - t_{m-\frac{1}{2}}\right) u_{tt}(c_m, x_j) \left((k+\frac{1}{2})\tau - s \right)^{-\alpha} ds \right]$$

$$= \frac{1}{\Gamma(1-\alpha)} \sum_{m=1}^{k} u_{tt}(c_m, x_j) \left[\int\limits_{(m-1)\tau}^{m\tau} \left(s - t_{m-\frac{1}{2}}\right) \left((k+\frac{1}{2})\tau - s \right)^{-\alpha} ds \right]$$

$$\begin{aligned}
&= \frac{1}{\Gamma(1-\alpha)} \sum_{m=1}^{k} u_{tt}(c_m, x_j) \left[\frac{(k-m+1)}{1-\alpha} \left(\left(k - m + \frac{3}{2} \right)^{1-\alpha} \right.\right. \\
&\quad \left. - \left(k - m + \frac{1}{2} \right)^{1-\alpha} \right) - \frac{1}{2-\alpha} \left(\left(k - m + \frac{3}{2} \right)^{2-\alpha} \right. \\
&\quad \left.\left. - \left(k - m + \frac{1}{2} \right)^{2-\alpha} \right) \tau^{2-\alpha} \right] \\
\leq\ & \frac{\max\limits_{1 \leq j \leq n} |u_{tt}(c_m, x_j)|}{\Gamma(1-\alpha)} \left\{ \frac{k\,(2k+1)^{1-\alpha}}{2^{1-\alpha}(1-\alpha)} \right. \\
&\quad - \frac{(2k-1)^{1-\alpha} + (2k-3)^{1-\alpha} + ... + 3^{1-\alpha} + 1^{1-\alpha}}{2^{1-\alpha}(1-\alpha)} \\
&\quad \left. - \frac{(2k+1)^{2-\alpha} - 1^{2-\alpha}}{2^{2-\alpha}(2-\alpha)} \tau^{2-\alpha} \right\} \\
\leq\ & \frac{\max\limits_{1 \leq j \leq n} |u_{tt}(c_m, x_j)|}{\Gamma(1-\alpha)} \tau^{2-\alpha}.
\end{aligned}$$

Setting $\sigma = \dfrac{1}{\Gamma(2-\alpha)} \frac{1}{\tau^{\alpha}}$ and $w_j = \sigma \left((j+1/2)^{1-\alpha} - (j-1/2)^{1-\alpha} \right)$, finally we have obtained the following approximation:

$$\begin{aligned}
\frac{\partial^{\alpha} u(t_{k+\frac{1}{2}}, x_j)}{\partial t^{\alpha}} &= \left[w_1 u^k + \sum_{m=1}^{k-1} \left(w_{k-m+1} - w_{k-m} \right) u^m \right. \\
&\quad \left. - w_k u^0 + \sigma \frac{(u_j^{k+1} - u_j^k)}{2^{1-\alpha}} \right] + O(\tau^{2-\alpha}).
\end{aligned} \tag{2.1}$$

On the other hand, we have

$$\frac{\partial^2 u(t_{k+\frac{1}{2}}, x_j)}{\partial x^2} = \frac{1}{2}\left[\frac{u_{j+1}^{k+1} - 2u_j^{k+1} + u_{j-1}^{k+1}}{h^2} + \frac{u_{j+1}^{k} - 2u_j^{k} + u_{j-1}^{k}}{h^2}\right] + O(h^2). \tag{2.2}$$

3. The Crank-Nicholson difference scheme

Using the equations (2.1) and (2.2), we obtain the following difference scheme which is accurate of order $O(\tau^{2-\alpha}+h^2)$:

$$\begin{cases} \left[w_1 u_j^k + \sum\limits_{m=1}^{k-1} (w_{k-m+1} - w_{k-m})\, u_j^m - w_k u_j^0 + \sigma \dfrac{(u_j^{k+1} - u_j^k)}{2^{1-\alpha}}\right] \\ -\left[\dfrac{u_{j+1}^{k+1} - 2u_j^{k+1} + u_{j-1}^{k+1}}{2h^2} + \frac{u_{j+1}^{k} - 2u_j^{k} + u_{j-1}^{k}}{2h^2}\right] = f(t_k + \frac{\tau}{2}, x_j), \\ 0 \le k \le N-1, \quad 1 \le j \le M-1, \\ u_j^0 = r(x_j), \quad 1 \le j \le M-1, \\ u_0^k = 0, \quad u_M^k = 0, \quad 0 \le k \le N. \end{cases} \tag{3.1}$$

We can arrange the system above, so to obtain

$$\begin{cases} \left[\left(-\frac{1}{2h^2}\right) u_{j+1}^{k+1} + \left(\frac{\sigma}{2^{1-\alpha}} + \frac{1}{h^2}\right) u_j^{k+1} + \left(-\frac{1}{2h^2}\right) u_{j-1}^{k+1}\right] \\ + \left[\left(-\frac{1}{2h^2}\right) u_{j+1}^{k} + \left(-\frac{\sigma}{2^{1-\alpha}} + \frac{1}{h^2}\right) u_j^{k} + \left(-\frac{1}{2h^2}\right) u_{j-1}^{k}\right] \\ + \left[w_1 u_j^k + \sum\limits_{m=1}^{k-1} (w_{k-m+1} - w_{k-m})\, u_j^m - w_k u_j^0\right] \\ = f(t_k + \frac{\tau}{2}, x_j), \quad 0 \le k \le N-1, \ 1 \le j \le M, \\ u_j^0 = r(x_j), \ 1 \le j \le M, \\ u_0^k = 0, \quad u_M^k = 0, \ 0 \le k \le N. \end{cases} \tag{3.2}$$

