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SURVEY PAPER

LIOUVILLE AND RIEMANN-LIOUVILLE FRACTIONAL DERIVATIVES VIA CONTOUR INTEGRALS

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Abstract

We study the fractional integral (fI) and fractional derivative (fD), attained by the analytic continuation (AC) of Liouville's fI (LfI) and Riemann-Liouville fI (RLfI). On the AC of RLfI, we give a detailed summary of Lavoie et al's review. The ACs of RLfI are expressed by means of contour integrals. Two of them use the Cauchy contour, and one is using the Pochhammer contour. In this case, the latter is AC of all the others for the functions treated. For the AC of LfI, one can find studies in Campos' papers and in Nishimoto's books, where the AC is using only the Cauchy contour. Here we present also an AC using a modified Pochhammer's contour. In this case, we see that any of these two ACs is not the AC of the other for all the functions treated. This fact leads to difficulties, if a careful study taking care of the domains of existence of each AC is not adopted. By taking account of this fact, we resolve the difficulties which occur in Nishimoto's formalism.

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1. Introduction

The notions and tools of fractional derivative (fD) and fractional differential equation (fDE) have been discussed since long time ago; see e.g. [4, 9, 13, 14]. When we have an initial value problem for a fDE where the initial values at a finite point are given, usually the Riemann-Liouville fD (RLfD) is used. When the solution decays to zero at ∞ or $-\infty$, the Liouville fD (LfD) is used. These fD are defined in terms of the respective fractional integrals (fI), which are the RLfI and LfI. In the present paper, we survey some studies on the analytic continuations (AC) of RLfI and LfI.

For AC of RLfI, we have a review by Lavoie et al. [6]. There, the AC are expressed by contour integrals. Two of them, which we call CfD and C′fI, use generalized Cauchy contours. The other, which we call PfD, uses the Pochhammer contour which appears in the AC of the beta function. We recall it in Section **2**. It is important to note, for a function of variable z in the form z^a or $z^\gamma \cdot \sum_{k=0}^{\infty} a_k z^k$, that CfD and C′fI are AC of RLfI, and PfD is an AC of CfD as well as of C′fI.

For the AC of LfI, we have the papers [2, 3], where Campos studied the AC using the Cauchy contour, which he called Hankel's contour. We call this AC of LfI as HfD. In a series of books [11, 12], Nishimoto gave a survey of his work on fractional calculus, where he started with HfD using the Cauchy contour and then defined his fractional differentiation and integral, which he called the differintegration (fDI). His fDI is defined as the AC of HfD.

In these works on the AC of LfI, there appears HfD using the Cauchy contour, but no comments are given on the one related with the Pochhammer contour. In Section **3**, we discuss the LfI, HfD and also the AC using a modified Pochhammer contour, which we call mPfD. Taking account of these, we clarify what happens with the AC of the LfI.

For the power function of the form z^a, HfD is an AC of LfI, and mPfD is an AC of HfD. When the function is the exponential function in the form e^{-az}, HfD is an AC of LfI, but mPfD is not an AC of HfD.

We note that the situation is simple for the AC of RLfI, since for all the functions treated, PfD is the AC of all the other AC of RLfI and may be called RLfD. The situation is not simple for the AC of LfI, since for all the functions treated, any of mPfD and HfD is not the AC of the other.

In Sections **4**$\sim$**6**, we give additional studies on the AC of LfI. In Section **4**, we discuss the index law of the AC of LfI and of RLfD. We have to pay special attention to the condition for which the index law of the mPfD is valid. Then a comment is given on the use of the index law in solving a simple fDE. As to RLfD, we call attention to the relation with distribution theory. In Section **5**, remarks are given of the fD of the cosine and sine functions. In Section **6**, remarks are given on the definition of mPfD.

We use the notations $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ to denote the sets of all integers, of all real numbers and of all complex numbers, respectively. We also use

$$\begin{aligned}
&\mathbb{Z}_{>0} := \{n \in \mathbb{Z}|\ n > 0\}, \quad \mathbb{Z}_{\le 0} := \mathbb{Z}\backslash\mathbb{Z}_{>0}, \quad \mathbb{Z}_{\ge 0} := \{n \in \mathbb{Z}|\ n \ge 0\},\\
&\mathbb{Z}_{<0} := \mathbb{Z}\backslash\mathbb{Z}_{\ge 0}, \quad \mathbb{R}_{>0} := \{x \in \mathbb{R}|\ x > 0\},\\
&{}_{-}\mathbb{C} := \{z \in \mathbb{C}|\ \mathrm{Re}\ z < 0\}, \quad {}_{+}\mathbb{C} := \{z \in \mathbb{C}|\ \mathrm{Re}\ z > 0\}.
\end{aligned}$$

Here $\mathbb{Z}\backslash\mathbb{Z}_{>0}$ denotes the set $\{n \in \mathbb{Z}| n \notin \mathbb{Z}_{>0}\}$. Let X be a path on the complex plane. We use notations $f \in \mathcal{L}^1(X)$ and $f \in \mathcal{L}^1_{loc}(X)$ to denote that a function f is integrable and locally integrable, respectively, on X. The notation $f^{(n)}(z)$ is used to represent $\frac{d^n}{dz^n}f(z)$ for $n \in \mathbb{Z}_{\ge 0}$, as usual.

2. Riemann-Liouville fI and its Analytic Continuations

Let $c, z \in \mathbb{C}$, and let $P(c, z)$ be the path from c to z, as shown in Fig. 1(a). For $f(z) \in \mathcal{L}^1(P(c, z))$, the RLfI is defined in [5, 6] by

$$ {}_{RL}D_c^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_c^z (z-\zeta)^{\lambda-1} f(\zeta) d\zeta. \tag{2.1}$$

When $f(z)$ is analytic on a neighborhood of $P(c, z)$, the ${}_{RL}D_c^{\nu} f(z)$ defined by (2.1) for $\lambda = -\nu$ is analytic as a function of ν in the domain ${}_{-}\mathbb{C}$; see e.g. [19, Section 5.31]. In [5, 6], three analytic continuations expressed by contour integrals are considered. In Introduction, they are called resp. CfD, C′fI and PfD.

The first one, CfD, is the fD given by

$$ {}_{C}D_c^{\nu} f(z) = \frac{\Gamma(\nu+1)}{2\pi i} \int_{C(c,z^+)} (\zeta - z)^{-\nu-1} f(\zeta) d\zeta, \tag{2.2}$$

for $\nu \in \mathbb{C}\backslash\mathbb{Z}_{<0}$, where the contour $C(c, z^+)$ is shown in Fig. 1(b), which starts from c, encircles the point z counterclockwise, and then goes back to c, without crossing the path $P(c, z)$. When $-n \in \mathbb{Z}_{<0}$, we put ${}_{C}D_c^{-n} f(z) = \lim_{\nu\to-n} {}_{C}D_c^{\nu} f(z)$, which is confirmed to be equal to ${}_{RL}D_c^{-n} f(z)$.

By this definition, we confirm that ${}_{C}D_c^{\nu} f(z)$ is an analytic continuation of ${}_{RL}D_c^{\nu} f(z)$ as a function of ν, so that for every ν for which the latter exists, the former also exists and they are equal with each other.

