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RESEARCH PAPER

EXISTENCE OF POSITIVE SOLUTIONS TO A HIGHER ORDER SINGULAR BOUNDARY VALUE PROBLEM WITH FRACTIONAL Q-DERIVATIVES

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Abstract

The authors study the singular boundary value problem with fractional q-derivatives

$$-(D_q^{\nu}u)(t) = f(t,u), \quad t \in (0,1),$$

$$(D_q^i u)(0) = 0,\ i = 0,\ldots,n-2, \quad (D_q u)(1) = \sum_{j=1}^{m} a_j (D_q u)(t_j) + \lambda,$$

where $q \in (0,1)$, $m \geq 1$ and $n \geq 2$ are integers, $n-1 < \nu \leq n$, $\lambda \geq 0$ is a parameter, $f : (0,1] \times (0,\infty) \to [0,\infty)$ is continuous, $a_i \geq 0$ and $t_i \in (0,1)$ for $i = 1,\ldots,m$, and D_q^{ν} is the q-derivative of Riemann-Liouville type of order ν. Sufficient conditions are obtained for the existence of positive solutions. Their analysis is mainly based on a nonlinear alternative of Leray-Schauder.

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1. Introduction and preliminaries on fractional q-calculus

In the recent years, the subject of fractional calculus has gained considerable popularity and importance mainly due to its demonstrated applications in numerous seemingly diverse and widespread fields of science and engineering. The monographs [30], [31], [32], and [33] provide excellent sources for the theory and applications of fractional calculus. Among all the topics, the existence of positive solutions of boundary value problems (BVPs) for fractional differential equations is currently undergoing active investigation; see for example, [3], [4], [5], [6], [12], [16], [21], [22], [27], [28], [29], [35], [36] and the references therein.

Many efforts have also been made to develop the theory of discrete fractional calculus in various directions. For some recent works, we refer the reader to [9], [10], [11], [13], [14], [15], [23], [24], [25].

Early work on fractional q-calculus can be found in [1] and [7]. Recently, there seems to be new interest in the study of this subject and many new developments have been made in the theory of fractional q-calculus (see e.g. [8], [19], [20], [34]).

To the best of our knowledge, there are few results available in the literature to study the important problem for existence of *positive solutions* for BVPs with fractional q-derivatives; the only papers we know of are by El-Shahed and Al-Askar [17], El-Shahed and Hassan [18], Ferreira [19], [20], and Graef and Kong [26]. In this paper, we will study the existence of positive solutions of a class of higher order *singular* BVPs with fractional q-derivatives, expecting these results to find good potential for applications.

To make the paper self-contained, below we recall some known facts on fractional q-calculus, that can be found, for example, in [1], [19], [20], [30], [34].

For $q \in (0,1)$, define

$$[a]_q = \frac{1-q^a}{1-q}, \quad a \in \mathbb{R}.$$

The q-analog of the Pochhammer symbol (the q-shifted factorial) is defined by

$$(a;q)_0 = 1, \quad (a;q)_k = \prod_{i=0}^{k-1}(1-aq^i), \quad k \in \mathbb{N} \cup \{\infty\}.$$

The q-analogue of the power function $(a-b)^k$ with $k \in \mathbb{N}_0 := \{0,1,2,\ldots\}$ is

$$(a-b)^{(0)} = 1, \quad (a-b)^{(k)} = \prod_{i=0}^{k-1}(a-bq^i), \quad k \in \mathbb{N},\ a,b \in \mathbb{R}.$$

The following relation between them holds

$$(a-b)^{(k)} = a^k (b/a;q)_k, \quad a \neq 0.$$

Their natural extensions to the reals are

$$(a;q)_\gamma = \frac{(a;q)_\infty}{(aq^\gamma;q)_\infty} \quad \text{and} \quad (a-b)^{(\gamma)} = a^\gamma \frac{(b/a;q)_\infty}{(q^\gamma b/a;q)_\infty}, \quad \gamma \in \mathbb{R}.$$

Clearly,

$$(a-b)^{(\gamma)} = a^\gamma (b/a;q)_\gamma, \quad a \neq 0,$$

and if $b = 0$, then $a^{(\gamma)} = a^\gamma$. We also use the notation $0^{(\gamma)} = 0$ for $\gamma > 0$. The q-gamma function is defined by

$$\Gamma_q(x) = (q;q)_{x-1}(1-q)^{1-x}, \quad x \in \mathbb{R} \setminus \{0,-1,-2,\ldots\}.$$

Obviously, $\Gamma_q(x+1) = [x]_q \Gamma_q(x)$.