4. Stability of the finite difference methods

We analyze the stability of the difference scheme by a Fourier analysis. Let U_j^k be the approximate solution and define $\rho_j^k = u_j^k - U_j^k;\ k = 0, 1, ..., N$, $j = 0, 1, ..., M$. Then, we obtain the following roundoff error equation

$$\begin{cases} \left[\left(-\frac{1}{2h^2}\right) \rho_{j+1}^{k+1} + \left(\frac{\sigma}{2^{1-\alpha}} + \frac{1}{h^2}\right) \rho_j^{k+1} + \left(-\frac{1}{2h^2}\right) \rho_{j-1}^{k+1}\right] \\ = \left[\left(\frac{1}{2h^2}\right) \rho_{j+1}^{k} + \left(\frac{\sigma}{2^{1-\alpha}} - \frac{1}{h^2}\right) \rho_j^{k} + \left(\frac{1}{2h^2}\right) \rho_{j-1}^{k}\right] \\ - \left[w_1 \rho^k + \sum\limits_{m=1}^{k-1} (w_{k-m+1} - w_{k-m})\, \rho^m - w_k \rho^0\right] \\ \rho_0^k = \rho_M^k = 0. \end{cases} \tag{4.1}$$

We now define the grid functions:

$$\rho^k(x) = \begin{cases} \rho_j^k, & \text{when} \quad x_{j-\frac{h}{2}} < x < x_{j+\frac{h}{2}} \\ 0, & \text{when} \quad 0 \le x \le \frac{h}{2} \ \text{ or } \ L - \frac{h}{2} < x \le L, \end{cases}$$

then $\rho^k(x)$ can be expanded in a Fourier series

$$\rho^k(x) = \sum_{l=-\infty}^{\infty} d_k(l)\, e^{\frac{i2\pi lx}{L}} \quad , \ k = 1, 2, ..., N,$$

where $d_k(l) = \frac{1}{L}\int_0^L \rho^k(x)\, e^{-\frac{i2\pi lx}{L}}$. Introducing the following norm

$$\left\|\rho^k\right\|_2 = \left(\sum_{j=1}^{M-1} h \left|\rho_j^k\right|^2\right)^{1/2} = \left[\int_0^L \left|\rho_j^k\right|^2 dx\right]^{1/2}$$

and applying the Parseval equality $\int_0^L \left|\rho^k(x)\right|^2 dx = \sum_{l=-\infty}^{\infty} |d_k(l)|^2$, we obtain

$$\left\|\rho^k\right\|_2^2 = \sum_{l=-\infty}^{\infty} |d_k(l)|^2 .$$

Based on the above analysis we can suppose that the solution of (4.1) has the following form $\rho_j^k = d_k e^{ijh\beta}$ where $\beta = 2\pi l/L$ and $L = 1$. Substituting the above expression into (4.1), we obtain

$$\begin{aligned}
&\left(-\tfrac{1}{2h^2}\right) d_{k+1} e^{i(j+1)h\beta} + \left(\tfrac{\sigma}{2^{1-\alpha}} + \tfrac{1}{h^2}\right) d_{k+1} e^{ijh\beta} + \left(-\tfrac{1}{2h^2}\right) d_{k+1} e^{i(j-1)h\beta} \\
&= \tfrac{1}{2h^2} d_k e^{i(j+1)h\beta} + \left(\tfrac{\sigma}{2^{1-\alpha}} - \tfrac{1}{h^2}\right) d_k e^{ijh\beta} + \tfrac{1}{2h^2} d_k e^{i(j-1)h\beta} \\
&- \left[w_1 d_k e^{ijh\beta} + \sum_{m=1}^{k-1} \left(w_{k-m+1} - w_{k-m}\right) d_m e^{ijh\beta} - w_k d_0 e^{ijh\beta}\right].
\end{aligned}$$

After simplifications, we write

$$\begin{aligned}
& d_{k+1}\left(-\frac{1}{2h^2}\left[e^{ih\beta} + e^{-ih\beta}\right] + \frac{\sigma}{2^{1-\alpha}} + \frac{1}{h^2}\right) \\
= \ & d_k\left(\frac{1}{2h^2}\left[e^{ih\beta} + e^{-ih\beta}\right] + \frac{\sigma}{2^{1-\alpha}} - \frac{1}{h^2}\right) \\
& - \left[w_1 d_k + \sum_{m=1}^{k-1} \left(w_{k-m+1} - w_{k-m}\right) d_m - w_k d_0\right].
\end{aligned}$$

Then, we obtain

$$d_{k+1}\left(-\frac{1}{h^2}\cos\beta h+\frac{\sigma}{2^{1-\alpha}}+\frac{1}{h^2}\right) \tag{4.2}$$
$$= d_k\left(\frac{1}{h^2}\cos\beta h+\frac{\sigma}{2^{1-\alpha}}-\frac{1}{h^2}\right)$$
$$-\left[w_1 d_k+\sum_{m=1}^{k-1}\left(w_{k-m+1}-w_{k-m}\right)d_m-w_k d_0\right].$$

PROPOSITION **4.1**. *If $\frac{3}{2}\le 3^\alpha$, then $|d_k|\le|d_0|$, $k=1,2,...,N$, where d_k is the solution of (4.2).*

P r o o f. We will use mathematical induction for the proof. We start with $k=0$, then

$$|d_1| = \left|\frac{\frac{1}{h^2}\cos\beta h+\frac{\sigma}{2^{1-\alpha}}-\frac{1}{h^2}}{-\frac{1}{h^2}\cos\beta h+\frac{\sigma}{2^{1-\alpha}}+\frac{1}{h^2}}\right||d_0| = \left|\frac{\frac{\cos\beta h-1}{h^2}+\frac{\sigma}{2^{1-\alpha}}}{\frac{1-\cos\beta h}{h^2}+\frac{\sigma}{2^{1-\alpha}}}\right||d_0|$$
$$= \left|\frac{\frac{-(1-\cos\beta h)}{h^2}+\frac{\sigma}{2^{1-\alpha}}}{\frac{1-\cos\beta h}{h^2}+\frac{\sigma}{2^{1-\alpha}}}\right||d_0|\le|d_0|,$$

therefore $|d_1|\le|d_0|$.

