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RESEARCH PAPER

A CHARACTERISTIC OF FRACTIONAL RESOLVENTS

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Abstract

In this paper we define and develop a theory of the Riemann-Liouville fractional semigroups and show that they are equivalent to the fractional resolvents of [K. Li, J. Peng, Applied Mathematics Letters, 25 (2012), 808–812].

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1. Introduction and Preliminaries

Due to the applications in physics, chemistry, materials and engineering, the fractional linear systems and the evolution differential equations have attracted an increasing interest in the last few decades. The derivatives and integrals of fractional order are more suitable for describing the properties of various real materials and phenomena, and their theory has gained recently a rapid development (see e.g. Anh and Leonenko [1], Eidelman and Kochubei [3], Hilfer [5], Kilbas et al. [7], Niu and Xie [9], Podlubny [11]).

Assume that X is a Banach space and $A : D(A) \subset X \to X$ is a closed linear operator. Generally, for $0 < \alpha < 1$, there are two types of Cauchy problems for fractional differential equations that are commonly considered.

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1) Abstract Cauchy problem with the Caputo fractional derivative:

$$\begin{cases} {}^C D_t^\alpha u(t) = Au(t), \quad t > 0, \\ u(0) = x \in X, \end{cases} \tag{1.1}$$

where ${}^C D_t^\alpha$ is the Caputo fractional differential operator defined for $0 < \alpha < 1$ as follows:

$${}^C D_t^\alpha u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\sigma)^{-\alpha} u'(\sigma) d\sigma.$$

2) Abstract Cauchy problem with the Riemann-Liouville fractional derivative:

$$\begin{cases} D_t^\alpha u(t) = Au(t), \\ (g_{1-\alpha} * u)(0) = \lim_{s\to 0^+} \displaystyle\int_0^s \frac{(s-\sigma)^{-\alpha}}{\Gamma(1-\alpha)} u(\sigma) d\sigma = x \in X, \end{cases} \tag{1.2}$$

where D_t^α is the Riemann-Liouville fractional differential operator defined for $0 < \alpha < 1$ by

$$D_t^\alpha u(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\sigma)^{-\alpha} u(\sigma) d\sigma.$$

Let us mention that for some applications, Hilfer [5] studied Riemann-Liouville fractional differential equations of the following form

$$D_t^\alpha f(r,t) = C_\alpha \Delta f(r,t) \tag{1.3}$$

with initial conditions given as fractional integrals $g_{1-\alpha}(t) * f(r,t)|_{t\to 0^+}$, where $f(r,t)$ is unknown field, C_α is the constant of fractional diffusion. The solution of (1.3) is obtained in terms of H-functions.

The practical problems usually require definitions of fractional derivatives allowing the utilization of physically interpretable initial conditions. So is the case of the Caputo fractional derivative when the initial conditions are expressed in terms of integer order derivatives. However, the Riemann-Liouville derivatives are also suitable for modeling some real phenomena. For example, in the field of viscoelasticity, Heymans and Podlubny [4] demonstrated that it is possible to attribute physical meaning to the initial conditions expressed in terms of the Riemann-Liouville fractional derivatives, and it is possible to obtain the initial values for such conditions by appropriate measurements.

Recently, Li and Peng [8] defined the notion of fractional resolvents as follows.

Definition **1.1**. ([8]) Let $0 < \alpha < 1$. A family $\{T(t)\}_{t>0}$ of bounded linear operators on Banach space X is called an α-order fractional resolvent, if it satisfies the following assumptions:

(P1) for any $x \in X$, $T(\cdot)x \in C((0,\infty),X)$, and

$$\lim_{t\to 0+} \Gamma(\alpha)t^{1-\alpha}T(t)x = x \quad \text{for all} \quad x \in X; \tag{1.4}$$

(P2) $T(s)T(t) = T(t)T(s)$ for all $t, s > 0$;
(P3) for all $t, s > 0$, there holds

$$T(t)J_s^\alpha T(s) - J_t^\alpha T(t)T(s) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} J_s^\alpha T(s) - \frac{s^{\alpha-1}}{\Gamma(\alpha)} J_t^\alpha T(t), \tag{1.5}$$

where $J_t^\alpha f(t)$ denotes the Riemann-Liouville integral of order α:

$$J_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\sigma)^{\alpha-1} f(t)dt.$$

Remark **1.1**. The reason why we do not consider the value of $T(\cdot)$ at 0 is that the function $t \mapsto \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ has a singularity at 0.