The q-derivative of a function h is defined by

$$(D_q h)(x) = \frac{h(x) - h(qx)}{(1-q)x} \quad \text{for } x \neq 0 \quad \text{and} \quad (D_q h)(0) = \lim_{x \to 0} (D_q h)(x),$$

and q-derivatives of higher order are given by

$$(D_q^0 h)(x) = h(x) \quad \text{and} \quad (D_q^k h)(x) = D_q(D_q^{k-1} h)(x), \ k \in \mathbb{N}.$$

The q-integral of a function h defined on the interval $[0,b]$ is given by

$$(I_q h)(x) = \int_0^x h(s) d_q s = x(1-q) \sum_{i=0}^\infty h(xq^i) q^i, \quad x \in [0,b].$$

If $a \in [0,b]$ and h is defined in the interval $[0,b]$, then its integral from a to b is defined by

$$\int_a^b h(s) d_q s = \int_0^b h(s) d_q s - \int_0^a h(s) d_q s.$$

Similarly to derivatives, an operator I_q^k is given by

$$(I_q^0 h)(x) = h(x) \quad \text{and} \quad (I_q^k h)(x) = I_q(I_q^{k-1} h)(x), \ k \in \mathbb{N}.$$

The fundamental theorem of calculus applies to these operators D_q and I_q, i.e.,

$$(D_q I_q h)(x) = h(x),$$

and if h is continuous at $x = 0$, then

$$(I_q D_q h)(x) = h(x) - h(0).$$

Definition **1.1**. Let $\nu \geq 0$ and h be a function defined on $[0,1]$. The fractional q-integral of Riemann-Liouville type is defined by $(I_q^0 h)(x) = h(x)$ and

$$(I_q^\nu h)(x) = \frac{1}{\Gamma_q(\nu)} \int_0^x (x - qs)^{(\nu-1)} h(s) d_q s, \quad \nu > 0,\ t \in [0,1].$$

Definition **1.2**. The fractional q-derivative of Riemann-Liouville type of order $\nu \geq 0$ is defined by $(D_q^0 h)(x) = h(x)$ and

$$(D_q^\nu h)(x) = (D_q^l I_q^{l-\nu} h)(x), \quad \nu > 0,$$

where l is the smallest integer greater than or equal to ν.

The rest of the paper is organized as follows. In Section **2**, we introduce our problem and present our main results. The proofs of the main results are given in Section **3**.

2. Fractional boundary value problems

In this section, we are concerned with positive solutions of the higher order singular BVP with fractional q-derivatives consisting of the equation

$$-(D_q^\nu u)(t) = f(t,u), \quad t \in (0,1), \tag{2.1}$$

and the boundary condition (BC)

$$(D_q^i u)(0) = 0,\ i = 0, \dots, n-2, \quad (D_q u)(1) = \sum_{j=1}^m a_j (D_q u)(t_j) + \lambda, \tag{2.2}$$

where $q \in (0,1)$, $m \geq 1$ and $n \geq 2$ are integers, $n - 1 < \nu \leq n$, $\lambda \geq 0$ is a parameter, $f : (0,1] \times (0,\infty) \to [0,\infty)$ is continuous, $a_j \geq 0$ and $t_j \in (0,1)$ for $j = 1, \dots, m$. By a *positive solution* of BVP (2.1), (2.2), we mean a function $u \in C[0,1]$ such that $u(t)$ satisfies (2.1) and (2.2), and $u(t) > 0$ on $(0,1]$.

