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RESEARCH PAPER

NONPOLYNOMIAL COLLOCATION APPROXIMATION OF SOLUTIONS TO FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract

We propose a non-polynomial collocation method for solving fractional differential equations. The construction of such a scheme is based on the classical equivalence between certain fractional differential equations and corresponding Volterra integral equations. Usually, we cannot expect the solution of a fractional differential equation to be smooth and this poses a challenge to the convergence analysis of numerical schemes. In this paper, the approach we present takes into account the potential non-regularity of the solution, and so we are able to obtain a result on optimal order of convergence without the need to impose inconvenient smoothness conditions on the solution. An error analysis is provided for the linear case and several examples are presented and discussed.

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1. Introduction

There has been considerable recent attention from the scientific community devoted to the analysis and numerical approximation of fractional differential equations. This follows mainly from the advantages of using

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derivatives of non integer order in the modeling of various phenomena in physics, chemistry, the life sciences, engineering and finance. Because the fractional differential operator is non-local, and derivatives of non-integer order depend on the entire past history of a function from the base time, it has become evident that models for certain processes and materials (especially those possessing memory properties) could be improved if the order of the differential equations used in the models was not limited to integer values. As Miller and Ross point out in their book, [17], there is hardly a field of science or engineering that cannot benefit from this insight. For those who are interested in applications of Fractional Calculus, we also refer to the books [5], [15], [19] and [20].

The theory and numerical analysis of fractional differential equations has also been widely developed in the last decades, see for example [4], [6], [7], [8], [10], [12], [14], [16], [18] and the references therein.

In this paper we consider fractional initial value problems (FIVPs for short) of the form:

$$D^{\alpha}_{*0}y(t) = f(t, y(t)), \quad t \geq 0, \tag{1.1}$$

$$y(0) = y_0, \tag{1.2}$$

where we assume $0 < \alpha < 1$ and the fractional derivative is defined in the Caputo sense, that is, D^{α}_{*0} denotes the Caputo differential operator of order $\alpha \notin \mathbb{N}$, defined by ([3]):

$$D^{\alpha}_{*0}y(t) := D^{\alpha}(y - T[y])(t),$$

where $T[y]$ is the Taylor polynomial of degree $\lfloor\alpha\rfloor$ for y, centered at 0, and D^{α} is the Riemann-Liouville derivative of order α [20]. The latter is defined by $D^{\alpha} := D^{\lceil\alpha\rceil}J^{\lceil\alpha\rceil-\alpha}$, with J^{β} being the Riemann-Liouville integral operator,

$$J^{\beta}y(t) := \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}y(s)ds$$

and $D^{\lceil\alpha\rceil}$ is the classical integer order derivative, where $\lceil\alpha\rceil$ is the smallest integer greater than or equal to α. Analogously, $\lfloor\alpha\rfloor$ denotes the biggest integer smaller than α.

Our approach is to construct a simple collocation method to approximate the solution of (1.1)-(1.2) with an optimal convergence order. The paper is organized in the following way: in Section **2** we recall some useful results on fractional differential equations. In Section **3** we provide a description of the numerical scheme. In Section **4** we provide an error analysis for our method in the linear case. In Section **5**, we illustrate the performance of the proposed method by considering some examples. We will also show that this method can be used on a wide range of fractional differential equations, including those of order $\alpha > 1$, multi-term fractional differential equations and terminal (or boundary) value problems. Finally, we conclude with some details of our plans for future work.

2. Existence, uniqueness and smoothness of the solution

Before we proceed to the description of our numerical approach, we recall some known results on the existence, uniqueness and smoothness of solutions of fractional differential equations.

The first result provides an equivalence between FIVPs of the form (1.1)-(1.2) and Volterra integral equations (see, for example [5]):

LEMMA **2.1**. *If the function f is continuous, the initial value problem (1.1)-(1.2) is equivalent to the following Volterra integral equation of the second kind:*

$$y(t) = y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s)) ds. \tag{2.1}$$

This result is useful not only for the construction of numerical methods, but also to obtain results on the existence and uniqueness of the solution. That was, in fact, the approach used in [8], where the authors proved the following two theorems establishing sufficient conditions for the existence and uniqueness of solutions to FIVPs.

THEOREM **2.1**. *Assume that $D = [0, \chi] \times [y_0 - \eta, y_0 + \eta]$ with some $\chi > 0$ and some $\eta > 0$, and let the function $f : D \to \mathbb{R}$ be continuous. Define $\chi^* = \min \left\{ \chi, \left(\frac{\eta \Gamma(\alpha+1)}{\|f\|_\infty} \right)^{\frac{1}{\alpha}} \right\}$. Then, there exists a function $y : [0, \chi^*] \to \mathbb{R}$ solving (1.1)-(1.2).*

THEOREM **2.2**. *Assume that $D = [0, \chi] \times [y_0 - \eta, y_0 + \eta]$ with some $\chi > 0$ and some $\eta > 0$. Furthermore, let the function $f : D \to \mathbb{R}$ be bounded on D and fulfill a Lipschitz condition with respect to the second variable*

$$|f(t, y) - f(t, z)| \leq L\,|y - z|$$

for some constant $L > 0$ independent of t, y and z. Then, there exists at most one function $y : [0, \chi^*] \to \mathbb{R}$ solving (1.1)-(1.2).