Remark **1.2**. Denote $[0,\infty)$ by R^+. By (P1) of Definition **1.1**, it follows that $T(\cdot)x \in L^1_{loc}(R^+, X)$, for any $x \in X$. In fact, (1.4) implies that there exists a constant $\delta > 0$, such that

$$\|T(t)x\| \le \frac{3}{2}\frac{t^{\alpha-1}}{\Gamma(\alpha)}\|x\|, \quad \forall\ 0 < t < \delta.$$

Thus, for any $\tau_0 > 0$, we have,

$$\int_0^{\tau_0} \|T(t)x\|dt \le \frac{3}{2}\int_0^{\tau_0} \frac{t^{\alpha-1}}{\Gamma(\alpha)}dt\|x\| = \frac{3}{2}\frac{\tau_0^\alpha}{\Gamma(\alpha+1)}\|x\|, \text{ if } \tau_0 \le \delta,$$

and

$$\begin{aligned}\int_0^{\tau_0} \|T(t)x\|dt &= \int_0^{\delta} \|T(t)x\|dt + \int_\delta^{\tau_0} \|T(t)x\|dt \\ &\le \frac{3}{2}\int_0^{\delta} \frac{t^{\alpha-1}}{\Gamma(\alpha)}dt\|x\| + \int_\delta^{\tau_0} \|T(t)x\|dt \\ &\le \frac{3}{2}\frac{\delta^\alpha}{\Gamma(\alpha+1)}\|x\| + \int_\delta^{\tau_0} \|T(t)x\|dt, \text{ if } \tau_0 > \delta.\end{aligned}$$

Therefore, $T(\cdot)x \in L^1_{loc}(R^+, X)$. This property implies that the integrals in (1.5) and (2.2) make sense.

The linear operator A defined by

$$D(A) = \{x \in X : \lim_{t\to 0^+} \frac{t^{1-\alpha}T(t)x - \frac{1}{\Gamma(\alpha)}x}{t^{2\alpha}} \ \text{ exists}\}$$

and

$$Ax = \lim_{t\to 0^+} \frac{t^{1-\alpha}T(t)x - \frac{1}{\Gamma(\alpha)}x}{t^{2\alpha}} \quad \text{for} \;\; x \in D(A)$$

is the generator of the α-order fractional resolvent $\{T(t)\}_{t>0}$, $D(A)$ is the domain of A.

In [8], Li and Peng related the fractional resolvent to the Riemann-Liouville fractional order abstract Cauchy problem (1.2) with

$$\lim_{t\to 0^+} \Gamma(\alpha)t^{1-\alpha}u(t) = x, \tag{1.6}$$

replacing the initial value condition by $(g_{1-\alpha} * u)(0) = x$. By the dominated convergence theorem, it is easy to obtain that (1.6) implies

$$(g_{1-\alpha} * u)(0) = \lim_{t\to 0^+} \frac{1}{\Gamma(1-\alpha)} \int_0^1 (1-\sigma)^{-\alpha}\sigma^{\alpha-1}(t\sigma)^{1-\alpha}u(t\sigma)d\sigma = x.$$

Hence, by [8, Theorem 3.1 and Theorem 3.2], the following theorem holds.

Theorem **1.1**. *Assume that A is the generator of the a-order fractional resolvent $\{T(t)\}_{t>0}$ on Banach space X. Then,*

(I) for every $x \in D(A)$, $T(t)x$ is a strong solution of (1.2), that is, $T(\cdot) \in C((0,\infty), D(A))$, $t \mapsto J_t^{1-\alpha}T(t)x \in C^1((0,\infty), X)$ and (1.2) holds;

(II) for every $x \in X$, $T(t)x$ is a mild solution of (1.2), that is, $t \mapsto J_t^{\alpha}T(t)x \in D(A)$ and $T(t)x = \frac{t^{\alpha-1}}{\Gamma(\alpha)}x + AJ_t^{\alpha}T(t)x$, $t > 0$.