Now, assume that $|d_n|\le|d_0|$, $n=1,2,...,k$. We need to prove this for $n=k+1$. Indeed,

$$|d_{k+1}| \le \left(\left|\frac{\frac{\cos\beta h-1}{h^2}+\frac{\sigma}{2^{1-\alpha}}-w_1}{\frac{1-\cos\beta h}{h^2}+\frac{\sigma}{2^{1-\alpha}}}\right|\right.$$
$$\left.+\frac{1}{\frac{1-\cos\beta h}{h^2}+\frac{\sigma}{2^{1-\alpha}}}\left[\sum_{m=1}^{k-1}|w_{k-m}-w_{k-m+1}|+w_k\right]\right)|d_0|$$
$$= \left(\left|\frac{\frac{\cos\beta h-1}{h^2}+\frac{\sigma}{2^{1-\alpha}}-w_1}{\frac{1-\cos\beta h}{h^2}+\frac{\sigma}{2^{1-\alpha}}}\right|+\frac{w_1}{\frac{1-\cos\beta h}{h^2}+\frac{\sigma}{2^{1-\alpha}}}\right)|d_0|.$$

If $\frac{\cos\beta h-1}{h^2}+\frac{\sigma}{2^{1-\alpha}}-w_1>0$, then

$$|d_{k+1}| \le \left(\frac{\frac{\cos\beta h-1}{h^2}+\frac{\sigma}{2^{1-\alpha}}-w_1+w_1}{\frac{1-\cos\beta h}{h^2}+\frac{\sigma}{2^{1-\alpha}}}\right)|d_0| = \left(\frac{\frac{\cos\beta h-1}{h^2}+\frac{\sigma}{2^{1-\alpha}}}{\frac{1-\cos\beta h}{h^2}+\frac{\sigma}{2^{1-\alpha}}}\right)|d_0|$$
$$\le |d_0|.$$

If $\frac{\cos\beta h-1}{h^2}+\frac{\sigma}{2^{1-\alpha}}-w_1<0$, then

$$\begin{aligned}|d_{k+1}| &\leq \left(\frac{w_1-\frac{\sigma}{2^{1-\alpha}}-\frac{\cos\beta h-1}{h^2}+w_1}{\frac{1-\cos\beta h}{h^2}+\frac{\sigma}{2^{1-\alpha}}}\right)|d_0| \\ &= \left(\frac{2w_1-\frac{\sigma}{2^{1-\alpha}}-\frac{\cos\beta h-1}{h^2}}{\frac{1-\cos\beta h}{h^2}+\frac{\sigma}{2^{1-\alpha}}}\right)|d_0|.\end{aligned}$$

Here,

$$\begin{aligned}|d_{k+1}| &\leq |d_0| \\ &\Leftrightarrow \frac{2w_1-\frac{\sigma}{2^{1-\alpha}}-\frac{\cos\beta h-1}{h^2}}{\frac{1-\cos\beta h}{h^2}+\frac{\sigma}{2^{1-\alpha}}}<1 \\ &\Leftrightarrow 2w_1-\frac{\sigma}{2^{1-\alpha}}-\frac{\cos\beta h-1}{h^2}\leq\frac{1-\cos\beta h}{h^2}+\frac{\sigma}{2^{1-\alpha}} \\ &\Leftrightarrow w_1\leq\frac{\sigma}{2^{1-\alpha}}\Leftrightarrow\left(1+\frac{1}{2}\right)^{1-\alpha}-\left(1-\frac{1}{2}\right)^{1-\alpha}\leq\frac{1}{2^{1-\alpha}} \\ &\Leftrightarrow \frac{3}{2}\leq 3^{\alpha}.\end{aligned}$$

□

THEOREM **4.1**. *The finite difference scheme (3.1) is stable, if* $\frac{3}{2}\leq 3^{\alpha}$.

P r o o f. Assume $\frac{3}{2}\leq 3^{\alpha}$, then using Proposition **4.1** and the Parseval equality, we obtain

$$\left\|\rho^k\right\|_2\leq\left\|\rho^0\right\|_2, \qquad k=1,2,...,N,$$

which means the proposed difference scheme is conditionally stable. □

5. Convergence of the finite difference scheme

Let us denote the truncation error at $(t_{k+\frac{1}{2}},x_j)$ by R_j^k. From the equations (2.1) and (2.2), we have

$$\left|R_j^k\right|\leq C_{j,k}^{(1)}\left(\tau^{2-\alpha}+h^2\right), \quad j=1,2,...,M-1;k=0,2,...,N-1,$$

then we write

$$\left|R_j^k\right|\leq C_1\left(\tau^{2-\alpha}+h^2\right), \quad j=1,2,...,M-1;k=0,2,...,N-1,$$

where $C_1=\max\limits_{1\leq j\leq M-1,1\leq k\leq N}\left\{C_{j,k}^{(1)}\right\}$.