When $n = 3$, $a_j = 0$ for $j = 1, \dots, m$, and $f(t,x)$ is nonsingular in x, BVP (2.1), (2.2) has been studied by Ferreira [19]. The well known Krasnosel'skii fixed point theorem was applied there to obtain an existence criterion for positive solutions. Very recently, in [26], the present authors studied the nonsingular version of BVP (2.1), (2.2) and obtained sufficient conditions for the uniqueness, existence, and nonexistence of positive solutions in terms of different values of λ.

In this paper, we establish a new general existence criterion for positive solutions of BVP (2.1) and (2.2). One application of our theorem to a special problem is also presented. Here, we allow the nonlinear term $f(t,x)$ to be singular at $x = 0$. The proof will employ a nonlinear alternative of Leray-Schauder; see Lemma **3.3** in Section **3**. Our results extend and complement recent results on this subject in the literature, especially those in [19] and [26].

Recall that the characteristic function χ on an interval $I \subseteq \mathbb{R}$ is given by

$$\chi_I(t) = \begin{cases} 1, & t \in I, \\ 0, & t \notin I. \end{cases}$$

Define a function $G : [0,1] \times [0,1] \to \mathbb{R}$ by

$$G(t,s) = \frac{t^{\nu-1}}{\left(1 - \sum_{j=1}^m a_j t_j^{\nu-2}\right) \Gamma_q(\nu)} \Bigg((1-s)^{(\nu-2)} - \sum_{j=1}^m a_j (t_j - s)^{(\nu-2)} \chi_{[0,t_j]}(s) \Bigg) - \frac{1}{\Gamma_q(\nu)} (t-s)^{(\nu-1)} \chi_{[0,t]}(s). \quad (2.3)$$

Note that, for the special case where $a_j = 0$ for $j = 1, \ldots, m$, $G(t,s)$ can be written as

$$G(t,s) = \frac{1}{\Gamma_q(\nu)} \begin{cases} (1-s)^{(\nu-2)} t^{\nu-1} - (t-s)^{(\nu-1)}, & 0 \le s \le t \le 1, \\ (1-s)^{(\nu-2)} t^{\nu-1}, & 0 \le t \le s \le 1. \end{cases}$$

We now present the first existence result in this paper. We will need the following hypotheses:

(H1) $0 \le \sum_{j=1}^m a_j t_j^{\nu-2} < 1$;

(H2) there exist continuous, nonnegative functions $g(x)$, $h(x)$, and $\phi(t)$ such that $g(x) > 0$ is nonincreasing on $(0,\infty)$, $h(x)/g(x)$ is nondecreasing on $(0,\infty)$,

$$f(t,x) \le \phi(t)(g(x) + h(x)) \quad \text{for } (t,x) \in (0,1] \times (0,\infty),$$

and

$$\int_0^1 \phi(s) g(s^{\nu-1}) d_q s < \infty;$$

(H3) there exists $\delta > 0$ such that

$$g(xy) \le \delta g(x) g(y) \quad \text{for all } x,\ y \in (0,\infty);$$

(H4) for each $r > 0$, there exists a continuous nonnegative function $\psi_r(t)$ such that

$$f(t,x) \ge \psi_r(t) \quad \text{for } (t,x) \in (0,1] \times (0,r]$$

and

$$0 < \int_0^1 G(1,qs) \psi_r(s) d_q s < \infty;$$

(H5) there exists $R > 0$ such that

$$R > \delta g(R)\left(1 + \frac{h(R)}{g(R)}\right)\int_0^1 G(1, qs)\phi(s)g(s^{\nu-1})d_q s$$

with δ given in (H3).

Theorem **2.1**. *Assume that conditions (H1)–(H5) hold and let*

$$\Lambda_R = \left(R - \delta g(R)\left(1 + \frac{h(R)}{g(R)}\right)\int_0^1 G(1, qs)\phi(s)g(s^{\nu-1})d_q s\right) \times \left(1 - \sum_{j=1}^{m} a_j t_j^{v-2}\right)[\nu - 1]_q. \tag{2.4}$$

Then, for each $\lambda \in [0, \Lambda_R)$, BVP (2.1), (2.2) has at least one positive solution $u(t)$ satisfying $\max_{t\in[0,1]} u(t) < R$.