In order to define Caputo type fractional semigroups and their relation to the fractional resolvents from Definition **1.1**, we mention the work of Peng and Li, noting that in this case the Caputo fractional resolvents satisfy the (Caputo)-fractional semigroup characterization:

$$\begin{aligned} &\int_0^{t+s} \frac{T(\tau)}{(t+s-\tau)^{\alpha}}d\tau - \int_0^{t} \frac{T(\tau)}{(t+s-\tau)^{\alpha}}d\tau - \int_0^{s} \frac{T(\tau)}{(t+s-\tau)^{\alpha}}d\tau \\ =&\alpha \int_0^t \int_0^s \frac{T(r_1)T(r_2)}{(t+s-r_1-r_2)^{1+\alpha}}dr_1 dr_2, \quad t, s \geq 0. \end{aligned} \tag{1.7}$$

Peng and Li [10] proved that the α-order fractional semigroup is closely related to the solution operator of the Caputo fractional abstract Cauchy problem (1.1) (for the definition of solution operator, we refer to [2]).

Moreover, the equality (1.7) is equivalent to the following equality

$$T(t)J_s^{\alpha}T(s) - J_t^{\alpha}T(t)T(s) = J_t^{\alpha}T(t) - J_s^{\alpha}T(s), \quad t, s \geq 0.$$

In the special case when $T(\cdot)$ is exponentially bounded (hence it is Laplace transformable), taking Laplace transform on both sides of (1.5) with respect to s and t, we derive that

$$(\lambda^{-\alpha} - \mu^{-\alpha})\hat{T}(\mu)\hat{T}(\lambda) = \mu^{-\alpha}\lambda^{-\alpha}(\hat{T}(\lambda) - \hat{T}(\mu)). \tag{1.8}$$

By [10, (2.5)], for the right side of (1.7), there holds

$$\begin{aligned}&\int_0^\infty e^{-t\mu}\int_0^\infty e^{-s\lambda}\alpha\int_0^t\int_0^s \frac{T(r_1)T(r_2)}{(t+s-r_1-r_2)^{1+\alpha}}dr_1dr_2dsdt\\ =&\frac{\Gamma(1-\alpha)}{\lambda-\mu}(\lambda^\alpha - \mu^\alpha)\hat{T}(\mu)\hat{T}(\lambda).\end{aligned} \tag{1.9}$$

The combination of (1.8) and (1.9) implies that

$$\begin{aligned}&\int_0^\infty e^{-t\mu}\int_0^\infty e^{-s\lambda}\alpha\int_0^t\int_0^s \frac{T(r_1)T(r_2)}{(t+s-r_1-r_2)^{1+\alpha}}dr_1dr_2dsdt\\ &= \Gamma(1-\alpha)\frac{\hat{T}(\mu)-\hat{T}(\lambda)}{\lambda-\mu}.\end{aligned}$$

On the other hand, we observe that, by similar proof of [6, (4.2)], there holds

$$\int_0^\infty e^{-\mu t}\int_0^\infty e^{-\lambda s}T(t+s)dsdt = \frac{\hat{T}(\mu)-\hat{T}(\lambda)}{\lambda-\mu}.$$

Hence,

$$\Gamma(1-\alpha)T(t+s) = \alpha\int_0^t\int_0^s \frac{T(r_1)T(r_2)}{(t+s-r_1-r_2)^{1+\alpha}}dr_1dr_2dsdt. \tag{1.10}$$

Therefore, it is reasonable to conjecture that (1.10) holds without the assumption that $\{T(t)\}_{t>0}$ is exponentially bounded. Conversely, we wonder whether the equality (1.10) essentiality describes a fractional resolvent or not. This is the task for the next section.

2. Riemann-Liouville Fractional Semigroups

In this section we develop the theory of the Riemann-Liouville fractional semigroups. Moreover, the equivalence of the Riemann-Liouville fractional semigroups and fractional resolvents is proved. Without lost of generality, our discussion is restricted to the case $0 < \alpha < 1$. Our definition is motivated by the conjecture in the previous section.

Definition **2.1**. We call a family $\{T(t)\}_{t>0}$ of bounded linear operators to be a Riemann-Liouville α-order fractional semigroup on a Banach space X, if the following conditions are satisfied:

i) for any $x \in X$, $t \mapsto T(t)x$ is continuous over $(0,\infty)$ and

$$\lim_{t\to 0+} \Gamma(\alpha)t^{1-\alpha}T(t)x = x; \tag{2.1}$$

ii) for all $t, s > 0$, there holds

$$\Gamma(1-\alpha)T(t+s) = \alpha \int_0^t \int_0^s \frac{T(r_1)T(r_2)}{(t+s-r_1-r_2)^{1+\alpha}} dr_1 dr_2, \tag{2.2}$$

where the integrals are in the sense of strong operator topology.