Let $\varepsilon_j^k = u(x_j, t_k) - u_j^k$ denote the error at (t_k, x_j) and we write the following error equation:

$$\begin{aligned}
&\left[\left(-\tfrac{1}{2h^2}\right)\varepsilon_{j+1}^{k+1} + \left(\tfrac{\sigma}{2^{1-\alpha}} + \tfrac{1}{h^2}\right)\varepsilon_j^{k+1} + \left(-\tfrac{1}{2h^2}\right)\varepsilon_{j-1}^{k+1}\right] \\
&+ \left[\left(-\tfrac{1}{2h^2}\right)\varepsilon_{j+1}^{k} + \left(-\tfrac{\sigma}{2^{1-\alpha}} + \tfrac{1}{h^2}\right)\varepsilon_j^{k} + \left(-\tfrac{1}{2h^2}\right)\varepsilon_{j-1}^{k}\right] \\
&+ \left[w_1\varepsilon_k + \sum_{m=1}^{k-1}(w_{k-m+1} - w_{k-m})\,\varepsilon_m - w_k\varepsilon_0\right] \\
&= R_j^k, \quad 0 \le k \le N-1,\ 1 \le j \le M.
\end{aligned}$$

The error equation is obtained by subtracting the equation (3.2) from the following equation

$$\begin{aligned}
&\left[\left(-\tfrac{1}{2h^2}\right)u(t_{k+1}, x_{j+1}) + \left(\tfrac{\sigma}{2^{1-\alpha}} + \tfrac{1}{h^2}\right)u(t_{k+1}, x_j) + \left(-\tfrac{1}{2h^2}\right)u(t_{k+1}, x_{j-1})\right] \\
&+ \left[\left(-\tfrac{1}{2h^2}\right)u(t_k, x_{j+1}) + \left(-\tfrac{\sigma}{2^{1-\alpha}} + \tfrac{1}{h^2}\right)u(t_k, x_j) + \left(-\tfrac{1}{2h^2}\right)u(t_k, x_{j-1})\right] \\
&+ \left[w_1 u(t_k, x_j) - \sum_{m=1}^{k-1}(w_{k-m+1} - w_{k-m})u(t_m, x_j) - w_k u(t_0, x_j)\right] \\
&= f\left(t_n + \tfrac{\tau}{2}\right) + R_j^k, \quad 0 \le k \le N-1,\ 1 \le j \le M.
\end{aligned}$$

Notice that the error equation satisfies the boundary conditions $\varepsilon_0^k = \varepsilon_M^k = 0$, $k = 1, 2, ..., N$ and the initial condition $\varepsilon_j^0 = 0$, $j = 0, 1, ..., M$.

We now analyze the convergence of the difference scheme by using the Fourier analysis. Similarly to the stability analysis, we define the grid functions

$$\varepsilon^k(x) = \begin{cases} \varepsilon_j^k, \text{ when } & x_{j-\frac{h}{2}} < x < x_{j+\frac{h}{2}},\ j = 1, 2, ..., M-1 \\ 0, \text{ when } & 0 \le x \le \frac{h}{2} \text{ or } L - \frac{h}{2} < x \le L, \end{cases}$$

$$R^k(x) = \begin{cases} R_j^k, \text{ when } & x_{j-\frac{h}{2}} < x < x_{j+\frac{h}{2}},\ j = 1, 2, ..., M-1 \\ 0, \text{ when } & 0 \le x \le \frac{h}{2} \text{ or } L - \frac{h}{2} < x \le L. \end{cases}$$

Thus $\varepsilon^k(x)$ and $R^k(x)$ have Fourier series expressions

$$\varepsilon^k(x) = \sum_{l=-\infty}^{\infty} \xi_k(l)\, e^{i2\pi lx/l}, \quad k = 1, 2, ..., N,$$

$$R^k(x) = \sum_{l=-\infty}^{\infty} \eta_k(l)\, e^{i2\pi lx/l}, \quad k = 1, 2, ..., N,$$

where

$$\xi_k(l) = \frac{1}{L}\int_0^L \varepsilon^k(x)\, e^{-i2\pi lx/l}dx \quad \text{and} \quad \eta_k(l) = \frac{1}{L}\int_0^L R^k(x)\, e^{-i2\pi lx/l}dx.$$

We now let

$$\varepsilon^k = \left[\varepsilon_1^k, \varepsilon_2^k, ..., \varepsilon_{M-1}^k\right]^T, \quad k = 1, 2, ..., N,$$

$$R^k = \left[R_1^k, R_2^k, ..., R_{M-1}^k\right]^T, \quad k = 1, 2, ..., N,$$

and introduce the following norms:

$$\left\|\varepsilon^k\right\|_2 = \left(\sum_{j=1}^{M-1} h \left|\varepsilon_j^k\right|^2\right)^{1/2} = \left[\int_0^L \left|\varepsilon^k(x)\right|^2 dx\right]^{1/2}, k = 1, 2, ..., N,$$

$$\left\|R^k\right\|_2 = \left(\sum_{j=1}^{M-1} h \left|R_j^k\right|^2\right)^{1/2} = \left[\int_0^L \left|R^k(x)\right|^2 dx\right]^{1/2}, k = 1, 2, ..., N.$$

Using the Parseval equality:

$$\int_0^L \left|\varepsilon^k(x)\right|^2 dx = \sum_{l=-\infty}^{\infty} \left|\xi_k(l)\right|^2, k = 1, 2, ..., N,$$

$$\int_0^L \left|R^k(x)\right|^2 dx = \sum_{l=-\infty}^{\infty} \left|\eta_k(l)\right|^2, k = 1, 2, ..., N,$$

we also have

$$\left\|\varepsilon^k\right\|_2^2 = \sum_{l=-\infty}^{\infty} \left|\xi_k(l)\right|^2, k = 1, 2, ..., N, \tag{5.1}$$

$$\left\|R^k\right\|_2^2 = \sum_{l=-\infty}^{\infty} \left|\eta_k(l)\right|^2, k = 1, 2, ..., N. \tag{5.2}$$