Let $f_1(z)$ be analytic on a neighborhood of $P(0, z)$, and let $\gamma \in \mathbb{R}$. If $f(z) = z^{\gamma} f_1(z)$ and $\gamma \notin \mathbb{Z}$, the second one C′fI is defined by

$$ {}_{T}D_0^{\nu} f(z) = e^{-i\pi\gamma} \frac{1}{2i\Gamma(-\nu)\sin(\pi\gamma)} \int_{C(z,0^+)} (\zeta - z)^{-\nu-1} f(\zeta) d\zeta, \tag{2.3}$$

for $\nu \in {}_{-}\mathbb{C}$, and the third one PfD by

$$
{}_{P}D_0^{\nu} f(z) = e^{-i\pi\gamma} \frac{\Gamma(\nu+1)}{4\pi \sin(\pi\gamma)} \int_{C_P(z)} (\zeta - z)^{-\nu-1} f(\zeta) d\zeta, \tag{2.4}
$$

for $\nu \in \mathbb{C}\backslash\mathbb{Z}_{<0}$. Here the contour $C_P(z)$ is z times of the contour $C_P(1)$ which is shown in Fig. 2. When $-n \in \mathbb{Z}_{<0}$, we put ${}_{P}D_0^{-n} f(z) =$ $\lim_{\nu\to -n} {}_{P}D_0^{\nu} f(z) = {}_{T}D_0^{-n} f(z)$.

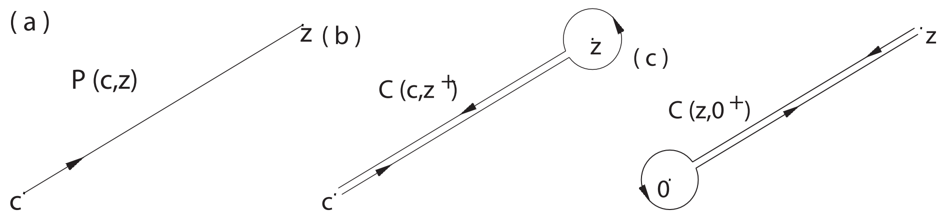

FIGURE 1. (a) Path of integration $P(c, z)$ in (2.1), (b) Cauchy contour $C(c, z^+)$ in (2.2), and (c) $C(z, 0^+)$ in (2.3).

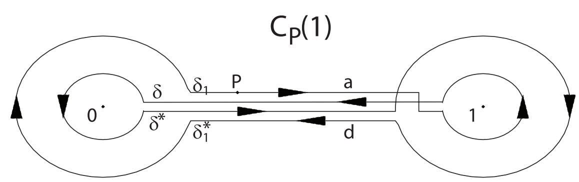

FIGURE 2. The Pochhammer contour $C_P(1)$ of integration. The four horizontal pathes are labeled by a, b, c and d from the top to the bottom.

When $f(z) = z^n f_1(z)$ for $n \in \mathbb{Z}_{\geq 0}$, we adopt

$$
{}_{T}D_0^{\nu} f(z) = \lim_{\gamma\to n} {}_{T}D_0^{\nu}[z^{\gamma-n} f(z)] = {}_{RL}D_0^{\nu} f(z),
$$

$$
{}_{P}D_0^{\nu} f(z) = \lim_{\gamma\to n} {}_{P}D_0^{\nu}[z^{\gamma-n} f(z)] = {}_{C}D_0^{\nu} f(z).
$$

We now confirm that ${}_{T}D_0^{\nu} f(z)$ and ${}_{P}D_0^{\nu} f(z)$ are analytic continuations of ${}_{RL}D_0^{\nu} f(z)$ and ${}_{C}D_0^{\nu} f(z)$, respectively, as functions of γ. We confirm also that ${}_{P}D_0^{\nu} f(z)$ is an analytic continuation of ${}_{T}D_0^{\nu} f(z)$ as a function of ν. Table 1 illustrates these relations.

Here we use symbols ${}_{*}D_c^{\nu} f(z)$ with different subscript $*$ which is either RL, C, T or P. Usually such a discrimination is not adopted. They are

	$\gamma+1 \in {}_+\mathbb{C}$	$\gamma \in \mathbb{C}\backslash\mathbb{Z}_{<0}$
$\nu \in {}_-\mathbb{C}$	${}_{RL}D_0^{\nu}f(z)$	${}_{T}D_0^{\nu}f(z)$
$\nu \in \mathbb{C}$	${}_{C}D_0^{\nu}f(z)$	${}_{P}D_0^{\nu}f(z)$

TABLE 1. The domains of ν and of γ in which ${}_{RL}D_0^{\nu}f(z)$ for $f(z)=z^{\gamma}f_1(z)$ and its analytic continuations exist.

simply denoted by ${}_{RL}D_c^{\nu}f(z)$ and called RLfD as a whole, even though when $\mathrm{Re}\,\nu<0$, it is equal to RLfI or to fI given by ${}_{T}D_c^{\nu}f(z)$.

In the discussion of RLfD, the basic function is usually the power function. When $f(z)=z^a$ for $a\in\mathbb{C}$, we have

$$ {}_{RL}D_0^{\nu}z^a = \frac{\Gamma(a+1)}{\Gamma(a-\nu+1)}z^{a-\nu}, \quad (\mathrm{Re}\,\nu<0,\ \mathrm{Re}\,a>-1). \tag{2.5} $$

This result is obtained with the aid of Euler's integral of the first kind for the beta function:

$$ B(\lambda,\kappa)=\frac{\Gamma(\lambda)\Gamma(\kappa)}{\Gamma(\lambda+\kappa)}=\int_0^1 t^{\lambda-1}(1-t)^{\kappa-1}dt, \quad (\lambda,\ \kappa\in{}_+\mathbb{C}). \tag{2.6} $$

By putting $\gamma=a$ and $f_1(z)=1$ in Table 1, we see the domains of ν and of a, where ${}_{R}D_0^{\nu}z^a$ and its analytic continuations exist.

In [19, Section 12.43], Pochhammer's formula for $B(\alpha,\beta)$ is given. When we put $\lambda=\alpha$ and $\kappa=\beta$, the formula is

$$ B(\lambda,\kappa)=-\frac{e^{-i\pi\lambda-i\pi\kappa}}{4\sin(\pi\lambda)\sin(\pi\kappa)}\int_{C_p(1)} t^{\lambda-1}(1-t)^{\kappa-1}dt. \tag{2.7} $$

The contour $C_p(1)$ is shown in Fig. 2, which consists of four pathes labeled by a, b, c and d. We may write this contour as $C(P,1^+,0^+,1^-,0^-)$ in a similar way to $C(c,z^+)$, that is shown in Fig. 1(b) for the Cauchy contour. By this formula, $B(\lambda,\kappa)$ is defined for $\lambda,\kappa\in\mathbb{C}\backslash\mathbb{Z}_{\le 0}$ as an analytic function of λ as well as of κ [19, Section 12.43], where $B(\lambda,\kappa)$ for $\lambda,\kappa\in\mathbb{Z}_{>0}$ is assumed to be defined by analytic continuation. By using this formula, ${}_{P}D_0^{\nu}z^a$ is shown to be the analytic continuation of ${}_{RL}D_0^{\nu}z^a$ in the domains shown in Table 1 for $\gamma=a$. In fact, the argument deriving (2.7) from (2.6) is used to derive the analytic continuation given by (2.4).