One of the features of fractional differential equations is that we cannot expect a solution to be smooth even if the right-hand side function f is smooth, a property that poses a challenge for the construction of numerical methods with a reasonable order of convergence. From [5] we have the following result concerning the smoothness of the solution of a fractional differential equation:

LEMMA **2.2**. *Assume that the solution of (1.1)-(1.2) exists and is unique on* $[0,T]$, *for a certain* $T>0$. *If* $\alpha=\frac{p}{q}$, *where* $p\geq 1$ *and* $q\geq 2$ *are two relatively prime integers and if the right-hand side function* f *can be written in the form* $f(t,y(t))=\overline{f}(t^{1/q},y(t))$, *where* $\overline{f}$ *is analytic in a neighborhood of* $(0,y_0)$, *then the unique solution of the problem (1.1)-(1.2) can be represented in terms of powers of* $t^{1/q}$: $y(t)=\sum_{i=0}^{\infty}a_i t^{i/q}, t\in[0,r)$, *where the* a_i *are constants.*

This means that assuming that the solution of (1.1)-(1.2) exists and is unique, then we can always write it as a sum $y=y_1+y_2$ where, for a fixed integer m, $y_1\in C^m([0,T])$ and y_2 is the nonsmooth part. Taking into account the above result, we will then follow the approach used in [1], also used in [2], to approximate the solution of (2.1) and, consequently, the solution of the initial value problem (1.1)-(1.2).

3. Numerical approach

Let us consider the finite dimensional space V_m^α, with dimension $\ell=\#V_m^\alpha$, of nonpolynomial functions defined by

$$V_m^\alpha=span\left\{t^{i+j\alpha}\ i,j\in\mathbb{N}_0,\ i+j\alpha<m\right\}. \tag{3.1}$$

In order to simplify the above notation we introduce the index set

$$W_{\alpha,m}=\{i+j\alpha\ i,j\in\mathbb{N}_0,\ i+j\alpha<m\}=\{\nu_k:k=1,...,\ell\}.$$

Using this notation we can write V_m^α as

$$V_m^\alpha=span\left\{t^{\nu_k},\ k=1,...,\ell\right\}. \tag{3.2}$$

Denoting by $\mathcal{P}_m$ the space of polynomials of degree $\leq m-1$, clearly $\mathcal{P}_m\subset V_m^\alpha$. Moreover, note that the space V_m^α contains nonpolynomial functions of the form t^{ν_k}, for some $k\in\{1,...,\ell\}$. Therefore, if we approximate the solution of problem (1.1)-(1.2) with a function spanned by the elements in this space, this may reflect the potential nonregularity of the solution of (1.1)-(1.2).

In order to do that, let us consider a partition of the interval $[0,T]$, the interval where the solution is sought,

$$\Delta_h=\{t_i=i\,h,\quad i=0,1,...,N\}, \tag{3.3}$$

into N subintervals of equal size $h = T/N$, $\sigma_0 = [0, t_1]$ and $\sigma_i = (t_i, t_{i+1}],\ i = 1, 2, \ldots, N-1$. At each one of these subintervals we define ℓ collocation points $t_{ik} = t_i + c_k\, h,\ k = 1, 2, \ldots, \ell$, where $c_k \in [0, 1]$, and the set

$$V^{\alpha}_{h,m} = \{v:\ v|_{\sigma_i} \in V^{\alpha}_{m},\ i = 0, 1, \ldots, N-1\}\,.$$

We will then seek a function $u \in V^{\alpha}_{h,m}$ such that

$$\begin{aligned} u(t_{ik}) &= y_0 + \frac{1}{\Gamma(\alpha)}\int_0^{t_{ik}} (t_{ik}-s)^{\alpha-1} f(s, u(s))ds, \\ & \qquad i = 0, 1, ..., N-1,\ k = 1, 2, ..., \ell. \end{aligned} \tag{3.4}$$

It will be convenient to introduce the following projection operator $P_h : C([0, T]) \to V^{\alpha}_{h,m}$, defined by (see [2]):

$$\left(P_h g\right)(s) = \sum_{k=1}^{\ell} L_{ik}(s) g(t_{ik}), \quad s \in \sigma_i,$$

where the Lagrange functions $L_{ik} \in V^{\alpha}_{m}$, are defined by

$$L_{ik}(s) = \sum_{p=1}^{\ell} \beta^{i}_{pk} t^{\nu_p},\ i = 0, 1, ..., N-1,\ k = 1, ..., \ell, \tag{3.5}$$

and the coefficients $[\beta^{i}_{pk}]_{p=1,\ldots,\ell}$ may be determined by solving the $(\ell \times \ell)$ linear system of equations $L_{ik}(t_{ij}) = \delta_{jk},\ k, j = 1, \ldots, \ell$.

Therefore, by definition, the operator P_h is such that $P_h(g)(t_{ik}) = g(t_{ik})$.

As an approximation of $f(s, u(s))$, on each of the subintervals $\sigma_i,\ i = 0, 1, ..., N-1$, we consider

$$f(s, u(s)) \approx (P_h f)(s) = \sum_{k=1}^{\ell} L_{ik}(s) f(t_{ik}, u(t_{ik})). \tag{3.6}$$

For $s \in [t_i, t_{ik}],\ i = 0, 1, ..., N-1,\ k = 1, ..., \ell$, we use

$$f(s, u(s)) \approx P_h(f)(s) = \sum_{\gamma=1}^{\ell} L^{k}_{i\gamma}(s) f(t_i + hc_\gamma c_k, u(t_i + hc_\gamma c_k)), \tag{3.7}$$

where $L^{k}_{i\gamma},\ i = 0, 1, ..., N-1,\ k = 1, ..., \ell,\ \gamma = 1, ..., \ell$ are the Lagrange functions associated with the points $t_i + hc_\gamma c_k$, defined similarly to (3.5).