Based on the above analysis, we also can suppose that

$$\varepsilon_j^k = \xi_k e^{i\beta jh} \qquad \text{and} \qquad R_j^k = \eta_k e^{i\beta jh} \tag{5.3}$$

respectively, where $\beta = 2\pi l/L$ and $L = 1$. Then, we have the error equation:

$$\begin{cases} \left(-\dfrac{1}{2h^2}\right)\varepsilon_{j+1}^{k+1} + \left(\dfrac{\sigma}{2^{1-\alpha}} + \dfrac{1}{h^2}\right)\varepsilon_j^{k+1} + \left(-\frac{1}{2h^2}\right)\varepsilon_{j-1}^{k+1} \\ = \left(\dfrac{1}{2h^2}\right)\varepsilon_{j+1}^k + \left(\dfrac{\sigma}{2^{1-\alpha}} - \dfrac{1}{h^2}\right)\varepsilon_j^k + \left(\frac{1}{2h^2}\right)\varepsilon_{j-1}^k \\ + \left[\sum\limits_{m=1}^{k-1}\left(w_{k-m} - w_{k-m+1}\right)\varepsilon_j^m + w_k\varepsilon^0 - w_1\varepsilon_j^k\right] + R_j^k, \\ 0 \le k \le N-1, \ 1 \le j \le M, \\ \varepsilon^0 = \varepsilon^N = 0, \quad 0 \le j \le M. \end{cases}$$

Using the formulas in (5.3), we obtain

$$\begin{aligned} & \left(-\frac{1}{2h^2}\right)\xi_{k+1}e^{i(j+1)h\beta}+\left(\frac{\sigma}{2^{1-\alpha}}+\frac{1}{h^2}\right)\xi_{k+1}e^{ijh\beta} \\ & +\left(-\frac{1}{2h^2}\right)\xi_{k+1}e^{i(j-1)h\beta} \\ = & \frac{1}{2h^2}\xi_k e^{i(j+1)h\beta}+\left(\frac{\sigma}{2^{1-\alpha}}-\frac{1}{h^2}\right)\xi_k e^{ijh\beta}+\frac{1}{2h^2}\xi_k e^{i(j-1)h\beta} \\ & +\left[\sum_{m=1}^{k-1}\left(w_{k-m}-w_{k-m+1}\right)\xi_m e^{ijh\beta}-w_1\xi_k e^{ijh\beta}\right]+\eta_{k+1}e^{ijh\beta}. \end{aligned}$$

After simplifications, we write

$$\begin{aligned} & \left(-\frac{1}{2h^2}\right)\xi_{k+1}e^{ih\beta}+\left(\frac{\sigma}{2^{1-\alpha}}+\frac{1}{h^2}\right)\xi_{k+1}+\left(-\frac{1}{2h^2}\right)\xi_{k+1}e^{-ih\beta} \\ = & \frac{1}{2h^2}\xi_k e^{ih\beta}+\left(\frac{\sigma}{2^{1-\alpha}}-\frac{1}{h^2}\right)\xi_k+\frac{1}{2h^2}\xi_k e^{-ih\beta} \\ & +\left[\sum_{m=1}^{k-1}\left(w_{k-m}-w_{k-m+1}\right)\xi_m-w_1\xi_k\right]+\eta_{k+1}. \end{aligned}$$

Then, we get the following equality:

$$\begin{aligned} \xi_{k+1} = & \frac{\frac{\cos\beta h-1}{h^2}+\frac{\sigma}{2^{1-\alpha}}}{\frac{1-\cos\beta h}{h^2}+\frac{\sigma}{2^{1-\alpha}}}\xi_k \\ & +\frac{1}{\frac{1-\cos\beta h}{h^2}+\frac{\sigma}{2^{1-\alpha}}}\left[\sum_{m=1}^{k-1}\left(w_{k-m}-w_{k-m+1}\right)\xi_m-w_1\xi_k+\eta_{k+1}\right]. \end{aligned} \tag{5.4}$$

PROPOSITION **5.1**. *Assuming that ξ_k $(k=1,2,...,N)$ is the solution of equation (5.4), then there exists a positive constant C_2, so that $|\xi_k| \le C_2\left|\eta_1\right|, \ k=1,2,...,N$.*

P r o o f. Noticing that $\varepsilon^0=0$, we have $\xi_0=0$.

By the convergence of the series in the right hand side of (5.2), then there is a positive constant C_k, such that

$$\left|\eta_k\right| \equiv \left|\eta_k(l)\right| \le C_k\left|\eta_1(l)\right| \equiv \left|\eta_1\right|, \ k=1,2,...,N$$

and then, we have

$$\left|\eta_k\right| \le C_2\left|\eta_1\right|, \quad k=1,2,...,N, \text{ where } C_2=\max_{1\le k\le N}\left\{C_k\right\}.$$

We prove the proposition by using mathematical induction.

For $k=0$, $\xi_1 = \dfrac{1}{\frac{1-\cos\beta h}{h^2}+\frac{\sigma}{2^{1-\alpha}}}\eta_1$, and we can say $|\xi_1| < |\eta_1| < C_2 |\eta_1|$.

Assume that $|\xi_n| \le C_2 |\eta_1|, \quad n = 1,2,...,k$. Now we need to show that it is also true for $n = k+1$. From (5.4), we write