The RLfD of $\sum_{n=0}^{\infty} a_n z^n$ and $z^\gamma \cdot \sum_{n=0}^{\infty} a_n z^n$ are calculated by means of term-by-term integration; see e.g. [6]. For instance,

$$\begin{aligned} {}_C D_0^\nu e^z &= \sum_{n=0}^{\infty} \frac{1}{n!} \cdot {}_C D_0^\nu z^n = \sum_{n=0}^{\infty} \frac{1}{\Gamma(n-\nu+1)} z^{n-\nu} \\ &= \frac{z^{-\nu}}{\Gamma(1-\nu)} \cdot {}_1F_1(1; 1-\nu; z), \qquad \nu \in \mathbb{C} \backslash \mathbb{Z}_{>0}. \end{aligned} \tag{2.8}$$

Here ${}_1F_1(\alpha; \beta; z)$ is a hypergeometric function.

3. Liouville's fI and its Analytic Continuations

3.1. Liouville's fI

Liouville [7, 8] started his study on fractional differentiation with the following formula for $\mu \in \mathbb{R}$ and $m, x \in \mathbb{R}$:

$$\frac{d^\mu}{dx^\mu} e^{mx} = m^\mu e^{mx}. \tag{3.1}$$

He presented its integral form for $\lambda \in {}_+\mathbb{C}$, by

$$\int^\lambda f(x) dx^\lambda = \frac{1}{(-1)^\lambda \Gamma(\lambda)} \int_0^\infty f(x+\alpha) \alpha^{\lambda-1} d\alpha. \tag{3.2}$$

Later, he presented also the following formula [8]:

$$\int^\lambda f(x) dx^\lambda = \frac{1}{\Gamma(\lambda)} \int_0^\infty f(x-\alpha) \alpha^{\lambda-1} d\alpha. \tag{3.3}$$

Formula (3.1) for $\mu \in {}_-\mathbb{C}$ follows from (3.2) or (3.3) for $\lambda = -\mu$, according to as $m < 0$ or $m > 0$, with the aid of Euler's integral of the second kind:

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \qquad z \in {}_+\mathbb{C}. \tag{3.4}$$

In the review of Liouville's works [8, Chapter VIII], it is mentioned that (3.2) is equal to

$$\int^\lambda f(x) dx^\lambda = \frac{1}{(-1)^\lambda \Gamma(\lambda)} \int_x^\infty (\zeta - x)^{\lambda-1} f(\zeta) d\zeta, \tag{3.5}$$

which is equal to ${}_{RL}D_c^{-\lambda} f(x)$ given by (2.1) if $c = \infty$.

For $z \in \mathbb{C}$ and $\phi \in \mathbb{R}$, let $P_\phi(z)$ be the path from z to $z + \infty \cdot e^{i\phi}$, as shown in Fig. 3. For $f(z) \in \mathcal{L}^1(P_\phi(z))$, we define ${}_L D_\phi^{-\lambda} f(z)$ by

$$\begin{aligned} {}_L D_\phi^{-\lambda} f(z) &= e^{i(\phi+\pi)\lambda} \frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1} f(z + te^{i\phi}) dt & (3.6) \\ &= e^{i(\phi+\pi)\lambda} \frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1} f(z - te^{i(\phi+\pi)}) dt. & (3.7) \end{aligned}$$

We call this Liouville's ϕ-dependent fI.

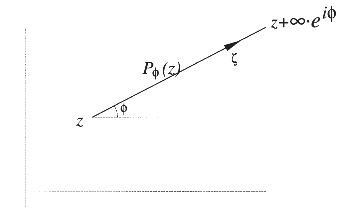

FIGURE 3. $P_\phi(z)$.

LfI given by (3.2) and (3.3) are now expressed resp. by ${}_LD_0^{-\lambda}f(x)$ and ${}_LD_{-\pi}^{-\lambda}f(x)$, for $x\in\mathbb{R}$.

In [16, Section 22.1], Samko et al. define LfI: $(I_{+,\phi}^{\lambda}f)(z)$ by

$$(I_{+,\phi}^{\lambda}f)(z)={}_{RL}D_c^{-\lambda}f(z)\quad\text{for}\quad c=z+\infty\cdot e^{i\phi}$$

and mention that it is equal to ${}_LD_\phi^{-\lambda}f(z)$ given by (3.6). There $(I_{-,\phi}^{\nu}f)(z)$ is defined by $(I_{-,\phi}^{\nu}f)(z)=e^{i\nu\pi}I_{+,\phi}^{\nu}f$. In some articles [9, Chapter VII], the right-hand side of (3.5) multiplied by $(-1)^\lambda$, that is equal to $(I_{-,\phi}^{\nu}f)(z)$, is called Weyl's fI, see [16, p. xxxii].

The right-hand sides of (3.6) and (3.7) are Mellin transforms as functions of λ, and hence they have two abscissas of convergence, [20, Chapter VI, Section 9]. The abscissas are 0 and the other, which we express by $-s_1[f]>0$. Here $s_1[f]$ is defined as follows.

DEFINITION 3.1. Let ϕ, $s\in\mathbb{R}$, $f(z)\in\mathcal{L}_{loc}^1(P_\phi(z))$, and let $s_1\in\mathbb{R}$ or $s_1=-\infty$ be the greatest lower bound of s for which $I_s:=\int_1^\infty t^{-s-1}|f(z+te^{i\phi})|dt$ converges. We then denote this s_1 by $s_1[f]$ or $s_1[f(z)]$.

When I_s converges, there exists a series $\{t_l\}_{l\in\mathbb{Z}_{>0}}$ of $t_l\in\mathbb{R}$ such that $t_l^{-s}|f(z+t_le^{i\phi})|\to0$ and $t_l\to\infty$ as $l\to\infty$. We express this fact simply by $t_l^{-s}|f(z+t_le^{i\phi})|\to0$ as $t_l\to\infty$. As a consequence, if $s_1[f]<0$, $|f(z+t_le^{i\phi})|\to0$ as $t_l\to\infty$.

DEFINITION 3.2. Let $\phi\in\mathbb{R}$, $f(z)\in\mathcal{L}_{loc}^1(P_\phi(z))$, and $s_1[f]<0$. Then for $\lambda\in\mathbb{C}$ satisfying $0<\operatorname{Re}\lambda<-s_1[f]$, we define Liouville's ϕ-dependent fI: ${}_LD_\phi^{-\lambda}f(z)$ by (3.6) or (3.7).

LEMMA 3.1. *${}_LD_\phi^{-\lambda}f(z)$ exists only when $|f(z+t_le^{i\phi})|\to0$ as $t_l\to\infty$.*

LEMMA 3.2. *${}_LD_\phi^{\nu}f(z)$ is an analytic function of ν in the domain $s_1[f]<\operatorname{Re}\nu<0$.*

Let $f(z) = e^{-az}$ for $a \in \mathbb{C}$ satisfying Re $(ae^{i\phi}) > 0$. Then $s_1[f] = -\infty$ and we obtain

$$ {}_LD^{\nu}_{\phi} e^{-az} = e^{-i\pi\nu} a^{\nu} e^{-az}, \quad \text{Re}\, \nu < 0, \tag{3.8}$$

by using the formula (3.4).