PROPOSITION **2.1**. *Let $\{T(t)\}_{t>0}$ be a Riemann-Liouville α-order fractional semigroup, then it is commutative, i.e. $T(t)T(s) = T(s)T(t)$ for all $t, s > 0$.*

P r o o f. The symmetry of the left side of (2.2) with respect to t and s implies that

$$\int_0^t \int_0^s \frac{T(r_1)T(r_2)}{(t+s-r_1-r_2)^{1+\alpha}} dr_1 dr_2 = \int_0^s \int_0^t \frac{T(r_1)T(r_2)}{(t+s-r_1-r_2)^{1+\alpha}} dr_1 dr_2,$$

for any $t,\ s > 0$. Then, the commutativity is proved by the same procedure of [10, Proposition 1]. □

THEOREM **2.1**. *Assume that $\{T(t)\}_{t>0}$ is an α-order fractional resolvent on Banach space X. Then, it satisfies the equality (2.2) and is therefore a Riemann-Liouville α-order fractional semigroup.*

P r o o f. Denote by $L(t,s)$ and $R(t,s)$ the left and right sides of equality (2.2), respectively. Obviously, what we need is to prove that $L(t,s) = R(t,s)$ for all $t, s > 0$. For brevity, we introduce the following notations. Let

$$\begin{aligned} H(t,s) =& T(t)J_s^\alpha T(s) - J_t^\alpha T(t)T(s), \\ K(t,s) =& \frac{s^{\alpha-1}}{\Gamma(\alpha)} J_t^\alpha T(t) - \frac{t^{\alpha-1}}{\Gamma(\alpha)} J_s^\alpha T(s), \quad t, s > 0. \end{aligned}$$

Moreover, for sufficiently large $b > 0$ denote by $g_b(t)$ the truncation of $T(t)$ at b, and by $R_b(t,s)$, $L_b(t,s)$, $H_b(t,s)$ and $K_b(t,s)$ the quantities resulted by replacing $T(t)$ with $g_b(t)$ in $R(t,s)$, $L(t,s)$, $H(t,s)$ and $K(t,s)$, respectively.

By [10, (2.5)], it follows that the Laplace transform of $R_b(t,s)$ with respect to t and s is given by

$$\hat{R}_b(\mu,\lambda) = \frac{\Gamma(1-\alpha)}{\lambda-\mu}(\lambda^\alpha - \mu^\alpha)\hat{g}_b(\mu)\hat{g}_b(\lambda). \tag{2.3}$$

The Laplace transform of $\hat{L}_b(t,s)$ with respect to s and t can be obtained in a routine matter:

$$\hat{L}_b(\mu,\lambda) = \Gamma(1-\alpha)\frac{\hat{g}_b(\mu)-\hat{g}_b(\lambda)}{\lambda-\mu}. \tag{2.4}$$

We set

$$P_b(t,s) = \alpha\int_0^t\int_0^s \frac{H_b(r_1,r_2)}{(t+s-r_1-r_2)^{1+\alpha}}dr_1dr_2$$

and

$$Q_b(t,s) = \alpha\int_0^t\int_0^s \frac{K_b(r_1,r_2)}{(t+s-r_1-r_2)^{1+\alpha}}dr_1dr_2.$$

By (1.5), $H(t,s) = K(t,s)$ for any $t,s>0$. Thus, for all $t,s>0$,

$$\lim_{b\to\infty} P_b(t,s) = \lim_{b\to\infty} K_b(t,s). \tag{2.5}$$

The Laplace transform of $P_a(t,s)$ is

$$\widehat{P_b}(\mu,\lambda) = (\mu^{-\alpha}-\lambda^{-\alpha})\widehat{R_a}(\mu,\lambda). \tag{2.6}$$

Therefore,

$$\begin{aligned}\hat{Q}_b(t,\lambda) &= \alpha\int_0^\infty e^{-\lambda s}\int_0^t\int_0^s \frac{\frac{r_2^{\alpha-1}}{\Gamma(\alpha)}J^\alpha_{r_1}g_b(r_1) - \frac{r_1^{\alpha-1}}{\Gamma(\alpha)}J^\alpha_{r_2}g_b(r_2)}{(t+s-r_1-r_2)^{1+\alpha}}dr_1dr_2ds\\ &= \alpha\int_0^t\int_0^\infty e^{-\lambda s}\int_0^s \frac{\frac{r_2^{\alpha-1}}{\Gamma(\alpha)}J^\alpha_{r_1}g_b(r_1) - \frac{r_1^{\alpha-1}}{\Gamma(\alpha)}J^\alpha_{r_2}g_b(r_2)}{(t+s-r_1-r_2)^{1+\alpha}}dr_1dsdr_2\\ &= \alpha\int_0^t\left(\frac{r_2^{\alpha-1}}{\Gamma(\alpha)}\lambda^{-\alpha}\hat{g}_b(\lambda) - \lambda^{-\alpha}J^\alpha_{r_2}g_b(r_2)\right)\widehat{\Pi}(t-r_2,\lambda)dr_2,\end{aligned}$$