$$
\begin{aligned}
|\xi_{k+1}| &\le \left|\frac{\frac{\cos\beta h-1}{h^2}+\frac{\sigma}{2^{1-\alpha}}-w_1}{\frac{1-\cos\beta h}{h^2}+\frac{\sigma}{2^{1-\alpha}}}\right| |\xi_k| + \left|\frac{1}{\frac{1-\cos\beta h}{h^2}+\frac{\sigma}{2^{1-\alpha}}}\right| \\
&\quad \times \left(\sum_{m=1}^{k-1} |(w_{k-m}-w_{k-m+1})|\,|\xi_m|\right) + \left|\frac{1}{\frac{1-\cos\beta h}{h^2}+\frac{\sigma}{2^{1-\alpha}}}\right| |\eta_{k+1}| \\
&\le \left(\left|\frac{\frac{\cos\beta h-1}{h^2}+\frac{\sigma}{2^{1-\alpha}}-w_1}{\frac{1-\cos\beta h}{h^2}+\frac{\sigma}{2^{1-\alpha}}}\right| + \left|\frac{1}{\frac{1-\cos\beta h}{h^2}+\frac{\sigma}{2^{1-\alpha}}}\right|\right. \\
&\quad \left. \times \sum_{m=1}^{k-1} (w_{k-m}-w_{k-m+1})\right) C_2 |\eta_1| + \left|\frac{1}{\frac{1-\cos\beta h}{h^2}+\frac{\sigma}{2^{1-\alpha}}}\right| C_2 |\eta_1| \\
&= \left(\left|\frac{\frac{\cos\beta h-1}{h^2}+\frac{\sigma}{2^{1-\alpha}}-w_1}{\frac{1-\cos\beta h}{h^2}+\frac{\sigma}{2^{1-\alpha}}}\right| + \frac{w_1-w_k}{\frac{1-\cos\beta h}{h^2}+\frac{\sigma}{2^{1-\alpha}}}\right. \\
&\quad \left. + \frac{1}{\frac{1-\cos\beta h}{h^2}+\frac{\sigma}{2^{1-\alpha}}}\right) C_2 |\eta_1| .
\end{aligned}
$$

If $\dfrac{\cos\beta h-1}{h^2}+\frac{\sigma}{2^{1-\alpha}}-w_1 > 0$, then $|\xi_{k+1}| \le \left[\dfrac{\frac{\cos\beta h-1}{h^2}+\frac{\sigma}{2^{1-\alpha}}-w_k+1}{\frac{1-\cos\beta h}{h^2}+\frac{\sigma}{2^{1-\alpha}}}\right] C_2 |\eta_1|$, since it is always satisfied that $\dfrac{\frac{\cos\beta h-1}{h^2}+\frac{\sigma}{2^{1-\alpha}}-w_k+1}{\frac{1-\cos\beta h}{h^2}+\frac{\sigma}{2^{1-\alpha}}} \le 1$ and we have $|\xi_{k+1}| \le C_2 |\eta_1|$. If $\dfrac{\cos\beta h-1}{h^2}+\frac{\sigma}{2^{1-\alpha}}-w_1 < 0$, then

$$
|\xi_{k+1}| \le \left[\frac{w_1-\frac{\cos\beta h-1}{h^2}-\frac{\sigma}{2^{1-\alpha}}+w_1-w_k+1}{\frac{1-\cos\beta h}{h^2}+\frac{\sigma}{2^{1-\alpha}}}\right] C_2 |\eta_1| .
$$

Here, we need to check that $\dfrac{2w_1-\frac{\cos\beta h-1}{h^2}-\frac{\sigma}{2^{1-\alpha}}-w_k+1}{\frac{1-\cos\beta h}{h^2}+\frac{\sigma}{2^{1-\alpha}}} \le 1$ and this inequality holds when $\frac{3}{2} \le 3^\alpha$. This completes proof. □

THEOREM **5.1**. *The finite difference scheme (3.1) is L_2-convergent, and the convergence order is $O\left(\tau^{2-\alpha}+h^2\right)$, if $\frac{3}{2} \le 3^\alpha$.*

P r o o f. Applying Proposition **5.1** and noticing (5.1) and (5.2), we obtain

$$\left\|\varepsilon^k\right\|_2 \le C_2 \left\|R^1\right\|_2 \le C_2 C_1 \left(\tau^{2-\alpha} + h^2\right).$$

So, $\left\|\varepsilon^k\right\|_2 \le C\left(\tau^{2-\alpha} + h^2\right)$ where $C = C_2 C_1$. This completes the proof. □

6. The solvability of the difference scheme

The difference scheme (3.2) can be written in matrix form

$$\begin{cases} AU^1 = BU^0 + \varphi^0, \quad n = 0, \\ AU^{n+1} = BU^n - w_1 U^n + \sum\limits_{j=1}^{n-1} \left(w_{n-j} - w_{n-j+1}\right) U^j \\ \qquad\qquad + w_n U^0 + \varphi^n, \quad 1 \le n \le N-1 \\ U_i^0 = r(x_i), \ 1 \le i \le M-1, \\ U_0^n = 0, \ \ U_M^n = 0, \ 0 \le n \le N, \end{cases} \tag{6.1}$$

where $\varphi^n = [\varphi_0^n, \varphi_1^n, \varphi_2^n, ..., \varphi_M^n]^T$, $\varphi_i^0 = r(x_i)$, $\varphi_i^n = f(t_{n+1/2}, x_i), 1 \le n \le N$, $1 \le i \le M$, and $U^n = [U_0^n, U_1^n, U_2^n, ..., U_M^n]^T$. Here, A and B are 3-diagonal matrices with the size $(M-1) \times (M-1)$ of the form:

$$A = \begin{bmatrix} \frac{\sigma}{2^{1-\alpha}} + \frac{1}{h^2} & \frac{-1}{2h^2} & & & \\ \frac{-1}{2h^2} & \frac{\sigma}{2^{1-\alpha}} + \frac{1}{h^2} & \frac{-1}{2h^2} & & \\ & \ddots & \ddots & \ddots & \\ & & \frac{-1}{2h^2} & \frac{\sigma}{2^{1-\alpha}} + \frac{1}{h^2} & \frac{-1}{2h^2} \\ & & & \frac{-1}{2h^2} & \frac{\sigma}{2^{1-\alpha}} + \frac{1}{h^2} \end{bmatrix},$$

$$B = \begin{bmatrix} \frac{\sigma}{2^{1-\alpha}} - \frac{1}{h^2} & \frac{1}{2h^2} & & & \\ \frac{1}{2h^2} & \frac{\sigma}{2^{1-\alpha}} - \frac{1}{h^2} & \frac{1}{2h^2} & & \\ & \ddots & \ddots & \ddots & \\ & & \frac{1}{2h^2} & \frac{\sigma}{2^{1-\alpha}} - \frac{1}{h^2} & \frac{1}{2h^2} \\ & & & \frac{1}{2h^2} & \frac{\sigma}{2^{1-\alpha}} - \frac{1}{h^2} \end{bmatrix}.$$