Substituting (3.6) and 3.7 in (3.4), we obtain the following approximation of $u(t_{jk})$:

$$\widehat{u}(t_{ik}) = y_0 + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{i-1} \sum_{\gamma=1}^{\ell} \sum_{p=1}^{\ell} \int_{t_j}^{t_{j+1}} (t_{ik}-s)^{\alpha-1} s^{\nu_p} ds \beta^{i}_{\gamma p} f(t_{j\gamma}, \widehat{u}(t_{j\gamma}))$$

$$+ \frac{1}{\Gamma(\alpha)} \sum_{\gamma=1}^{\ell} \sum_{p=1}^{\ell} \int_{t_i}^{t_{ik}} (t_{ik}-s)^{\alpha-1} s^{\nu_p} ds \beta^{ik}_{\gamma p} f(t_i + hc_k c_\gamma, \widehat{u}(t_i + hc_k c_\gamma))$$

$$
\begin{aligned}
&= \; y_0 + \sum_{j=0}^{i-1}\sum_{\gamma=1}^{\ell}\sum_{p=1}^{\ell} w_{ik}^{j,p}\beta_{\gamma p}^{j} f(t_{j\gamma}, \widehat{u}(t_j + hc_\gamma)) \\
&+ \; \sum_{\gamma=1}^{\ell}\sum_{p=1}^{\ell} w_{ik}^{i,p}\beta_{\gamma p}^{ik} f(t_i + hc_k c_\gamma, \widehat{u}(t_i + hc_k c_\gamma)), \\
&\quad\; i = 0, 1, \dots, N-1, \; k = 1, \dots, \ell,
\end{aligned} \tag{3.8}
$$

where

$$
\begin{aligned}
w_{ik}^{j,p} &= \frac{1}{\Gamma(\alpha)} \int_{t_j}^{t_{j+1}} (t_{ik} - s)^{\alpha-1} s^{\nu_p} ds, \quad j < i, \\
w_{ik}^{i,p} &= \frac{1}{\Gamma(\alpha)} \int_{t_i}^{t_{ik}} (t_{ik} - s)^{\alpha-1} s^{\nu_p} ds, \; i = 0, 1, ..., N-1, \; k = 1, ..., \ell.
\end{aligned}
$$

If the right-hand side function f is linear, then we have to solve $N-1$ linear systems of ℓ equations; if not, in order to solve each system, we apply Newton's method using, for example, as an initial approximation the ℓ-dimensional vector $[y_0, \; y_0, \dots, y_0]^T$ for $i = 0$ and $[y_{i-1\;\ell}, \; y_{i-1\;\ell}, \dots, y_{i-1\;\ell}]^T$ for $i = 1, \dots, N-1$, where $y_{jk} = \widehat{u}(t_{jk})$. After solving (3.8) an approximation to the solution of (1.1)-(1.2) will be given by:

$$
y(s) \approx \sum_{k=1}^{\ell} y_{jk} L_{jk}(s), \quad s \in \sigma_j, \; j = 0, 1, ..., N-1.
$$

Remark **3.1**. Since the potential nonsmooth behavior of the solution is present only at the origin (see Lemma **2.2**), we could have used a nonpolynomial approximation only on the first subinterval $[0, t_1]$ (or on the first subintervals, depending on the stepsize h), and on the remaining subintervals a classical polynomial approximation could have been considered. This would obviously result in a lower computational cost since the set of spanning polynomial functions would have lower dimension than the one considered. But this investigation is beyond the scope of this paper.

4. Error Analysis

We consider next the error analysis of our collocation method in the linear case, that is, the case where the right-hand side function f can be written in the form

$$
f(t, y(t)) = \beta y(t) + g(t), \tag{4.1}
$$

for a certain continuous function g and $\beta \in \mathbb{R}$.

If this is the case, it will not be necessary to approximate $f(s,u)$ by an element of the space V_m^α and equations (2.1) and (3.4) become

$$y(t) = y_0 + \frac{\beta}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}g(s)ds, \quad (4.2)$$

$$\begin{aligned} u(t_{ik}) &= y_0 + \frac{\beta}{\Gamma(\alpha)}\int_0^{t_{ik}} (t_{ik}-s)^{\alpha-1}u(s)ds + \frac{1}{\Gamma(\alpha)}\int_0^{t_{ik}} (t_{ik}-s)^{\alpha-1}g(s)ds \\ &= y_0 + \frac{1}{\Gamma(\alpha)}\int_0^{t_{ik}} (t_{ik}-s)^{\alpha-1}g(s)ds \\ &+ \frac{\beta}{\Gamma(\alpha)}\left(\sum_{j=0}^{i-1}\int_{t_j}^{t_{j+1}} (t_{ik}-s)^{\alpha-1}u(s)ds + \int_{t_i}^{t_{ik}} (t_{ik}-s)^{\alpha-1}u(s)ds\right). \end{aligned} \quad (4.3)$$

Since $u \in V_m^\alpha$ equation (4.3) can be rewritten as

$$\begin{aligned} u(t_{ik}) &= y_0 + \frac{1}{\Gamma(\alpha)}\int_0^{t_{ik}} (t_{ik}-s)^{\alpha-1}g(s)ds \\ &+ \beta\sum_{j=0}^{i-1}\sum_{\gamma=1}^{\ell}\left(\sum_{p=1}^{\ell} w_{ik}^{j,p}\beta_{\gamma p}^{j}\right)u(t_{j\gamma}) + \sum_{\gamma=1}^{\ell}\left(\sum_{p=1}^{\ell} w_{ik}^{i,p}\beta_{\gamma p}^{i}\right)u(t_{i\gamma}), \\ & i = 0,1,...,N-1, \ \ k = 1,2,...,\ell. \end{aligned} \quad (4.4)$$

In order to obtain a global convergence result, we analyze the error,

$$e(t) = y(t) - u(t),$$

at the collocation points $t = t_{ik}$, $i = 0,\dots,N-1$, $k = 1,\dots,\ell$. For convenience, we also define the vectors:

$$\begin{aligned} \mathbf{e}_i &= \left[e\left(t_{i1}\right)\dots e\left(t_{i\ell}\right)\right]^T \\ &= \left[e\left(t_{ik}\right)\right]_{k=1,\dots,\ell}^T, \quad i = 0,1,...,N-1. \end{aligned}$$

For the error analysis we will need the following two auxiliary lemmas from [2] and [11], respectively:

LEMMA **4.1**. *Let L_{ik} be the Lagrange functions defined by (3.5). There exists a positive constant c such that*

$$\|L_{ik}\|_{\sigma_i} \le c, \quad k = 1,\dots,\ell. \quad (4.5)$$

Furthermore, for $f = f_1 + f_2$, where $f_1 \in C^m([0,T])$ and $f_2 \in V_m^\alpha$, we have

$$\|f - P_h f\|_{\sigma_i} \le \bar{c}h^m \left\|f_1^{(m)}\right\|_{\sigma_i}, \quad (4.6)$$

for some positive constant $\bar{c}$.