where $\widehat{\Pi}(t,\lambda)$ stands for the Laplace transform of the function $(t+s)^{-\alpha-1}$ of s. Thus, we can obtain

$$\begin{aligned}\hat{Q}_b(\mu,\lambda) &= \alpha\lambda^{-\alpha}\mu^{-\alpha}(\hat{g}_b(\lambda)-\hat{g}_b(\mu))\widehat{\Pi}(\mu,\lambda)\\ &= \alpha\lambda^{-\alpha}\mu^{-\alpha}(\hat{g}_b(\lambda)-\hat{g}_b(\mu))\int_0^\infty e^{-\mu t}\int_0^\infty e^{-\lambda s}\frac{1}{(t+s)^{1+\alpha}}dsdt\\ &= \alpha\lambda^{-\alpha}\mu^{-\alpha}(\hat{g}_b(\lambda)-\hat{g}_b(\mu))\frac{\Gamma(1-\alpha)}{\alpha(\lambda-\mu)}(\lambda^\alpha-\mu^\alpha)\\ &= (\mu^{-\alpha}-\lambda^{-\alpha})\hat{L}_b(\mu,\lambda).\end{aligned} \tag{2.7}$$

By virtue of Laplace transform, we obtain from (2.6) and (2.7) that

$$P_b(t,s) = (J^\alpha_s - J^\alpha_t)R_b(t,s) \text{ and } Q_b(t,s) = (J^\alpha_s - J^\alpha_t)L_b(t,s),\ \ \forall t,s>0. \tag{2.8}$$

The combination of (2.5) and (2.8) implies that

$$(J_s^\alpha - J_t^\alpha)L(t,s) = (J_s^\alpha - J_t^\alpha)R(t,s), \quad \forall t,s>0. \tag{2.9}$$

Therefore, $R(t,s) = L(t,s)$, $\forall t,s>0$. This completes the proof. □

Below we prove that the converse of Theorem **2.1** also holds, that is, a Riemann-Liouville fractional semigroup indicates a fractional resolvent. To do this, we begin with the definition of the generator of a Riemann-Liouville α-order fractional semigroup.

DEFINITION **2.2**. Let $\{T(t)\}_{t>0}$ be a Riemann-Liouville α-order fractional semigroup on Banach space X. Denote by $D(A)$ the set of all $x \in X$ such that the limit

$$\lim_{t\to 0^+} \Gamma(\alpha+1)t^{-\alpha}J_t^{1-\alpha}\left(T(t)x - \frac{t^{\alpha-1}}{\Gamma(\alpha)}x\right)$$

exists. Then, the operator $A : D(A) \to X$ defined by

$$Ax = \lim_{t\to 0^+} \Gamma(\alpha+1)t^{-\alpha}J_t^{1-\alpha}\left(T(t)x - \frac{t^{\alpha-1}}{\Gamma(\alpha)}x\right)$$

is called the generator of $\{T(t)\}_{t>0}$.

PROPOSITION **2.2**. *Let $\{T(t)\}_{t>0}$ be a Riemann-Liouville α-order fractional semigroup on Banach space X with generator A. Then,*

(a) For any $x \in X$ and $t>0$, there holds $J_t^\alpha T(t)x \in D(A)$ and

$$T(t)x = \frac{t^{\alpha-1}}{\Gamma(\alpha)}x + AJ_t^\alpha T(t)x. \tag{2.10}$$

(b) $T(t)D(A) \subset D(A)$ and $T(t)Ax = AT(t)x$, for all $x \in D(A)$.

(c) For all $x \in D(A)$, we have

$$T(t)x = \frac{t^{\alpha-1}}{\Gamma(\alpha)}x + J_t^\alpha T(t)Ax.$$

(d) A is equivalently defined by

$$Ax = \Gamma(2\alpha)\lim_{t\to 0^+} \frac{t^{1-\alpha}T(t)x - \frac{1}{\Gamma(\alpha)}x}{t^\alpha}$$

and $D(A)$ is consisting just of those $x \in X$ such that the above limit exists.