Let $f(z) = z^a$ for $a \in \mathbb{C}$. Then we have $s_1[f] = \text{Re}\, a$ and

$$ {}_LD^{\nu}_{\phi} z^a = e^{-i\pi\nu} \frac{\Gamma(\nu - a)}{\Gamma(-a)} z^{a-\nu}, \quad \text{if Re}\, a < \text{Re}\, \nu < 0. \tag{3.9}$$

To obtain this, we use the following formula for the beta function $B(\lambda, \kappa)$:

$$B(\lambda, \kappa) = \frac{\Gamma(\lambda)\Gamma(\kappa)}{\Gamma(\lambda+\kappa)} = \int_0^{\infty} x^{\lambda-1}(1+x)^{-\lambda-\kappa} dx, \quad (\lambda,\ \kappa \in {}_+\mathbb{C}). \tag{3.10}$$

This formula is obtained from (2.6), by the change of variable $x = \frac{t}{1-t}$.

In Sections **3.2** and **3.3**, we define ${}_HD^{\nu}_{\phi} f(z)$ and ${}_MD^{\nu}_{\phi} f(z)$, which are analytic continuations of ${}_LD^{\nu}_{\phi} f(z)$, called HfD and mPfD in Section **1**, and study their basic properties.

3.2. Contour integral with Hankel's contour

The gamma-function $\Gamma(z)$ defined by (3.4) exists in the domain $z \in {}_+\mathbb{C}$. We know Hankel's formula which defines $\Gamma(z)$ in the whole complex plane, [19, Section 12.22]. We write the formula as

$$\Gamma(z) = e^{i\pi z} \frac{1}{2i \sin \pi z} \int_{C_H} \zeta^{z-1} e^{-\zeta} d\zeta, \quad z \in \mathbb{C}. \tag{3.11}$$

Here C_H is the limit of $X, Y \to \infty$ of the contour C_{H^*} shown in Fig. 4.

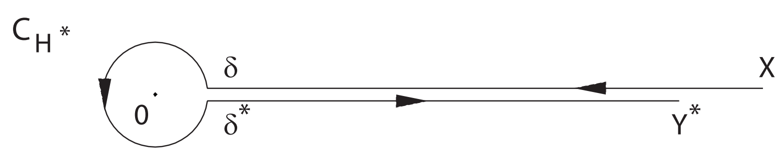

FIGURE 4. The contour of integration C_{H^*}, from X, to δ, to $\delta^* = \delta e^{2i\pi}$, and then to $Y^* = Y e^{2i\pi}$, where $\delta, X, Y \in \mathbb{R}_{>0}$ satisfy $\delta < X,\ Y$.

By applying the method of Hankel to the integral given in (3.6), we define ${}_HD^{\nu}_{\phi} f(z)$ as follows.

Definition 3.3. Let $f(\zeta)$ be analytic on a neighborhood of $P_\phi(z)$. Then for $\nu \in \mathbb{C}\backslash\mathbb{Z}_{<0}$ satisfying $\mathrm{Re}\,\nu > s_1[f]$, we define the ϕ-dependent fD, ${}_H D_\phi^\nu f(z)$, by:

$$ {}_H D_\phi^\nu f(z) = e^{-i\phi\nu}\frac{\Gamma(\nu+1)}{2\pi i}\int_{C_H}\eta^{-\nu-1}f(z+\eta e^{i\phi})d\eta, \tag{3.12}$$

which can be expressed also as (3.13) given below. When $-n > s_1[f]$ for $n \in \mathbb{Z}_{>0}$, we put ${}_H D_\phi^{-n} f(z) = \lim_{\nu\to -n}[{}_H D_\phi^\nu f(z)]$.

The proofs of the theorems given in Sections **3.2** and **3.3** are presented in Appendix A.

Theorem **3.1**. *If ${}_H D_\phi^\nu f(z)$ for $\mathrm{Re}\,\nu < 0$ exists, then ${}_L D_\phi^\nu f(z)$ also exists and ${}_H D_\phi^\nu f(z) = {}_L D_\phi^\nu f(z)$ for $\mathrm{Re}\,\nu < 0$.*

Lemma 3.3. *${}_H D_\phi^\nu f(z)$ is an analytic function of ν in the domain* $\mathrm{Re}\,\nu > s_1[f]$.

This follows from the fact that the integral in (3.12) is an analytic function of ν when it converges [19, Sections 5.31 and 5.32]. We also use Theorem **3.1** and Lemma 3.2.

Usually ${}_H D_\phi^\nu f(z)$ is defined as a generalization of the Cauchy integral formula of differentiation [11, 12, 2, 3], by

$$ {}_H D_\phi^\nu f(z) = \frac{\Gamma(\nu+1)}{2\pi i}\int_{z+\infty\cdot e^{i\phi}}^{(z+)}\frac{f(\zeta)}{(\zeta-z)^{\nu+1}}d\zeta. \tag{3.13}$$

Campos [2] called the contour in this expression the Hankel contour, where he defined ${}_H D_\phi^\nu f(z)$ only for $\nu \notin \mathbb{Z}_{<0}$.

Theorem **3.2**. *If ${}_H D_\phi^n f(z)$ for $n \in \mathbb{Z}_{\geq 0}$ exists, ${}_H D_\phi^n f(z) = f^{(n)}(z)$.*

3.3. Contour integral with a modified Pochhammer's contour

In Section **2**, we use Pochhammer's formula (2.7) which gives the beta function $B(\lambda,\kappa)$ in the whole complex plane as a function of λ as well as of κ. That formula is obtained as the analytic continuation of Euler's integral of the first kind (2.6). We now give a modified Pochhammer's formula which corresponds to (3.10).

Formula (3.10) is obtained from (2.6) by the change of variable $x = \frac{t}{1-t}$. We now give a formula from (2.7) by the same change of variable $\eta = \frac{t}{1-t}$.

As a result, we obtain the modified Pochhammer's formula for $B(\lambda,\kappa)$:

$$B(\lambda,\kappa) = -e^{-i\pi\lambda-i\pi\kappa}\frac{1}{4\sin(\pi\lambda)\sin(\pi\kappa)}\int_{\tilde{C}}\eta^{\lambda-1}(1+\eta)^{-\lambda-\kappa}d\eta, \quad (3.14)$$

which applies for all $\lambda,\kappa\in\mathbb{C}\backslash\mathbb{Z}_{\leq 0}$. Here $\tilde{C}$ is the closed contour shown in Fig. 5.