The following auxiliary result is a singular discrete Gronwall inequality, proved in [11]:

LEMMA **4.2**. *Let x_i, $0 \le i \le N$, be a sequence of non-negative real numbers satisfying*

$$x_i \le \psi_i + Mh^{1-\beta} \sum_{j=0}^{i-1} \frac{x_j}{(i-j)^{\beta}}, \quad 0 \le i \le N, \tag{4.7}$$

where $0 < \beta < 1$, $M > 0$ is bounded independently of h, and ψ_i, $0 \le i \le N$, is a monotonic increasing sequence of non-negative real numbers. Then

$$x_i \le \psi_i \, E_{1-\beta}\left(M\Gamma(1-\beta)(ih)^{1-\beta}\right), \quad 0 \le i \le N,$$

where $E_\alpha(x) = \sum_{i=0}^{\infty} \frac{x^k}{\Gamma(\alpha k+1)}$ denotes the Mittag-Leffler function of order α.

In what follows, our concern will be to obtain an upper bound for $\|\mathbf{e}_i\|_\infty$, $i = 0, 1, ..., N$. From (4.2) and (4.4), we have:

$$e(t_{ik}) = \frac{\beta}{\Gamma(\alpha)} \left(\sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} \frac{(y(s)-u(s))}{(t_{ik}-s)^{1-\alpha}} ds + \int_{t_i}^{t_{ik}} \frac{(y(s)-u(s))}{(t_{ik}-s)^{1-\alpha}} ds \right), \tag{4.8}$$

and then

$$\begin{aligned} |e(t_{ik})| \quad &\le \quad \frac{|\beta|}{\Gamma(\alpha)} \left(\sum_{j=0}^{i-1} \|y-u\|_{\sigma_j} \int_{t_j}^{t_{j+1}} (t_{ik}-s)^{\alpha-1} \, ds \right. \\ &+ \quad \left. \|y-u\|_{\sigma_i} \int_{t_i}^{t_{ik}} (t_{ik}-s)^{\alpha-1} \, ds \right). \end{aligned} \tag{4.9}$$

Next, we obtain an upper bound for $\|y-u\|_{\sigma_i}$, $i = 0, 1, ..., N-1$.

LEMMA **4.3**. *Let y be the unique solution of equation (1.1)-(1.2), with the forcing function f defined by (4.1), and let $u \in V_m^\alpha$ be the approximate solution of y obtained by the nonpolynomial collocation method (4.4). Then*

$$\|y-u\|_{\sigma_i} \le C_1 h^m + C_2 \|\mathbf{e}_i\|_\infty, \quad i = 0, 1, ..., N-1, \tag{4.10}$$

where C_1 and C_2 are positive constants that do not depend on N.

P r o o f. First, note that

$$\|y-u\|_{\sigma_i} \le \|y - P_h y\|_{\sigma_i} + \|P_h y - u\|_{\sigma_i}, \quad i = 0, 1, ..., N-1. \tag{4.11}$$

From Lemma **2.2**, we assume that for a fixed $m \in \mathbb{N}$, the solution of (1.1)-(1.2) can be written as $y = y_1 + y_2$ where $y_1 \in C^m([0,T])$ and $y_2 \in V_m^\alpha$. Therefore taking Lemma **4.1** into account, we conclude that

$$\|y - P_h y\|_{\sigma_i} \leq \bar{c} h^m \left\| y_1^{(m)} \right\|_{\sigma_i} \leq C_1 h^m, \quad i = 0, 1, ..., N-1, \tag{4.12}$$

where C_1 is a positive constant given by $C_1 = \bar{c} \max_{s \in [0,T]} \left\| y_1^{(m)}(s) \right\|$.

On the other hand, $u \in V_m^\alpha$ which implies $u = P_h u$ and then

$$\begin{aligned} \|P_h y - u\|_{\sigma_i} &= \|P_h(y-u)\|_{\sigma_i} = \left\| \sum_{j=1}^{\ell} L_{ij}(s) \left(y\left(t_{ij}\right) - u\left(t_{ij}\right) \right) \right\|_{\sigma_i} \\ &\leq \max_{j \in \{1,\dots,\ell\}} \|L_{ij}\| \sum_{j=1}^{\ell} \left| y\left(t_{ij}\right) - u\left(t_{ij}\right) \right| \\ &\leq c \sum_{j=1}^{\ell} \left| e\left(t_{ij}\right) \right| \leq C_2 \|\mathbf{e}_i\|_\infty, \end{aligned} \tag{4.13}$$

with c defined on Lemma **4.1**, (4.5).

Hence, the result (4.10) follows immediately from (4.11), (4.12) and (4.13). □

Using the result (4.10) from Lemma **4.3** on (4.9), we obtain

$$\begin{aligned} |e\left(t_{ik}\right)| \leq \frac{|\beta|}{\Gamma(\alpha)} \Bigg(& C_1 h^m \int_0^{t_{ik}} \left(t_{ik} - s\right)^{\alpha-1} ds \\ + & C_2 \sum_{j=0}^{i-1} \|\mathbf{e}_j\|_\infty \int_{t_j}^{t_{j+1}} \left(t_{ik} - s\right)^{\alpha-1} ds \\ & + C_2 \|\mathbf{e}_i\|_\infty \int_{t_i}^{t_{ik}} \left(t_{ik} - s\right)^{\alpha-1} ds \Bigg). \end{aligned} \tag{4.14}$$

It is straightforward to prove that

$$\begin{aligned} & \int_{t_j}^{t_{j+1}} \left(t_{ik} - s\right)^{\alpha-1} ds \leq D h^\alpha (i-j)^{\alpha-1}, \\ & \quad j < i, \ i = 0, 1, ..., N-1, \ k = 1, ..., \ell, \end{aligned} \tag{4.15}$$

for some positive constant D.