THEOREM **6.1**. *The difference equation (6.1) is uniquely solvable.*

P r o o f. Because $\dfrac{\sigma}{2^{1-\alpha}} > 0$, the coefficient matrix of the difference equation (6.1) is a strictly diagonally dominant matrix. Therefore, A is a nonsingular matrix; this proves Theorem **6.1**. □

7. Numerical analysis

Example **7.1**. Let $h(t) = \exp(t)$. Here, we present **Table 1** for the various values of τ and α to show the order of the proposed difference method using the max norm of the errors with the formula

$$Error(\alpha,\tau) = \max_{1\leq n\leq N}\left|\frac{\partial^\alpha h(t_{n+1/2})}{\partial t^\alpha} - H(t_{n+1/2})\right|,$$

where $H(t_{n+1/2})$ is the approximation to $\dfrac{\partial^\alpha h(t_{n+1/2})}{\partial t^\alpha}$ obtained by (2.1) at $t_n = 1$.

	$\alpha = 0.1$		$\alpha = 0.5$		$\alpha = 0.9$	
N	$Error(\alpha,\tau)$	Rate	$Error(\alpha,\tau)$	Rate	$Error(\alpha,\tau)$	Rate
4	0.0269867	-	0.0414008	-	0.0295844	-
8	0.0065363	4.1	0.0124283	3.3	0.0110504	2.7
16	0.0016336	4.0	0.0039162	3.2	0.0045109	2.4
32	0.0004149	3.9	0.0012748	3.1	0.0019471	2.3

Table 1. Error table for $h(t) = \exp(t)$

Example **7.2**.

$$\begin{cases} \dfrac{\partial^{1/2}u(t,x)}{\partial t^{1/2}} = \dfrac{\partial^2 u(t,x)}{\partial x^2} + x(1-x)\dfrac{6t^{3-1/2}}{\Gamma(4-1/2)} + 2(t^3+1), \\ (0 < x < 1,\ 0 < t < 1), \\ u(0,x) = (1-x)x, \quad 0 \leq x \leq 1, \\ u(t,0) = 0, \quad u(t,1) = 0,\ 0 \leq t \leq 1. \end{cases}$$

The exact solution of this problem is $U(t,x) = (1-x)x(t^3+1)$. The solution by the proposed scheme is given in **Figure 1**. The errors when solving this problem are listed in **Table 2** for various values of time and space nodes. The errors in the table are calculated by the formula

$$\max_{\substack{0\leq n\leq M \\ 0\leq k\leq N}}\left|u(t_k,x_n) - U_n^k\right|.$$

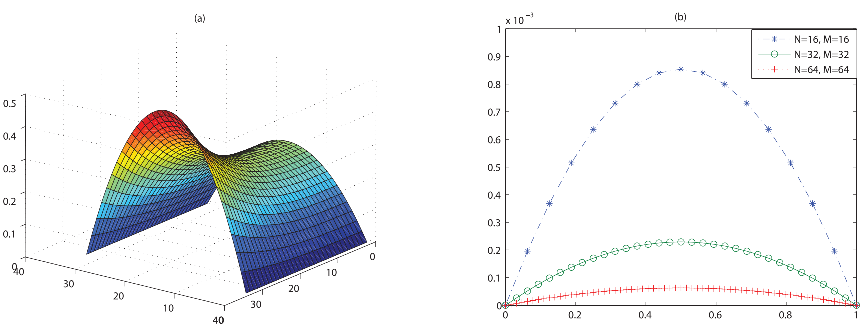

FIGURE 1. (a) The solutions by the proposed method when $N = 32$, $M = 32$. (b) The errors for some values of M and N when $t = 1$.

Space Nodes(M)	Time Nodes(N)	Error
32	4	0.0124648486
32	8	0.0032352895
32	16	0.0008536728
32	32	0.0002291988
32	64	0.0000628150
32	128	0.0000176561

TABLE 2. Error table for Example 2.

It can be concluded from the tables and the figures that when the step size is reduced by a factor of $1/2$, the error decreases by about $(1/2)^{2-\alpha}$. The numerical results support the claim about the order of the convergence.

8. Conclusion

In this work, the Crank-Nicholson difference scheme is successfully extended to solve a time fractional heat equation with Caputo derivative. A sufficient condition is presented to prove the proposed method is stable and the numerical solution converges to the exact one with the order $O(\tau^{2-\alpha} + h^2)$. The numerical results are in good agreement with the theoretical results.

References

[1] A.M. Abu-Saman, A.M. Assaf, Stability and convergence of Crank-Nicholson method for fractional advection dispersion equation. *Advances in Appl. Math. Analysis* **2**, No 2 (2007), 117–125.

[2] B. Baeumer, D.A. Benson, M.M. Meerschaert, Advection and dispersion in time and space. *Physica A* **350** (2005), 245–262.

[3] R.L. Bagley, P.J. Torvik, A theoretical basis for the application of fractional calculus to viscoelasticity. *Journal of Rheology* **27**, No 3 (1983), 201–210.

[4] D.A. Benson, S. Wheatcraft, M.M. Meerschaert, Application of a fractional advection–dispersion equation. *Water Resour. Res.* **36** (2000), 1403–1412.

[5] M. Caputo, Linear models of dissipation whose Q is almost frequency independent - II. *Geophys. J. Royal Astr. Soc.* **13** (1967), 529–539; Reprinted in: *Fract. Calc. Appl. Anal.* **11**, No 1 (2008), 3–14; at http://www.math.bas.bg/∼fcaa.