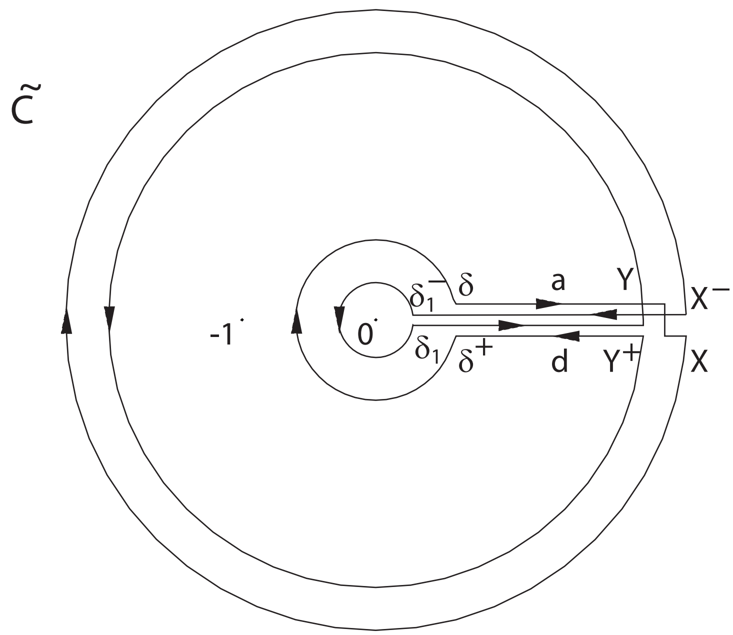

FIGURE 5. The contour of integration $\tilde{C}$, from δ, to X, to $X^- = Xe^{-2i\pi}$, $\delta_1^- = \delta_1 e^{-2i\pi}$, δ_1, Y, $Y^+ = Ye^{2i\pi}$, $\delta^+ = \delta e^{2i\pi}$, and then back to δ. The four horizontal pathes are called a, b, c and d from the top to the bottom.

In this place, we give a definition of ${}_MD^\nu_\phi f(z)$ in a restricted condition.

CONDITION A. $f(z)$ is expressed as $f(z) = z^\gamma f_1(z)$, where $\gamma\in\mathbb{R}$ and $f_1(z)$ is an entire function.

In Section **6**, we introduce more general conditions, which are Conditions B and C.

DEFINITION 3.4. Let Condition A or Condition B (see Section **6**) be satisfied for $\gamma\in\mathbb{R}$. If $\gamma\notin\mathbb{Z}$, $\nu\in\mathbb{C}\backslash\mathbb{Z}_{<0}$ and $\nu-\gamma\notin\mathbb{Z}$, we define ${}_MD^\nu_\phi f(z)$

of order ν by

$$ {}_M D_\phi^\nu f(z) = e^{-i\phi\nu - i\pi\nu + i\pi\gamma} \frac{\Gamma(\nu+1)}{4\pi\sin(\pi(\gamma-\nu))} \int_{\tilde{C}} \eta^{-\nu-1} f(z+\eta e^{i\phi}) d\eta, \quad (3.15) $$

where $\max(\delta, \delta_1) < |z| < \min(X, Y)$ and $(\arg z + \phi - \pi) \not\equiv 0 \pmod{2\pi}$ are assumed for the contour $\tilde{C}$. If $\gamma \notin \mathbb{Z}$, the value of ${}_M D_\phi^\nu f(z)$ at $\nu \in \mathbb{C}$ satisfying $\nu \in \mathbb{Z}_{<0}$ or $\nu - \gamma \in \mathbb{Z}$ should be determined by analytic continuation. If $\gamma = n \in \mathbb{Z}$, it is defined by

$$ {}_M D_\phi^\nu f(z) = \lim_{\gamma' \to n} {}_M D_\phi^\nu [z^{\gamma' - n} f(z)]. \quad (3.16) $$

Lemma 3.4. *${}_M D_\phi^\nu f(z)$ is analytic as a function of $\nu \in \mathbb{C}$, as far as $\nu \notin \mathbb{Z}_{<0}$ and $\gamma - \nu \notin \mathbb{Z}$.*

This follows from the fact that the integral in (3.15) is analytic as a function of ν, as is confirmed by [19, Section 5.21].

In discussing the relation between ${}_M D_\phi^\nu f(z)$ and ${}_H D_\phi^\nu f(z)$, $s_2[f]$ defined below plays an important role.

Definition 3.5. We denote by $s_2[f]$, the least of $s_2 \in \mathbb{R}$ such that $\sup_{-\pi<\theta\le\pi} t_l^{-s} |f(z + t_l e^{i\theta})| \to 0$ as $t_l \to \infty$ for all $s \in \mathbb{R}$ satisfying $s > s_2$.

Theorem **3.3**. *Let $s_2[f] < \infty$ and let ${}_M D_\phi^\nu f(z)$ exist. Then ${}_H D_\phi^\nu f(z)$ also exists, and ${}_M D_\phi^\nu f(z) = {}_H D_\phi^\nu f(z)$, for $\nu \in \mathbb{C}$ satisfying* $\mathrm{Re}\,\nu > s_1[f]$.

Theorem **3.4**. *Let $f(z)$ be an entire function. Then,* (i) ${}_M D_\phi^\nu f(z) = 0$ *for $\nu \in \mathbb{C} \backslash \mathbb{Z}$.* (ii) *If $s_2[f] < \infty$, ${}_H D_\phi^\nu f(z) = 0$ for* $\mathrm{Re}\,\nu > s_1[f]$.

Theorem **3.5**. *If ${}_M D_\phi^n f(z)$ exists for $n \in \mathbb{Z}_{\ge 0}$, ${}_M D_\phi^n f(z) = f^{(n)}(z)$.*

3.4. Comparison of Liouville's fI and its analytic continuations

In Table 2, we summarize the domains of ν in which ${}_L D_\phi^\nu f(z)$ and its analytic continuations exist and are analytic as a function of ν. The rows for ${}_L D_\phi^\nu f(z)$ and ${}_H D_\phi^\nu f(z)$ are due to Lemmas 3.2 and 3.3, respectively. The row for ${}_M D_\phi^\nu f(z)$ for $s_2[f] < \infty$ is due to Lemma 3.4 and Theorem **3.3**.

Let $f(z) = e^{-az}$ for $a \in \mathbb{C}$ satisfying $\mathrm{Re}\,(ae^{i\phi}) > 0$. Then $s_1[f] = -\infty$ and $s_2[f] = \infty$. By using (3.8), Table 2 and Theorem **3.4**, we obtain the column for e^{-az} in Table 3.

	$s_2[f] < \infty$	$s_2[f] = \infty$
${}_L D^\nu_\phi f(z)$	$s_1[f] < \operatorname{Re}\nu < 0$	$s_1[f] < \operatorname{Re}\nu < 0$
${}_H D^\nu_\phi f(z)$	$\operatorname{Re}\nu > s_1[f]$	$\operatorname{Re}\nu > s_1[f]$
${}_M D^\nu_\phi f(z)$	$\nu \in \mathbb{C}$, $\nu \notin \{n \in \mathbb{Z}_{<0} \vert n \le s_1[f]\}$, $\nu \notin \{\nu' \in \mathbb{C} \vert \operatorname{Re}\nu' \le s_1[f],\ b - \nu' \in \mathbb{Z}\}$	$-----$

TABLE 2. Domains of ν in which ${}_L D^\nu_\phi f(z)$, ${}_H D^\nu_\phi f(z)$ and ${}_M D^\nu_\phi f(z)$ exist and are analytic.