(e) A is closed and densely defined.

(f) A admits at most one Riemann-Liouville α-order fractional semigroup.

P r o o f. (a) Let $x \in X$ and $t > 0$ be fixed. Denote by $g_t(\cdot)$ the truncation of $T(\cdot)$ at t, that is, $g_t(\sigma) = T(\sigma), 0 < \sigma \le t$ and $g_t(\sigma) = 0$ otherwise. Define the function $H_t(r,s)$ for $r, s > 0$ by

$$H_t(r,s) = \left(g_t(r) - \frac{r^{\alpha-1}}{\Gamma(\alpha)} I\right) J_s^\alpha g_t(s)x. \tag{2.11}$$

Obviously, for $0 < r \le t$,

$$H_t(r,t) = \left(T(r) - \frac{r^{\alpha-1}}{\Gamma(\alpha)}\right) J_t^\alpha T(t)x. \tag{2.12}$$

Taking Laplace transform with respect r and s successively for both sides of (2.11), we derive

$$\hat{H}_t(\mu,\lambda) = \lambda^{-\alpha}\hat{g}_t(\mu)\hat{g}_t(\lambda)x - \lambda^{-\alpha}\mu^{-\alpha}\hat{g}_t(\lambda)x. \tag{2.13}$$

The combination of (2.3), (2.4) and (2.13) implies that

$$\begin{aligned}\hat{H}_t(\mu,\lambda) =& \mu^{-\alpha}\hat{g}_t(\mu)\hat{g}_t(\lambda)x - \mu^{-\alpha}\lambda^{-\alpha}\hat{g}_t(\mu)x \\ &+ \frac{\lambda^{-\alpha}\mu^{-\alpha}(\lambda-\mu)}{\Gamma(1-\alpha)}(\hat{L}_t(\mu,\lambda) - \hat{R}_t(\mu,\lambda))x.\end{aligned}$$

By virtue of the Laplace transform, it follows that

$$\begin{aligned}H_t(r,s) =& \left(g_t(s) - \frac{s^{\alpha-1}}{\Gamma(\alpha)} I\right) J_r^\alpha g_t(r)x \\ &+ \frac{[({}^C D_s^{1-\alpha})J_r^\alpha - ({}^C D_r^{1-\alpha})J_s^\alpha]\cdot[L_t(r,s) - R_t(r,s)]x}{\Gamma(1-\alpha)}.\end{aligned}$$

Here the Laplace transform formulas

$$\widehat{{}^C D_t^\beta f}(\lambda) = \lambda^\beta \hat{f}(\lambda) - \lambda^{\beta-1} f(0), \quad 0 < \beta < 1, \; f \in C([0,\infty), X)$$

is used.

By the definition of g_t, it follows that $L_t(r,s) = R_t(r,s)$ for all $0 < s, r \le t$, we have that

$$H_t(r,s) = \left(T(s) - \frac{s^{\alpha-1}}{\Gamma(\alpha)} I\right) J_r^\alpha T(r)x, \quad \forall\, 0 < r, s \le t.$$

In particular, we obtain

$$H_t(r,t) = \left(T(t) - \frac{t^{\alpha-1}}{\Gamma(\alpha)} I\right) J_r^\alpha T(r)x, \quad \forall\, 0 < r \le t. \tag{2.14}$$

Combining (2.12) and (2.14), we derive that

$$\begin{aligned}
&\lim_{r\to 0^+}\Gamma(\alpha+1)r^{-\alpha}J_r^{1-\alpha}\left(T(r)-\frac{r^{\alpha-1}}{\Gamma(\alpha)}\right)J_t^{\alpha}T(t)x\\
=&\Gamma(\alpha+1)\lim_{r\to 0^+}r^{-\alpha}\left(T(t)-\frac{t^{\alpha-1}}{\Gamma(\alpha)}I\right)J_r^{1}T(r)x\\
=&\Gamma(\alpha+1)\left(T(t)-\frac{t^{\alpha-1}}{\Gamma(\alpha)}I\right)\lim_{r\to 0^+}r^{-\alpha}\int_0^r T(\sigma)xd\sigma\\
=&\Gamma(\alpha+1)\left(T(t)-\frac{t^{\alpha-1}}{\Gamma(\alpha)}I\right)\lim_{r\to 0^+}\int_0^1\frac{\sigma^{\alpha-1}}{\Gamma(\alpha)}\Big[\Gamma(\alpha)(r\sigma)^{1-\alpha}T(r\sigma)x\Big]d\sigma.
\end{aligned}$$