To demonstrate the applicability of Theorem **2.1**, we consider a special case of BVP (2.1), (2.2), namely, the BVP consisting of the equation

$$-(D_q^\nu u)(t) = c(t)u^{-\alpha} + \mu d(t)u^\beta, \quad t \in (0, 1), \tag{2.5}$$

and BC (2.2), where $\alpha \geq 0$ and $\beta \geq 0$ are constants, c and d are continuous functions on $[0, 1]$ with $c(t) > 0$ and $d(t) \geq 0$ on $[0, 1]$, and $\mu \geq 0$ is a parameter.

The following result is a direct consequence of Theorem **2.1**.

Corollary **2.1**. *Assume that (H1) holds and $\alpha < 1/(\nu - 1)$.*

(i) If $\beta < 1$, then BVP (2.5), (2.2) has at least one positive solution for each λ, $\mu \in [0, \infty)$.

(ii) If $\beta \geq 1$, then there exist $\overline{\lambda} \in (0, \infty)$ and $\overline{\mu} \in (0, \infty)$ such that BVP (2.5), (2.2) has at least one positive solution for each $\lambda \in [0, \overline{\lambda})$ and $\mu \in [0, \overline{\mu})$.

3. Proofs of the main results

Throughout this section, let $C[0, 1]$ be the Banach space of continuous functions equipped with the norm $||u|| = \max_{t\in[0,1]} |u(t)|$.

Lemma **3.1** below gives some properties of $G(t, s)$ and was proved in [26, Lemma 2.1].

Lemma **3.1**. *Assume (H1) holds. The function $G(t, s)$ defined by (2.3) has the following properties:*

(a) $G(t, qs) \geq 0$ for $t, s \in [0, 1]$;

(b) $G(t,qs) \le G(1,qs)$ *for* $t,s \in [0,1]$*;*
(c) $G(t,qs) \ge t^{\nu-1}G(1,qs)$ *for* $t,s \in [0,1]$.

The following lemma follows directly from [26, Lemma 2.2] and shows that $G(t,s)$ can be used to obtain the equivalent integral forms for some given BVPs.

LEMMA **3.2**. *Assume that (H1) holds and* $w \in C[0,1]$*. Then,* $u(t)$ *is a solution of the BVP consisting of the equation*

$$-(D_q^\nu u)(t) = w(t), \quad t \in (0,1),$$

and BC (2.2) *if and only if*

$$u(t) = \int_0^1 G(t,qs)w(s)d_qs + \frac{\lambda t^{\nu-1}}{\left(1 - \sum_{j=1}^m a_j t_j^{v-2}\right)[\nu-1]_q}.$$

We also need the following version of the well known nonlinear alternative of Leray–Schauder. We refer the reader to [2, Theorem 1.2.3] for this result.

LEMMA **3.3**. *Let* K *be a convex subset of a normed linear space* X*, and let* Ω *be an bounded open subset with* $\tilde{p} \in \Omega$*. Then every compact map* $N : \overline{\Omega} \to K$ *has at least one of the following two properties:*
(i) N *has at least one fixed point in* $\overline{\Omega}$*;*
(ii) there is $u \in \partial\Omega$ *and* $\mu \in (0,1)$ *such that* $u = (1-\mu)\tilde{p} + \mu N u$.

Now, we are in a position to prove our results.

P r o o f. (Proof of Theorem **2.1**.) Let $\lambda \in [0, \Lambda_R)$ be fixed. By (H5) and (2.4), there exists $k_0 \in \mathbb{N}$ such that

$$\begin{aligned} R \quad > \quad & \frac{1}{k_0} + \delta g(R)\left(1 + \frac{h(R)}{g(R)}\right)\int_0^1 G(1,qs)\phi(s)g(s^{\nu-1})d_qs \\ & + \frac{\lambda}{\left(1 - \sum_{j=1}^m a_j t_j^{v-2}\right)[\nu-1]_q}. \end{aligned} \tag{3.1}$$