	$f(z) = e^{-az}$		$f(z) = z^a$	
	$s_1[f] = -\infty$, $s_2[f] = \infty$		$s_1[f] = s_2[f] = \operatorname{Re} a$	
	Domain of ν	Domain of a	Domain of ν	Domain of a
${}_L D^\nu_\phi f(z)$	${}_-\mathbb{C}$	$ae^{i\phi} \in {}_+\mathbb{C}$	$\operatorname{Re} a < \operatorname{Re}\nu < 0$	${}_-\mathbb{C}$
${}_H D^\nu_\phi f(z)$	$\mathbb{C}$	$ae^{i\phi} \in {}_+\mathbb{C}$	$\operatorname{Re}\nu > \operatorname{Re} a$	$\mathbb{C}$
${}_M D^\nu_\phi f(z)$	${}_M D^\nu_\phi e^{-az} = 0$		$\mathbb{C}$, $a - \nu \notin \mathbb{Z}_{\ge 0}$	$\mathbb{C}$

TABLE 3. Domains of ν and a in which ${}_L D^\nu_\phi f(z)$, ${}_H D^\nu_\phi f(z)$ and ${}_M D^\nu_\phi f(z)$ exist and are analytic as functions of ν, for $f(z) = e^{-az}$, z^a.

Let $f(z) = z^a$ for $a \in \mathbb{C}$. Then $s_2[f] = s_1[f] = \operatorname{Re} a$, and in place of (3.9), in the case of $a \notin \mathbb{Z}_{\ge 0}$, we obtain

$$ {}_M D^\nu_\phi z^a = e^{-i\pi\nu} \frac{\Gamma(\nu - a)}{\Gamma(-a)} z^{a-\nu}, \qquad (\nu \in \mathbb{C},\ a \in \mathbb{C} \backslash \mathbb{Z}_{\ge 0}), \tag{3.17} $$

$$ {}_H D^\nu_\phi z^a = e^{-i\pi\nu} \frac{\Gamma(\nu - a)}{\Gamma(-a)} z^{a-\nu}, \qquad (\operatorname{Re}\nu > \operatorname{Re} a,\ a \in \mathbb{C} \backslash \mathbb{Z}_{\ge 0}). \tag{3.18} $$

Formula (3.17) is obtained with the aid of (3.14), and (3.18) follows from it by Theorem **3.3**. We note that ${}_M D^\nu_\phi z^a$ is uniquely determined by (3.17) for all $\nu \in \mathbb{C}$ if $a \notin \mathbb{Z}_{\ge 0}$.

In the case of $a = n \in \mathbb{Z}_{\ge 0}$, by using (3.17) in (3.16), we obtain

$$ {}_M D^\nu_\phi z^n = 0, \qquad \nu \in \mathbb{C} \backslash \mathbb{Z}, \tag{3.19} $$

$$ {}_M D^m_\phi z^n = \lim_{a' \to n} {}_M D^m_\phi z^{a'} = \begin{cases} \frac{n!}{(n-m)!} z^{n-m}, & (m \in \mathbb{Z},\ m \le n), \\ 0, & (m \in \mathbb{Z},\ m > n). \end{cases} \tag{3.20} $$

Then in particular, ${}_M D^0_\phi z^n = z^n$.

In Appendix B, a remark is given on nonstandard analysis related with this calculation.

By (3.19), (3.20) and Theorem **3.3**, we have

$$ {}_{H}D^{\nu}_{\phi} z^n = 0, \quad (n \in \mathbb{Z}_{\geq 0},\ \mathrm{Re}\,\nu > n). \tag{3.21}$$

Formulas (3.19) and (3.21) are in accordance with Theorem **3.4**.

Remark 3.1. When $n \in \mathbb{Z}_{\geq 0}$ and $m \in \mathbb{Z}$, in place of (3.20), we could adopt ${}_{\bar{M}}D^{m}_{\phi} z^n := \lim_{\nu\to m} {}_{M}D^{\nu}_{\phi} z^n$. Then by (3.19), we would get ${}_{\bar{M}}D^{m}_{\phi} z^n = 0$ and hence ${}_{\bar{M}}D^{0}_{\phi} z^n = 0$.

The column for z^a in Table 3 is obtained by using (3.9), (3.18), (3.21), (3.17), (3.19) and (3.20).

Remark 3.2. By definition of e^z, we have $e^{-az} = \sum_{k=0}^{\infty} \frac{1}{k!}(-az)^k$. By (3.8) and Table 3, ${}_{H}D^{\nu}_{\phi} e^{-az} = (-a)^{\nu} e^{-az}$ for $\mathrm{Re}\,(ae^{i\phi}) > 0$. On the other hand, by (3.21), we would have $\sum_{k=0}^{\infty}[\frac{1}{k!} \cdot {}_{H}D^{\nu}_{\phi}(-az)^k] = 0$ if there existed a $\nu \in \mathbb{C}$ satisfying $\mathrm{Re}\,\nu > k$ for all $k \in \mathbb{Z}_{\geq 0}$, but there exists no such a ν. If we ignored the condition $\mathrm{Re}\,\nu > n$ in (3.21), there would occur a conflict.

Let $f(z) = e^{-az}$. Then $s_1[f] = -\infty$ and $s_2[f] = \infty$. By Theorem **3.4**, we have ${}_{M}D^{\nu}_{\phi} e^{-az} = \sum_{k=0}^{\infty}[\frac{1}{k!} \cdot {}_{M}D^{\nu}_{\phi}(-az)^k] = 0$ for $\nu \in \mathbb{C}\backslash\mathbb{Z}$. When $\nu = m \in \mathbb{Z}_{\geq 0}$, we have ${}_{M}D^{m}_{\phi} e^{-az} = (-a)^m e^{-az} = \sum_{k=0}^{\infty}[\frac{1}{k!} \cdot {}_{M}D^{m}_{\phi}(-az)^k]$, by using Theorem **3.5** and (3.20). Theorem **3.3** states that ${}_{M}D^{\nu}_{\phi} e^{-az}$ and ${}_{H}D^{\nu}_{\phi} e^{-az}$ are not related analytically.

We find discussions on the consistency of analytic continuations of Liouville's fI of exponential function in [3, p. 360].

3.5. Remarks on Nishimoto's fDI

In a series of books [11, 12], Nishimoto gave a survey of his work on fractional calculus, where he defined the fractional differintegration (fDI), respresenting the fractional differentiation and integration as a whole, with the aid of a generalization of the Cauchy integral formula of differentiation, by (3.13), which is equivalent to (3.12),

He denote Nishimoto's fDI by $(f)_\nu$, which he defined as follows.

(i) When ${}_{H}D^{\nu}_{\phi} f(z)$ exists for $\phi = 0$ or $\phi = -\pi$, $(f)_\nu$ is equal to it.
(ii) When $s_1[f] \neq -\infty$ and the analytic continuation of ${}_{H}D^{\nu}_{\phi} f(z)$ as a function of ν for $\phi = 0$ or $\phi = -\pi$ exists, $(f)_\nu$ for $\nu \in \mathbb{C}\backslash\mathbb{Z}$ is equal to the analytic continuation.
(iii) When $m \in \mathbb{Z}_{>0}$, $(f)_{-m} = \lim_{\nu\to -m}(f)_\nu$.
(iv) When $m \in \mathbb{Z}_{\geq 0}$, $(f)_m = f^{(m)}(z)$.