Therefore, computing the integrals and using the estimate (4.15) in (4.14) it follows that

$$\begin{aligned} |e\,(t_{ik})| &\leq \frac{|\beta|}{\Gamma(\alpha)}\left(C_1\frac{t_{ik}^{\alpha}}{\alpha}h^m + C_3h^{\alpha}\sum_{j=0}^{i-1}(i-j)^{\alpha-1}\|\mathbf{e}_j\|_\infty + C_2\frac{c_k^{\alpha}}{\alpha}h^{\alpha}\|\mathbf{e}_i\|_\infty\right) \\ &\leq C_4h^m + C_5h^{\alpha}\sum_{j=0}^{i-1}\frac{\|\mathbf{e}_j\|_\infty}{(i-j)^{1-\alpha}} + \frac{C_2}{\alpha}h^{\alpha}\|\mathbf{e}_i\|_\infty, \end{aligned}$$

where $C_4 = \dfrac{|\beta|\,C_1T^{\alpha}}{\alpha\Gamma(\alpha)},\ C_5 = \dfrac{C_3\,|\beta|}{\Gamma(\alpha)} = \dfrac{C_2\,D\,|\beta|}{\Gamma(\alpha)}$.

Under the condition $\left(1 - \dfrac{C_2}{\alpha}h^{\alpha}\right) > 0$, which certainly holds for sufficiently small $h > 0$, we conclude that

$$\|\mathbf{e}_i\|_\infty \leq C_6h^m + \frac{C_7}{\Gamma(\alpha)}h^{\alpha}\sum_{j=0}^{i-1}\frac{\|\mathbf{e}_j\|_\infty}{(i-j)^{1-\alpha}},$$

for some positive constants $C_6,\ C_7$ that do not depend on N.

The weakly singular discrete Gronwall inequality (4.7), with $\beta = 1-\alpha$, leads to the following result

$$\|\mathbf{e}_i\|_\infty \leq C_6h^m\,E_\alpha\left(C_7(ih)^{\alpha}\right), \quad i = 0,1,...,N.$$

Thus, from the above inequality we conclude that, for sufficiently small $h > 0$, there exists a positive constant C that does not depend on N such that

$$\|\mathbf{e}_i\|_\infty \leq Ch^m, \quad i = 0,1,...,N. \tag{4.16}$$

We are now able to draw the following conclusion about the optimal order of convergence of the nonpolynomial collocation method applied to fractional initial value problems with f given by (4.1).

THEOREM **4.1**. *Let y be the solution of problem (1.1)-(1.2), with f given by (4.1), and u the approximate solution obtained with the nonpolynomial collocation method defined on a uniform mesh of stepsize h. Then, for sufficiently small h, there exists a positive constant C independent of h and of the choice of the collocation parameters, such that*

$$\|y-u\| \leq Ch^m. \tag{4.17}$$

P r o o f. The result (4.17) follows from Lemma **4.3** and (4.16). □

5. Numerical results

In this section we illustrate the theoretical results and the performance of the proposed collocation method using some examples. Throughout this section we will use the following notation:

$$\varepsilon_h = \max_{j=1,\ldots,N} \left| y(t_j) - y_N(t_j) \right|, \quad N = 1/h, \tag{5.1}$$

$$\epsilon_h = \max_{j=0,\ldots,N-1} \max_{k=1,\ldots,\ell} \left| y(t_{jk}) - y_N(t_{jk}) \right|, \quad N = 1/h, \tag{5.2}$$

$$p = \log\left(\frac{\varepsilon_h}{\varepsilon_{h/2}}\right) / \log(2), \tag{5.3}$$

$$q = \log\left(\frac{\epsilon_h}{\epsilon_{h/2}}\right) / \log(2), \tag{5.4}$$

where y_N denotes an approximation of the solution y, obtained by the nonpolynomial collocation method.

As a first example, we consider the following simple linear problem:

Example 1.

$$\begin{aligned} D_{*0}^{\frac{1}{2}} y(t) &= \frac{1}{2} y(t), \quad t > 0, \\ y(0) &= 1, \end{aligned} \tag{5.5}$$

whose analytical solution is $y(t) = E_{1/2}(0.5\sqrt{t}) = \sum_{k=0}^{\infty} 0.5^k \frac{t^{k/2}}{\Gamma(\frac{k}{2}+1)}$.

In Table 1 we list the maximum of the errors at the mesh points (ε_h) and the experimental rate of convergence p, and in Table 2 we present the maximum of the errors at the collocation points (ϵ_h) and the respective experimental rate of convergence, q.

The numerical results for the initial value problem (5.5) were obtained by application of the method proposed in Section **3** on the spaces $V_1^{1/2}$ ($\#V_1^{1/2} = 2$), $V_2^{1/2}$ ($\#V_2^{1/2} = 4$) and $V_3^{1/2}$ ($\#V_3^{1/2} = 6$), where the following collocation parameters

$c_1 = 0.25,\ c_2 = 0.75,$

$c_1 = 0.15,\ c_2 = 0.25,\ c_3 = 0.5,\ c_4 = 0.75,$

$c_1 = 0.05,\ c_2 = 0.15,\ c_3 = 0.25,\ c_4 = 0.5,\ c_5 = 0.75,\ c_6 = 0.9,$

were considered. In Figure 1 we present the graphs of the absolute error, considering a mesh with stepsize $h = 1/160$.