Let $\mathbb{N}^* = \{k_0, k_0+1, \ldots\}$. For any fixed $k \in \mathbb{N}^*$, consider the integral equation

$$\begin{aligned} u(t) &= \frac{1}{k} + \mu \int_0^1 G(t, qs) f_k(s, u(s)) d_q s + \frac{\lambda \mu t^{\nu-1}}{\left(1 - \sum_{j=1}^m a_j t_j^{v-2}\right) [\nu - 1]_q} \\ &= \frac{1}{k} + \mu T_k u(t), \end{aligned} \tag{3.2}$$

where $\mu \in (0, 1)$, $f_k(t, x) = f(t, \max\{x, 1/k\})$ for $(t, x) \in (0, 1) \times \mathbb{R}$, and

$$T_k u(t) = \int_0^1 G(t, qs) f_k(s, u(s)) d_q s + \frac{\lambda t^{\nu-1}}{\left(1 - \sum_{j=1}^m a_j t_j^{v-2}\right) [\nu - 1]_q}.$$

We now prove two claims.

Claim 1. For any $\mu \in (0, 1)$, any possible solution $u(t)$ of (3.2) satisfies

$$u(1) = ||u|| \quad \text{and} \quad u(t) \geq t^{\nu-1} ||u|| \quad \text{for } t \in [0, 1].$$

In fact, from parts (a) and (b) of Lemma **3.1**, we have

$$\begin{aligned} u(1) &= \frac{1}{k} + \mu \int_0^1 G(1, qs) f_k(s, u(s)) d_q s + \frac{\lambda \mu}{\left(1 - \sum_{j=1}^m a_j t_j^{v-2}\right) [\nu - 1]_q} \\ &\geq \frac{1}{k} + \mu \int_0^1 G(t, qs) f_k(s, u(s)) d_q s + \frac{\lambda \mu t^{\nu-1}}{\left(1 - \sum_{j=1}^m a_j t_j^{v-2}\right) [\nu - 1]_q} \\ &= u(t) \geq 0 \quad \text{on } [0, 1], \end{aligned}$$

so $u(1) = ||u||$. By parts (a) and (c) of Lemma **3.1**, it follows that

$$\begin{aligned} u(t) &\geq \frac{1}{k} + \mu t^{\nu-1} \int_0^1 G(1, qs) f_k(s, u(s)) d_q s \\ &\quad + \frac{\lambda \mu t^{\nu-1}}{\left(1 - \sum_{j=1}^m a_j t_j^{v-2}\right) [\nu - 1]_q} \\ &\geq t^{\nu-1} \left(\frac{1}{k} + \mu \int_0^1 G(1, qs) f_k(s, u(s)) d_q s \right. \\ &\quad \left. + \frac{\lambda \mu}{\left(1 - \sum_{j=1}^m a_j t_j^{v-2}\right) [\nu - 1]_q} \right) \\ &= t^{\nu-1} u(1) = ||u||. \end{aligned}$$

Thus, Claim 1 holds.

Claim 2. For any $\mu \in (0, 1)$, any possible solution of (3.2) satisfies $||u|| \neq R$.

Suppose the claim is not true and assume that $u(t)$ is a solution of (3.2) for some $\mu \in (0,1)$ with $||u|| = R$. By (3.2), $u(t) \geq 1/k$ on $[0,1]$. Then, $f_k(t,u(t)) = f(t,u(t))$ and (3.2) can be written as

$$u(t) = \frac{1}{k} + \mu \int_0^1 G(t,qs) f(s,u(s)) d_q s + \frac{\lambda \mu t^{\nu-1}}{\left(1 - \sum_{j=1}^m a_j t_j^{v-2}\right) [\nu-1]_q}.$$