As a consequence, we have $(e^{-az})_\nu = {}_HD^\nu_\phi e^{-az}$ by (3.8) and Table 3, for Re $(ae^{i\phi}) > 0$, and $(z^a)_\nu = {}_MD^\nu_\phi z^a$ given by (3.17) and $(z^n)_\nu = 0$ for $n \in \mathbb{Z}_{\geq 0}$ and for $\nu \in \mathbb{C}$ excluding $\nu = m$ satisfying $m \leq n$. Because of this choice, a consistency is lost as seen from Remark 3.2. Nishimoto's fractional calculus is very interesting in taking advantage of the analyticity and the index law, but there exist some inconsistencies to be clarified.

By (3.17) and (3.18), Nishimoto's fDI of $f(z) = z^a$ is given by $(z^a)_\nu = {}_MD^\nu_\phi z^a = {}_HD^\nu_\phi z^a$ for Re $(\nu - a) > 0$, and its analytic continuation to Re $(\nu - a) \leq 0$. We see that $(z^a)_\nu = {}_MD^\nu_\phi z^a$ for all ν, $a \in \mathbb{C}$, including $\nu = m \in \mathbb{Z}$ and $a = n \in \mathbb{Z}_{\geq 0}$.

Formula (3.19) shows that, if $n \in \mathbb{Z}_{\geq 0}$, ${}_MD^\nu_\phi z^n = 0$ for $\nu \in \mathbb{C}$ satisfying $n-\nu \notin \mathbb{Z}_{\geq 0}$. The singularities are isolated ones at $\nu = n, n-1, \cdots$. If we remove them, regarding them as removable singularities, we would obtain ${}_MD^\nu_\phi z^n = 0$ for all $\nu \in \mathbb{C}$. In the above definition, (iii) states that Nishimoto makes this choice at $\nu = -m$ for $m \in \mathbb{Z}_{>0}$, but he states that $(z^2)_{-1} = \lim_{\nu \to -1}(z^2)_\nu = \frac{1}{3}z^3$ in [12, p.47]. The result is consistent with (3.20), but the last equality does not hold.

4. Index Law of fD and ${}_HD^\nu_\phi g(t) = z^{-b}$

In this section, we use $D^n f(z)$ to represent $f^{(n)}(z)$ for $n \in \mathbb{Z}_{\geq 0}$. When $n = 0$ or $= 1$, we put $D^0 f(z) = f(z)$ and $D^1 f(z) = Df(z) = f'(z)$.

4.1. Index law of ${}_LD^{-\lambda}_\phi f(z)$

THEOREM **4.1**. *Let $\lambda, \kappa \in {}_+\mathbb{C}$, and let ${}_LD^{-\lambda-\kappa}_\phi f(z)$ exist. Then the index law ${}_LD^{-\lambda}_\phi[{}_LD^{-\kappa}_\phi f(z)] = {}_LD^{-\lambda-\kappa}_\phi f(z)$ holds.*

A customary proof is given at the end of Appendix A.

LEMMA 4.1. *If ${}_LD^{-\kappa}_\phi f(z)$ exists, ${}_LD^{-\kappa}_\phi f(z + t_l e^{i\phi}) \to 0$ as $t_l \to \infty$.*

This follows from Lemma 3.1 and Theorem **4.1**.

LEMMA 4.2. *If $n \in \mathbb{Z}_{>0}$ satisfies $-n > s_1[f]$, $D^n[{}_LD^{-n}_\phi f(z)] = f(z)$.*

P r o o f. When $n = 1$, this is confirmed by taking differentiation of ${}_LD^{-1}_\phi f(z) = -\int_z^{z+\infty\cdot e^{i\phi}} f(\zeta)d\zeta$, which is given by (3.6) for $\lambda = 1$, taking account of Assumption A. The same equation for $n > 1$ is then confirmed with the aid of Theorem **4.1**. □

4.2. Index law of ${}_H D_\phi^\nu f(z)$

We now adopt the following assumption.

ASSUMPTION B. Let $f^{(n)}(\zeta)$ for $n \in \mathbb{Z}_{>0}$ exist on $P_\phi(z)$. Then $s_1[f^{(n)}] \le s_1[f] - n$.

This implies that $t_l^{-s}|f^{(n)}(z+t_l e^{i\phi})| \to 0$ as $t_l \to \infty$ when $s > s_1[f] - n$.

LEMMA 4.3. *Let $n \in \mathbb{Z}_{>0}$, and let ${}_H D_\phi^\nu f(z)$ exist. Then ${}_H D_\phi^\nu f(z) = {}_H D_\phi^{\nu-n} f^{(n)}(z)$.*

P r o o f. We apply the partial integration to (3.12), and then we note that we can choose the boundary values of the integration to be 0 by Assumption B. □

THEOREM **4.2**. *Let $m \in \mathbb{Z}_{>0}$ satisfy $m > s_1[f]$, and $\nu \in \mathbb{C}$ satisfy $s_1[f] < \operatorname{Re}\nu < m$. Then*

$$ {}_H D_\phi^\nu f(z) = {}_L D_\phi^{\nu-m} f^{(m)}(z). \tag{4.1}$$

This is due to Lemma 4.3 and Theorem **3.1**.

LEMMA 4.4. *Let $n \in \mathbb{Z}_{\ge 0}$, and let $\phi \in \mathbb{R}$, $z \in \mathbb{C}$ satisfy $|\phi - \arg z| < \pi$. Then $s_1[z^n \log z] = n$, and*

$$ {}_H D_\phi^\nu (z^n \log z) = n!\, e^{-i\pi(\nu-n-1)}\Gamma(\nu-n) z^{-\nu+n}, \qquad \operatorname{Re}\nu > n. \tag{4.2}$$

P r o o f. When $\operatorname{Re}\nu > n$, by using Theorem **4.2**, we obtain ${}_H D_\phi^\nu (z^n \log z) = {}_L D_\phi^{\nu-n-1}[D^{n+1}(z^n \log z)] = {}_L D_\phi^{\nu-n-1}(n! z^{-1})$. By using (3.9) in the last member, we confirm (4.2). □

LEMMA 4.5. *If ${}_H D_\phi^\nu f$ exists, then ${}_H D_\phi^\nu f(z + t_l e^{i\phi}) \to 0$ as $t_l \to \infty$.*

This follows from Theorem **4.2** and Lemma 4.1.

THEOREM **4.3**. *Let $\phi \in \mathbb{R}$ and μ, $\nu \in \mathbb{C}$. If $\operatorname{Re}\nu > s_1[f]$ and $\operatorname{Re}(\mu + \nu) > s_1[f]$, then the index law*

$$ {}_H D_\phi^\mu [{}_H D_\phi^\nu f(z)] = {}_H D_\phi^{\mu+\nu} f(z) \tag{4.3}$$

holds. In particular, either if $s_1[f] < 0$ *and* $\mathrm{Re}\,\nu \le 0$*, or if* $\mathrm{Re}\,\nu > 0$ *and* $s_1[f] < -\mathrm{Re}\,\nu$*,*

$$ {}_H D_\phi^\nu[{}_H D_\phi^{-\nu} f(z)] = f(z). \tag{4.4}$$

P r o o f. By using Theorems **4.2** and **4.1** and Lemma 4.2, we obtain

$$\begin{aligned} {}_H D_\phi^\mu[{}_H D_\phi^\nu f(z)] &= {}_L D_\phi^{\mu-m} D^m[{}_L D_\phi^{\nu-n-m} D^{n+m} f(z)] \\ &= {}_L D_\phi^{\mu+\nu-n-m}[D^{n+m} f(z)] = {}_H D_\phi^{\mu+\nu} f(z), \end{aligned}$$

for $m = \max\{\lfloor \mathrm{Re}\,\mu \rfloor + 1, 0\}$ and $n = \max\{\lfloor \mathrm{Re}\,\nu \rfloor + 1, 0\}$. (4.4) is obtained from the index law (4.3) by using Theorem **3.2**. Here $\lfloor x \rfloor$ for $x \in \mathbb{R}$ denotes the greatest integer not greater than x. □

In [2], we find a proof of the index law (4.3) for $\mu, \nu \in {}_+\mathbb{C}$, where Campos stated that the case when $\nu \in \mathbb{Z}_{<0}$ is excluded in (4.3).