[6] C. Chen, F. Liu and K. Burrage, Finite difference methods and a fourier analysis for the fractional reaction–subdiffusion equation. *Appl. Math. Comput.* **198** (2008), 754–769.

[7] C. Chen, F. Liu, I. Turner and V. Anh, A Fourier method for the fractional diffusion equation describing sub-diffusion. *J. of Computational Physics* **227** (2007), 886–897.

[8] Z. Deng, V.P. Singh, L. Bengtsson, Numerical solution of fractional advection-dispersion equation. *J. Hydraulic Eng.* **130** (2004), 422–431.

[9] L. Galue, S.L. Kalla, B.N. Al-Saqabi, Fractional extensions of the temperature field problems in oil strata. *Appl. Math. Comput.* **186** (2007), 35–44.

[10] X. Hu, L. Zhang, Implicit compact difference schemes for fractional cable equation. *Appl. Math. Modelling* **36**, No 9 (2012), 4027–4043.

[11] I. Karatay, S.R. Bayramoglu, A. Sahin, Implicit difference approximation for the time fractional heat equation with the nonlocal condition. *Appl. Numerical Mathematics* **61** No 12, (2011), 1281–1288.

[12] I. Karatay, S.R. Bayramoglu, An efficient difference scheme for time fractional advection dispersion equations. *Appl. Math. Sciences* **6**, No 98 (2012), 4869–4878.

[13] I. Karatay, S.R. Bayramoglu, A characteristic difference scheme for time-fractional heat equations based on the Crank-Nicholson difference schemes. *Abstract and Applied Analysis* **2012**, Article ID 548292 (2012), 11 pages; doi: 10.1155/2012/548292.

[14] T.A.M. Langlands, B.I. Henry, The accuracy and stability of an implicit solution method for fractional diffusion equation. *J. of Computational Physics* **205** (2005), 719–736.

[15] X. Li, X. Han, X. Wang, Numerical modeling of viscoelastic flows using equal low-order finite elements. *Comput. Methods Appl. Mech. Engrg.* **199** (2010), 570–581.

[16] X. Li, M. Xu, X. Jiang, Homotopy perturbation method to time-fractional diffusion equation with a moving boundary. *Appl. Math. Comput.* **208** (2009), 434–439.

[17] V.E. Lynch, B.A. Carreras, D. del-Castillo-Negrete, K.M. Ferreira-Mejias, H.R. Hicks, Numerical methods for the solution of partial differential equations of fractional order. *J. Comput. Phys.* **192** (2003), 406–421.

[18] J.A.T. Machado, Discrete-time fractional-order controllers. *Fract. Calc. Appl. Anal.* **4**, No 1 (2001), 47–66.

[19] M.M. Meerschaert, H.-P. Scheffler, C. Tadjeran, Finite difference methods for two-dimensional fractional dispersion equation. *J. of Computational Physics* **211** (2006), 249–261.

[20] M.M. Meerschaert, C. Tadjeran, Finite difference approximations for two-sided space-fractional partial differential equations. *Appl. Numer. Math.* **56**, No 1 (2006), 80–90.

[21] M.M. Meerschaert, C. Tadjeran, Finite difference approximations for fractional advection-dispersion flow equations. *J. Comput. Appl. Math.* **172**, No 1 (2004), 65–77.

[22] S. Momani, Z. Odibat, Analytical solution of a time-fractional Navier–Stokes equation by Adomian decomposition method. *Appl. Math. Comput.* **177**, No 2 (2006), 488–494.

[23] Z. Odibat, S. Momani, Application of variational iteration method to nonlinear differential equations of fractional order. *Int. J. Nonlinear Sci. Numer. Simul.* **1**, No 7 (2006), 15–27.

[24] Z. Odibat, S. Momani, Numerical methods for nonlinear partial differential equations of fractional order. *Applied Mathematical Modelling* **32** (2008), 28–39.

[25] Z. Odibat, S. Momani, Approximate solutions for boundary value problems of time-fractional wave equation. *Appl. Math. Comput.* **181** (2006), 767–774.

[26] I. Podlubny, *Fractional Differential Equations*. Academic Press, New York (1999).

[27] I. Podlubny, *The Laplace Transform Method for Linear Differential Equations of Fractional Order*. Preprint of Slovak Academy of Science, Slovak Republic (1994).

[28] M. Raberto, E. Scalas, F. Mainardi, Waiting-times returns in high frequency financial data: an empirical study. *Physica A* **314** (2002), 749–755.

[29] S.S. Ray, R.K. Bera, Analytical solution of a fractional diffusion equation by Adomian decomposition method. *Appl. Math. Comput.* **174** (2006), 329–336.

[30] E. Scalas, R. Gorenflo, F. Mainardi, Fractional calculus and continuous-time finance. *Physica A* **284**, (2000), 376–384.

[31] L. Su, W. Wang, Z. Yang, Finite difference approximations for the fractional advection–diffusion equation. *Physics Letters A* **373** (2009), 4405–4408.

[32] C. Tadjeran, M.M. Meerschaert, H.-P. Scheffler, A second-order accurate numerical approximation for the fractional diffusion equation. *Journal of Comput. Physics* **213** (2006), 205–213.

[33] S.B. Yuste, L. Acedo, An explicit finite difference method and a new Von Neumann-type stability analysis for fractional diffusion equations. *J. Numer. Anal.* **42**, No 5 (2005), 1862–1874.

[34] Y.N. Zhang, Z.Z. Sun, H.W. Wu, Error estimates of Crank-Nicolson-type difference schemes for the subdiffusion equations. *J. Numer. Anal.* **49**, No 6 (2011), 2302-2322.

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