From (H2), (H3), and Claim 1, we see that

$$\begin{aligned} R &= ||u|| = u(1) = \frac{1}{k} + \mu \int_0^1 G(1,qs) f(s,u(s)) d_q s \\ &\quad + \frac{\lambda \mu}{\left(1 - \sum_{j=1}^m a_j t_j^{v-2}\right) [\nu-1]_q} \\ &\leq \frac{1}{k_0} + \int_0^1 G(1,qs)\phi(s) g(u(s)) \left(1 + \frac{h(u(s))}{g(u(s))}\right) d_q s \\ &\quad + \frac{\lambda}{\left(1 - \sum_{j=1}^m a_j t_j^{v-2}\right) [\nu-1]_q} \\ &\leq \frac{1}{k_0} + \left(1 + \frac{h(R)}{g(R)}\right) \int_0^1 G(1,qs)\phi(s) g(s^{\nu-1}R) d_q s \\ &\quad + \frac{\lambda}{\left(1 - \sum_{j=1}^m a_j t_j^{v-2}\right) [\nu-1]_q} \\ &\leq \frac{1}{k_0} + \delta g(R) \left(1 + \frac{h(R)}{g(R)}\right) \int_0^1 G(1,qs)\phi(s) g(s^{\nu-1}) d_q s \\ &\quad + \frac{\lambda}{\left(1 - \sum_{j=1}^m a_j t_j^{v-2}\right) [\nu-1]_q}. \end{aligned}$$

This contradicts (3.1), so Claim 2 holds.

Note that $1/k \leq 1/k_0 < R$ and (3.2) can be rewritten as

$$u(t) = (1-\mu)\frac{1}{k} + \mu N_k u(t),$$

where $N_k u(t) = T_k u(t) + 1/k$. Then, Lemma **3.3** implies that $N_k(t)$ has at least one fixed point u_k in $\overline{\Omega} = \{u \in C[0,1] \ : \ ||u|| \leq R\}$. Thus,

$$u_k(t) = \frac{1}{k} + \int_0^1 G(t,qs) f_k(s,u_k(s)) d_q s + \frac{\lambda t^{\nu-1}}{\left(1 - \sum_{j=1}^m a_j t_j^{v-2}\right) [\nu-1]_q}.$$

Then, $u_k(t) \geq 1/k$ on $[0,1]$, and so

$$u_k(t)=\frac{1}{k}+\int_0^1 G(t,qs)f(s,u_k(s))d_qs+\frac{\lambda t^{\nu-1}}{\left(1-\sum_{j=1}^m a_j t_j^{v-2}\right)[\nu-1]_q}. \quad (3.3)$$

Note that $||u_k|| \leq R$. Then, in view of (H4), there exists a continuous nonnegative function $\psi_R(t)$ such that

$$f(t,u_k(t)) \geq \psi_R(t) \quad \text{on } (0,1).$$

Let $l=\int_0^1 G(1,qs)\psi_R(s)d_qs$. Clearly, $l>0$ by (H4). From parts (a) and (c) of Lemma **3.1** and (3.3), we obtain

$$u_k(t) \geq t^{\nu-1}\int_0^1 G(1,qs)\psi_R(s)d_qs = lt^{\nu-1} \quad \text{on } [0,1]. \quad (3.4)$$

It is clear that the sequence $\{u_k\}_{k\in\mathbb{N}^*}$ is uniformly bounded. In the following, we show that it is also equicontinuous on $[0,1]$. For any $t,s \in [0,1]$, since $G(t,s)$ and $t^{\nu-1}$ are uniformly continuous in t, we see that for any $\epsilon>0$, there exists $\eta>0$ such that for any $t_1,t_2 \in [0,1]$ with $|t_1-t_2|<\eta$, we have

$$|G(t_1,qs)-G(t_2,qs)| < \frac{\epsilon}{2A}, \quad s \in [0,1],$$

and

$$|t_1^{\nu-1}-t_2^{\nu-1}| < \frac{\epsilon\left(1-\sum_{j=1}^m a_j t_j^{v-2}\right)[\nu-1]_q}{2\Lambda_R},$$

where

$$A=\delta g(l)\left(1+\frac{h(R)}{g(R)}\right)\int_0^1 \phi(s)g(s^{\nu-1})d_qs.$$