In this example, none of the collocation points is coincident with a mesh point. In this case, and in most of the numerical experiments that we have

h	Collocation in $V_{h,1}^{\alpha}$ ε_h	p	Collocation in $V_{h,2}^{\alpha}$ ε_h	p	Collocation in $V_{h,3}^{\alpha}$ ε_h	p
0.2	$5.004 \cdot 10^{-3}$	–	$2.437 \cdot 10^{-5}$	–	$1.754 \cdot 10^{-8}$	–
0.1	$2.210 \cdot 10^{-3}$	1.18	$5.261 \cdot 10^{-6}$	2.21	$1.846 \cdot 10^{-9}$	3.25
0.05	$1.015 \cdot 10^{-3}$	1.12	$1.189 \cdot 10^{-6}$	2.15	$2.051 \cdot 10^{-10}$	3.17
0.025	$4.791 \cdot 10^{-4}$	1.08	$2.770 \cdot 10^{-7}$	2.10	$2.362 \cdot 10^{-11}$	3.12
0.0125	$2.301 \cdot 10^{-4}$	1.06	$6.594 \cdot 10^{-8}$	2.07	$2.788 \cdot 10^{-12}$	3.08
0.00625	$1.119 \cdot 10^{-4}$	1.04	$1.593 \cdot 10^{-8}$	2.05	$3.462 \cdot 10^{-13}$	3.06

TABLE 1. Error norms and rates for collocation in $V_{h,m}^{\alpha}$ at the mesh points, $m = 1,\ 2,\ 3$, applied to Example 1 (FIVP (5.5)).

h	Collocation in $V_{h,1}^{\alpha}$ ϵ_h	q	Collocation in $V_{h,2}^{\alpha}$ ϵ_h	q	Collocation in $V_{h,3}^{\alpha}$ ϵ_h	q
0.2	$6.267 \cdot 10^{-4}$	–	$1.764 \cdot 10^{-6}$	–	$3.860 \cdot 10^{-9}$	–
0.1	$1.933 \cdot 10^{-4}$	1.70	$2.673 \cdot 10^{-7}$	2.72	$2.903 \cdot 10^{-10}$	3.73
0.05	$6.218 \cdot 10^{-5}$	1.64	$4.245 \cdot 10^{-8}$	2.65	$2.293 \cdot 10^{-11}$	3.66
0.025	$2.059 \cdot 10^{-5}$	1.59	$6.961 \cdot 10^{-9}$	2.61	$1.873 \cdot 10^{-12}$	3.61
0.0125	$6.952 \cdot 10^{-6}$	1.57	$1.167 \cdot 10^{-9}$	2.58	$1.565 \cdot 10^{-13}$	3.58
0.00625	$2.380 \cdot 10^{-6}$	1.55	$1.989 \cdot 10^{-10}$	2.55	$1.354 \cdot 10^{-14}$	3.53

TABLE 2. Error norms and rates for collocation in $V_{h,m}^{\alpha}$ at the collocation points, $m = 1,\ 2,\ 3$, applied to Example 1 (FIVP (5.5)).

carried out, the errors at the collocation points appear to converge to zero with a higher order than the errors at the mesh points. Therefore in the next example we explore what happens if some of the collocation points coincide with the mesh points.

As a second example we consider the following linear problem:

EXAMPLE 2.

$$\begin{aligned} D_{*0}^{\alpha} y(t) &= \frac{2}{\Gamma(3-\alpha)} t^{2-\alpha} - \frac{1}{\Gamma(2-\alpha)} t^{1-\alpha} + t^2 - t - y(t), \quad t > 0, \\ y(0) &= 0, \end{aligned} \tag{5.6}$$

whose analytical solution is $y(t) = t^2 - t$. Note that in this example $D_{*0}^{\alpha} y$ is not a smooth function.

The numerical results presented in Tables 3 and 4 illustrate the performance of the proposed method when applied to the problem (5.6), with

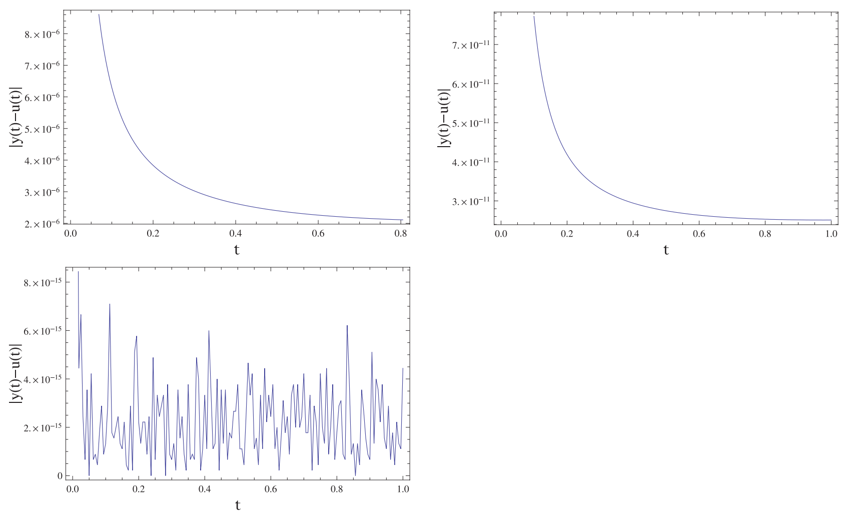

FIGURE 1. Example 1 (FIVP (5.5)) - Absolute error for the collocation method in the spaces $V^{\alpha}_{1/160,m}$ with $m = 1,\ 2,\ 3$. Left top: $m = 1$, Right top: $m = 2$ and Bottom: $m = 3$.