By the dominated convergence theorem, it follows that

$$\begin{aligned}
&\lim_{r\to 0^+}\Gamma(\alpha+1)r^{-\alpha}J_r^{1-\alpha}\left(T(r)-\frac{r^{\alpha-1}}{\Gamma(\alpha)}\right)J_t^{\alpha}T(t)x\\
=&\Gamma(\alpha+1)\left(T(t)-\frac{t^{\alpha-1}}{\Gamma(\alpha)}I\right)\int_0^1\frac{\sigma^{\alpha-1}}{\Gamma(\alpha)}\lim_{r\to 0^+}\Big[\Gamma(\alpha)(r\sigma)^{1-\alpha}T(r\sigma)x\Big]d\sigma\\
=&\Gamma(\alpha+1)\left(T(t)-\frac{t^{\alpha-1}}{\Gamma(\alpha)}I\right)\int_0^1\frac{\sigma^{\alpha-1}}{\Gamma(\alpha)}xd\sigma\\
=&T(t)x-\frac{t^{\alpha-1}}{\Gamma(\alpha)}x.
\end{aligned}$$

This implies that $J_t^{\alpha}T(t)x\in D(A)$ and

$$AJ_t^{\alpha}T(t)x=T(t)x-\frac{t^{\alpha-1}}{\Gamma(\alpha)}x.$$

(b) and (c) are directly obtained by Proposition **2.1** and (a).

(d) Denote by D the set of those $x\in X$ such that the limit

$$\lim_{t\to 0^+}\frac{t^{1-\alpha}T(t)x-\frac{1}{\Gamma(\alpha)}x}{t^{\alpha}}$$

exists. Let $x\in D(A)$. Then, by (b), we have that

$$\begin{aligned}
&\Gamma(2\alpha)\lim_{t\to 0^+}\frac{t^{1-\alpha}T(t)x-\frac{1}{\Gamma(\alpha)}x}{t^{\alpha}}\\
=&\Gamma(2\alpha)\lim_{t\to 0^+}\frac{J_t^{\alpha}Ax}{t^{2\alpha-1}}\\
=&\Gamma(2\alpha)\lim_{t\to 0^+}\frac{1}{\Gamma(\alpha)t^{2\alpha-1}}\int_0^t(t-\sigma)^{\alpha-1}T(\sigma)Axd\sigma\\
=&\frac{\Gamma(2\alpha)}{\Gamma(\alpha)^2}\lim_{t\to 0^+}\int_0^1\sigma^{\alpha-1}(1-\sigma)^{\alpha-1}\Big[\Gamma(\alpha)(t\sigma)^{1-\alpha}T(t\sigma)Ax\Big]d\sigma.
\end{aligned}$$

By the dominated convergence theorem, we obtain

$$\begin{aligned}
&\Gamma(2\alpha)\lim_{t\to 0^+}\frac{t^{1-\alpha}T(t)x-\frac{1}{\Gamma(\alpha)}x}{t^\alpha}\\
=&\frac{\Gamma(2\alpha)}{\Gamma(\alpha)^2}\int_0^1\sigma^{\alpha-1}(1-\sigma)^{\alpha-1}\lim_{t\to 0^+}\Big[\Gamma(\alpha)(t\sigma)^{1-\alpha}T(t\sigma)Ax\Big]d\sigma\\
=&\frac{\Gamma(2\alpha)}{\Gamma(\alpha)^2}B(\alpha,\alpha)Ax=Ax.
\end{aligned}$$