Thus, when $|t_1-t_2|<\eta$, from (H2), (H3), (3.3), and (3.4), it follows that

$$\begin{aligned}
|u_k(t_1)-u_k(t_2)| &\leq \int_0^1 |G(t_1,qs)-G(t_2,qs)|f(s,u_k(s))d_qs \\
&\quad +\frac{\Lambda_R|t_1^{\nu-1}-t_2^{\nu-1}|}{\left(1-\sum_{j=1}^m a_j t_j^{v-2}\right)[\nu-1]_q} \\
&< \frac{\epsilon}{2A}\int_0^1 \phi(s)g(u_k(s))\left(1+\frac{h(u_k(s))}{g(u_k(s))}\right)d_qs+\frac{\epsilon}{2} \\
&\leq \frac{\epsilon}{2A}\left(1+\frac{h(R)}{g(R)}\right)\int_0^1 \phi(s)g(s^{\nu-1}l)d_qs+\frac{\epsilon}{2} \\
&\leq \frac{\epsilon}{2A}\delta g(l)\left(1+\frac{h(R)}{g(R)}\right)\int_0^1 \phi(s)g(s^{\nu-1})d_qs+\frac{\epsilon}{2} \\
&< \frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon.
\end{aligned}$$

Thus, $\{u_k\}_{k\in\mathbb{N}^*}$ is equicontinuous.

Now, by the Arzelà-Ascoli theorem, $\{u_k\}_{k\in\mathbb{N}^*}$ has a subsequence, also denoted by $\{u_k\}_{k\in\mathbb{N}^*}$, that converges uniformly to a function $u \in C[0,1]$. Note that $u_k(t)$ satisfies (3.3) and (3.4). Then, letting $k \to \infty$, we see that

$$u(t) = \int_0^1 G(t,qs)f(s,u(s))d_q s + \frac{\lambda t^{\nu-1}}{\left(1-\sum_{j=1}^m a_j t_j^{v-2}\right)[\nu-1]_q} \tag{3.5}$$

and

$$u(t) \geq l t^{\nu-1} > 0 \quad \text{on } (0,1].$$

Then, by Lemma **3.2**, $u(t)$ is a positive solution of BVP (2.1), (2.2). Since $||u_k|| < R$ and $u = \lim_{k\to\infty} u_k$, we have $||u|| \leq R$. By an argument similar to the one used to show Claim 2, we have $||u|| < R$. This completes the proof of the theorem. □

P r o o f. (Proof of Corollary **2.1**.) We will apply Theorem **2.1**. To this end, let $f(t,x) = c(t)x^{-\alpha} + \mu d(t)x^{\beta}$, $g(x) = x^{-\alpha}$, $h(x) = \mu x^{\beta}$, and $\phi(t) = \max\{c(t),\ d(t)\}$. Then, (H2), (H3) with $\delta = 1$, and (H4) with $\psi_r(t) = r^{-\alpha}c(t)$ hold. Let

$$B = \int_0^1 G(1,qs)\phi(s)s^{-\alpha(\nu-1)}d_q s.$$

In view of the condition that $0 \leq \alpha < 1/(\nu-1)$, we have $0 < B < \infty$, Clearly, (H5) is equivalent to

$$\mu < C(R) \quad \text{for some } R > 0,$$

where

$$C(x) = \frac{x^{\alpha+1} - B}{Bx^{\alpha+\beta}}, \quad x > 0.$$

Then, we have that (H5) holds if

$$0 \leq \mu < \overline{\mu} := \sup_{x>0} C(x). \tag{3.6}$$

We now discuss two cases.

Case 1. $\beta < 1$. For this case, $C(x)$ is strictly increasing and $\overline{\mu} = \infty$. Thus, for any $\mu \in [0,\infty)$, by (3.6), there exists $R_1 > 0$ such that (H5) holds for all $R \geq R_1$. Note that, for Λ_R defined by (2.4), we have $\lim_{R\to\infty} \Lambda_R = \infty$. The conclusion in part (i) then follows from Theorem **2.1**.

Case 2. $\beta \geq 1$. For this case, $0 < \overline{\mu} < \infty$. Thus, for any $\mu \in [0,\overline{\mu})$, from (3.6), there exists $R_2 \in (0,\infty)$ such that (H5) holds with $R = R_2$. Let $\overline{\lambda} = \Lambda_{R_2}$. Then, it is easy to check that $0 < \overline{\lambda} < \infty$. The conclusion in part (ii) then also follows from Theorem **2.1**. This completes the proof of the corollary. □

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