$\alpha = 1/4,\ \alpha = 1/3$ and $\alpha = 1/2$, in the case $m = 1$. The collocation parameters that we have used are the Radau II points, for the several values of α:

$$c_1 = \frac{1}{3},\ c_2 = 1, \quad \text{for } \alpha = 1/2;$$
$$c_1 = \frac{1}{10}\left(4-\sqrt{6}\right),\ c_2 = \frac{1}{10}\left(4+\sqrt{6}\right),\ c_3 = 1, \quad \text{for } \alpha = 1/3;$$
$$c_1 = 0.088588,\ c_2 = 0.409467,\ c_3 = 0.787659,\ c_4 = 1, \quad \text{for } \alpha = 1/4.$$

h	$\alpha = 1/4$ ϵ_h	p	$\alpha = 1/3$ ϵ_h	p	$\alpha = 1/2$ ϵ_h	p
0.1	$4.130 \cdot 10^{-4}$	–	$5.739 \cdot 10^{-4}$	–	$1.211 \cdot 10^{-3}$	–
0.05	$1.094 \cdot 10^{-4}$	1.92	$1.530 \cdot 10^{-4}$	1.91	$3.236 \cdot 10^{-4}$	1.90
0.025	$2.877 \cdot 10^{-5}$	1.93	$4.034 \cdot 10^{-5}$	1.92	$8.501 \cdot 10^{-5}$	1.93
0.0125	$7.520 \cdot 10^{-6}$	1.94	$1.054 \cdot 10^{-5}$	1.94	$2.203 \cdot 10^{-5}$	1.95
0.00625	$1.954 \cdot 10^{-6}$	1.94	$2.732 \cdot 10^{-6}$	1.95	$5.655 \cdot 10^{-6}$	1.96

TABLE 3. Error norms and rates for collocation in $V^{\alpha}_{h,1}$, at the collocation points, applied to Example 2 (FIVP (5.6)).

The next example is a nonlinear one.

h	$\alpha = 1/4$		$\alpha = 1/3$		$\alpha = 1/2$	
	ε_h	p	ε_h	p	ε_h	p
0.1	$2.507 \cdot 10^{-5}$	–	$7.891 \cdot 10^{-4}$	–	$1.998 \cdot 10^{-4}$	–
0.05	$7.107 \cdot 10^{-5}$	1.82	$2.182 \cdot 10^{-4}$	1.84	$4.033 \cdot 10^{-5}$	2.26
0.025	$1.987 \cdot 10^{-6}$	1.84	$5.959 \cdot 10^{-6}$	1.87	$7.972 \cdot 10^{-6}$	2.31
0.0125	$5.479 \cdot 10^{-7}$	1.86	$1.607 \cdot 10^{-6}$	1.89	$1.558 \cdot 10^{-6}$	2.34
0.00625	$1.493 \cdot 10^{-7}$	1.88	$4.282 \cdot 10^{-7}$	1.91	$3.035 \cdot 10^{-7}$	2.36

TABLE 4. Error norms and rates for collocation in $V^{\alpha}_{h,1}$, at the mesh points, applied to Example 2 (FIVP (5.6)).

EXAMPLE 3.

$$D^{\alpha}_{*0}y(t) = \frac{40320}{\Gamma(9-\alpha)}t^{8-\alpha} - 3\frac{\Gamma(5+\frac{\alpha}{2})}{\Gamma(5-\frac{\alpha}{2})}t^{4-\frac{\alpha}{2}} + \left(\frac{3}{2}t^{\frac{\alpha}{2}} - t^4\right)^3 - (y(t))^{3/2}, \quad t > 0,$$
$$y(0) = 0. \tag{5.7}$$

The analytical solution is $y(t) = t^8 - 3t^{4+\alpha/2} + \frac{9}{4}t^{\alpha}$.

h	$\alpha = 1/3$		$\alpha = 1/2$		$\alpha = 2/3$	
	ϵ_h	ε_h	ϵ_h	ε_h	ϵ_h	ε_h
0.2	$3.099 \cdot 10^{-4}$	$1.833 \cdot 10^{-5}$	$1.876 \cdot 10^{-4}$	$4.345 \cdot 10^{-4}$	$4.628 \cdot 10^{-5}$	$1.267 \cdot 10^{-4}$
0.1	$1.909 \cdot 10^{-5}$	$1.202 \cdot 10^{-6}$	$1.170 \cdot 10^{-5}$	$4.143 \cdot 10^{-5}$	$2.364 \cdot 10^{-6}$	$5.622 \cdot 10^{-6}$
0.05	$1.143 \cdot 10^{-6}$	$7.753 \cdot 10^{-8}$	$6.682 \cdot 10^{-7}$	$3.232 \cdot 10^{-6}$	$1.189 \cdot 10^{-7}$	$2.093 \cdot 10^{-7}$
0.025	$6.717 \cdot 10^{-7}$	$4.897 \cdot 10^{-9}$	$3.602 \cdot 10^{-8}$	$2.286 \cdot 10^{-7}$	$5.937 \cdot 10^{-9}$	$7.176 \cdot 10^{-9}$
0.0125	$3.889 \cdot 10^{-9}$	$3.020 \cdot 10^{-10}$	$1.915 \cdot 10^{-9}$	$1.536 \cdot 10^{-8}$	$2.955 \cdot 10^{-10}$	$2.360 \cdot 10^{-10}$
0.2						
0.1	4.02	3.93	4.00	3.39	4.29	4.49
0.05	4.06	3.95	4.13	3.68	4.32	4.75
0.025	4.09	3.98	4.21	3.82	4.33	4.87
0.0125	4.11	4.02	4.23	3.90	4.33	4.93

TABLE 5. Error norms and rates for collocation in $V^{\alpha}_{h,2}$, at the collocation points and mesh points, applied to Example 3 (FIVP (5.7)).