This implies that $x\in D$ and then $D(A)\subset D$. Now we prove the converse inclusion. Let $x\in D$. The existence of limit

$$\lim_{t\to 0^+}\frac{t^{1-\alpha}T(t)x-\frac{1}{\Gamma(\alpha)}x}{t^\alpha}$$

implies that

$$\begin{aligned}
&\lim_{t\to 0^+}\Gamma(\alpha+1)t^{-\alpha}J_t^{1-\alpha}\left(T(t)x-\frac{t^{\alpha-1}}{\Gamma(\alpha)}x\right)\\
=&\lim_{t\to 0^+}\frac{\Gamma(\alpha+1)}{\Gamma(1-\alpha)}\int_0^1\sigma^{2\alpha-1}(1-\sigma)^{-\alpha}\frac{(t\sigma)^{1-\alpha}T(t\sigma)x-\frac{1}{\Gamma(\alpha)}x}{(t\sigma)^\alpha}d\sigma\\
=&\frac{\Gamma(\alpha+1)}{\Gamma(1-\alpha)}\int_0^1\sigma^{2\alpha-1}(1-\sigma)^{-\alpha}\lim_{t\to 0^+}\frac{(t\sigma)^{1-\alpha}T(t\sigma)x-\frac{1}{\Gamma(\alpha)}x}{(t\sigma)^\alpha}d\sigma\\
=&\frac{\Gamma(\alpha+1)}{\Gamma(1-\alpha)}B(2\alpha,1-\alpha)\lim_{t\to 0^+}\frac{t^{1-\alpha}T(t)x-\frac{1}{\Gamma(\alpha)}x}{t^\alpha}\\
=&\Gamma(2\alpha)\lim_{t\to 0^+}\frac{t^{1-\alpha}T(t)x-\frac{1}{\Gamma(\alpha)}x}{t^\alpha}.
\end{aligned}$$

Here the dominated convergence theorem is used. Hence, $x\in D(A)$ and

$$Ax=\Gamma(2\alpha)\lim_{t\to 0^+}\frac{t^{1-\alpha}T(t)x-\frac{1}{\Gamma(\alpha)}x}{t^\alpha}. \tag{2.15}$$

(e) The closedness and densely defined of A is followed directly from the combination of (d) and [8].

(f) Assume that both $\{T(t)\}_{t\geq 0}$ and $\{S(t)\}_{t\geq 0}$ are Riemann-Liouville α-order fractional semigroups generated by A. Then, by (c), for all $x\in D(A)$, we have

$$\begin{aligned}\frac{t^{\alpha-1}}{\Gamma(\alpha)} * T(t)x =& (S(t) - J_t^{\alpha} AS(t)) * T(t)x \\ =& S(t) * T(t)x - (J_t^{\alpha} AS(t)) * T(t)x \\ =& S(t) * (T(t)x - J_t^{\alpha} AT(t)x) \\ =& \frac{t^{\alpha-1}}{\Gamma(\alpha)} * S(t)x.\end{aligned}$$

By the Titchmarsh theorem, for any $t > 0$, $T(t) = S(t)$ on $D(A)$. The result is obtained by the density of $D(A)$. □

From the above theorem, we can derive that the inverse of Theorem **2.1** also holds.

THEOREM **2.2**. *Assume that $\{T(t)\}_{t>0}$ is a Riemann-Liouville α-order fractional semigroup on Banach space X. Then, $\{T(t)\}_{t>0}$ is an α-order fractional resolvent for system (1.2).*

Proof. Let $x \in X$. In (2.10), replacing x with $J_s^{\alpha} T(s)x$, we derive

$$T(t) J_s^{\alpha} T(s)x = \frac{t^{\alpha-1}}{\Gamma(\alpha)} J_s^{\alpha} T(s)x + AJ_t^{\alpha} T(t) J_s^{\alpha} T(s)x. \tag{2.16}$$

By Proposition **2.1** and replacing x with $J_s^{\alpha} T(s)x$ in (2.10), it follows that

$$\begin{aligned}T(t) J_s^{\alpha} T(s)x =& \frac{t^{\alpha-1}}{\Gamma(\alpha)} J_s^{\alpha} T(s)x + AJ_s^{\alpha} T(s) J_t^{\alpha} T(t)x \\ =& \frac{t^{\alpha-1}}{\Gamma(\alpha)} J_s^{\alpha} T(s)x + \left(T(s) - \frac{s^{\alpha-1}}{\Gamma(\alpha)}\right) J_t^{\alpha} T(t)x.\end{aligned} \tag{2.17}$$

Hence, (1.5) is obtained directly from (2.17). Hence, $\{T(t)\}_{t>0}$ is an α-order fractional resolvent. Moreover, by Proposition **2.2**, it follows that A is just the generator of $\{T(t)\}_{t>0}$, which is deemed to be a resolvent family. The proof is completed. □

The combination of Theorem **2.1** and Theorem **2.2** implies the equivalence of Riemann-Liouville fractional semigroups and fractional resolvent families.

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