Next we remark that the proposed collocation method can also be used on multi-term initial value problems or on initial value problems with order greater than one. In each of these cases, we just have to transform our original equation into an equivalent system of fractional differential equations with order smaller than one (see [9] for details). Let us the consider the following example.

h	$\alpha = 1/3$		$\alpha = 1/2$		$\alpha = 2/3$	
	ϵ_h	ε_h	ϵ_h	ε_h	ϵ_h	ε_h
0.2	$1950 \cdot 10^{-5}$	$7.099 \cdot 10^{-7}$	$2.486 \cdot 10^{-5}$	$7.476 \cdot 10^{-6}$	$3.538 \cdot 10^{-6}$	$1.038 \cdot 10^{-6}$
0.1	$1.370 \cdot 10^{-6}$	$6.328 \cdot 10^{-8}$	$1.394 \cdot 10^{-6}$	$3.291 \cdot 10^{-7}$	$2.070 \cdot 10^{-7}$	$6.847 \cdot 10^{-8}$
0.05	$8.269 \cdot 10^{-8}$	$4.623 \cdot 10^{-9}$	$7.570 \cdot 10^{-8}$	$1.823 \cdot 10^{-8}$	$1.048 \cdot 10^{-8}$	$3.762 \cdot 10^{-9}$
0.025	$4.857 \cdot 10^{-9}$	$3.151 \cdot 10^{-10}$	$4.056 \cdot 10^{-9}$	$9.949 \cdot 10^{-10}$	$5.225 \cdot 10^{-10}$	$1.961 \cdot 10^{-10}$
0.0125	$2.810 \cdot 10^{-10}$	$2.040 \cdot 10^{-11}$	$2.157 \cdot 10^{-10}$	$5.362 \cdot 10^{-11}$	$2.598 \cdot 10^{-11}$	$9.973 \cdot 10^{-12}$
0.2						
0.1	3.83	3.49	4.16	4.51	4.10	3.92
0.05	4.05	3.77	4.20	4.17	4.30	4.19
0.025	4.09	3.88	4.22	4.20	4.33	4.26
0.0125	4.11	3.95	4.23	4.21	4.33	4.30

TABLE 6. Error norms and rates for collocation in $V^{\alpha}_{h,3}$, at the collocation points and mesh points, applied to Example 3 (FIVP (5.7)).

EXAMPLE 4.

$$\begin{aligned} D^2 y(t) + D^{0.5} y(t) + y(t) &= t^3 + 6t + \frac{3.2 t^{2.5}}{\Gamma(0.5)}, \quad t > 0, \\ y(0) = 0, y'(0) &= 0, \end{aligned}$$

whose analytical solution is known and given by $y(t) = t^3$.

First, we convert this problem into the equivalent linear system of equations

$$\begin{cases} D^{0.5} y_1(t) = y_2(t) \\ D^{0.5} y_2(t) = y_3(t) \\ D^{0.5} y_3(t) = y_4(t) \\ D^{0.5} y_4(t) = -y_1(t) - y_2(t) + t^3 + 6t + \dfrac{3.2 t^{2.5}}{\Gamma(0.5)} \end{cases}, \tag{5.8}$$

together with the conditions

$$y_1(0) = 0, \quad y_2(0) = 0, \quad y_3(0) = 0, \quad y_4(0) = 0, \tag{5.9}$$

then rewrite this system of fractional equations as a system of integral equations and apply the proposed method to the system of integral equations. The numerical results that we have obtained are shown on Table 7.

Finally, we remark that this collocation method can also be combined with a shooting algorithm in order to approximate the solution of terminal (or boundary) value problems of the form

$$\begin{aligned} D^{\alpha}_{*0} y(t) &= f(t, y(t)), \ t > 0, \\ y(a) &= y_a, \end{aligned}$$

with $a > 0$.

h	Collocation in $V^{\alpha}_{h,1}$				Collocation in $V^{\alpha}_{h,2}$			
	ϵ_h	q	ε_h	p	ϵ_h	q	ε_h	p
0.2	$5.623 \cdot 10^{-2}$	–	$6.090 \cdot 10^{-2}$	–	$3.280 \cdot 10^{-5}$	–	$1.259 \cdot 10^{-4}$	–
0.1	$1.548 \cdot 10^{-2}$	1.86	$1.675 \cdot 10^{-2}$	1.86	$4.041 \cdot 10^{-6}$	3.02	$1.575 \cdot 10^{-5}$	3.0
0.05	$4.173 \cdot 10^{-3}$	1.89	$4.502 \cdot 10^{-3}$	1.90	$5.030 \cdot 10^{-7}$	3.01	$1.969 \cdot 10^{-6}$	3.0
0.025	$1.112 \cdot 10^{-3}$	1.91	$1.196 \cdot 10^{-3}$	1.91	$6.279 \cdot 10^{-8}$	3.00	$2.461 \cdot 10^{-7}$	3.0
0.0125	$2.940 \cdot 10^{-4}$	1.92	$3.152 \cdot 10^{-4}$	1.92	$7.845 \cdot 10^{-9}$	3.00	$3.077 \cdot 10^{-8}$	3.0

TABLE 7. Error norms and rates for collocation in $V^{\alpha}_{h,m}$, for $m = 1,\ 2$, at the collocation points and mesh points, applied to the system of fractional differential equations (5.8)-(5.9).

As has been done in [14], first we consider the initial value problem

$$\begin{aligned} D^{\alpha}_{*0}y(t) &= f(t, y(t)),\ t > 0 \\ y(0) &= y_0, \end{aligned}$$

for a certain value of y_0. Next we determine its numerical solution using the collocation method. Then we use an iterative scheme (the bisection method, for example) to find the appropriate y_0, for which the solution of the initial value problem passes through the point (a, y_a).

6. Conclusions and future work

In this paper we have described a nonpolynomial collocation method to approximate the solution of fractional differential equations. The construction of the method was based on the equivalence between fractional initial value problems and Volterra integral equations. We have provided an error analysis of the method in the linear case, and we have presented some numerical results that are in agreement with the theoretically obtained estimates. For future work, we intend to propose a nonpolynomial collocation method for terminal boundary problems, avoiding the combination of this nonpolynomial method with a shooting algorithm. We expect in this work to use the equivalence between terminal value problems and Fredholm integral equations (see [14]).

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