THEOREM **4.4**. *Either if* $s_1[f] < 0$ *and* $\mathrm{Re}\,\nu \le 0$*, or if* $\mathrm{Re}\,\nu > 0$ *and* $s_1[f] < -\mathrm{Re}\,\nu$*, then a particular solution of* ${}_H D_\phi^\nu g(z) = f(z)$ *is given by* $g(z) = {}_H D_\phi^{-\nu} f(z)$*.*

We confirm this by using (4.4).

DEFINITION 4.1. If $s_1[f] \le -1$, Liouville's fD for $\mathrm{Re}\,\nu \ge 0$ is usually defined by

$$ {}_L D_\phi^\nu f(z) = D^m[{}_L D_\phi^{\nu-m} f(z)], \tag{4.5}$$

if the righthand side exists, where $m = \lfloor \nu \rfloor + 1$.

LEMMA 4.6. *If* $s_1[f] \le -1$ *and* ${}_H D_\phi^\nu f(z)$ *exists, then* ${}_L D_\phi^\nu f(z)$ *defined by Definition 4.1 also exists, and* ${}_L D_\phi^\nu f(z) = {}_H D_\phi^\nu f(z)$ *for* $\mathrm{Re}\,\nu > s_1[f]$*.*

This is confirmed by using Theorem **4.3**, **3.2** and **3.1**.

4.3. Index law of ${}_M D_\phi^\nu f(z)$

LEMMA 4.7. *Let* $f(z)$ *satisfy Condition A or B for* $\gamma \notin \mathbb{Z}$*, and let* $g(z) = {}_M D_\phi^\nu f(z)$*. Then* $g(z)$ *satisfies the same condition with* γ *replaced by* $\gamma - \nu$*.*

P r o o f. If $f(ze^{2i\pi}) = e^{2i\pi\gamma} f(z)$, we confirm $g(ze^{2i\pi}) = e^{2i\pi(\gamma-\nu)} g(z)$ by using (3.15). □

Theorem **4.5**. *Let Condition A or Condition B (see Section* **6***) be satisfied for $\gamma \notin \mathbb{Z}$, let $s_2[f] < \infty$, and μ, $\nu \in \mathbb{C}$. If $\nu - \gamma \notin \mathbb{Z}$, $\mu + \nu - \gamma \notin \mathbb{Z}$ and $\mu, \nu, \mu + \nu \notin \mathbb{Z}_{<0}$, then the index law*

$$ {}_M D_\phi^\mu [{}_M D_\phi^\nu f(z)] = {}_M D_\phi^{\mu+\nu} f(z) \tag{4.6}$$

holds. In particular, if $\nu - \gamma \notin \mathbb{Z}$ and $\nu \notin \mathbb{Z}$,

$$ {}_M D_\phi^{-\nu} [{}_M D_\phi^\nu f(z)] \;=\; f(z), \tag{4.7}$$

$$ {}_M D_\phi^{-\nu} [{}_H D_\phi^\nu f(z)] \;=\; f(z), \qquad \text{if} \qquad \mathrm{Re}\,\nu > s_1[f]. \tag{4.8}$$

P r o o f. The index law is due to Lemmas 4.7, 3.4, Theorems **3.3** and **4.3**. (4.7) is due to Theorem **3.5**. (4.8) is due to Theorem **3.3**. □

Theorem **4.6**. *Let $f(z)$ satisfy Condition A or Condition B (see Section* **6***), and let $s_1[f] < 0$. Then a particular solution of ${}_H D_\phi^\nu g(z) = f(z)$ is given by $g(z) = {}_M D_\phi^{-\nu} f(z)$, if $s_2[g] < \infty$.*

P r o o f. The condition $s_1[f] < 0$ is required by Lemma 4.5. We replace f and γ by g and $\gamma + \nu$, respectively, in (4.8). □

4.4. Solution of ${}_H D_\phi^\nu g(t) = z^{-b}$

We study the problem of solving the linear equation:

$$ {}_H D_\phi^\nu g(z) = z^{-b}, \tag{4.9}$$

for $b \in \mathbb{C}$. When a solution exists, the solution $g(z)$ is given by a particular solution satisfying this equation, added by a complementary solution $h(z)$, which satisfies the homogeneous equation ${}_H D_\phi^\nu h(z) = 0$.

If $\mathrm{Re}\,\nu > 0$, by (3.21), a polynomial of degree $m = \lceil \mathrm{Re}\,\nu \rceil - 1$, $h(z) = p_m(z)$, satisfies ${}_H D_\phi^\nu h(z) = 0$. Here $\lceil x \rceil$ for $x \in \mathbb{R}$ denotes the least integer not less than x.

Theorem **4.7**. *A solution of* (4.9) *exists if and only if* $\mathrm{Re}\,b > 0$. *A particular solution $g_0(z)$ of* (4.9) *is then given by*

$$ g_0(z) = e^{i\pi\nu} \tfrac{\Gamma(b-\nu)}{\Gamma(b)} z^{\nu-b}, \qquad \text{if } \nu - b \notin \mathbb{Z}_{\geq 0}, \tag{4.10}$$

$$ g_0(z) = e^{i\pi(b-1)} \tfrac{1}{n!\,\Gamma(b)} z^n \log z, \qquad \text{if } \nu - b = n \in \mathbb{Z}_{\geq 0}. \tag{4.11}$$

These are confirmed by using (3.18) and Lemma 4.4, respectively.

We now give a derivation of (4.10) and (4.11) with the aid of Theorems **4.4** and **4.6**.

P r o o f. The condition Re $b > 0$ is required by Lemma 4.5. By Theorems **4.6** and (3.17), a particular solution of (4.9) is given by

$$g(z) = {}_M D_\phi^{-\nu} z^{-b} = e^{i\pi\nu} \frac{\Gamma(b-\nu)}{\Gamma(b)} z^{-b+\nu}, \qquad \text{if } \nu - b \notin \mathbb{Z}_{\geq 0}. \quad (4.12)$$

When Re $(\nu - b) < 0$, (4.10) is obtained by using $g(z) = {}_H D_\phi^{-\nu} z^{-b}$, which is due to Theorem **4.4** and (3.18). In deriving (4.11), we put $\nu - b = n + \epsilon$. Then (4.12) gives $g(z) = C(\epsilon) z^{n+\epsilon}$, where $C(\epsilon) = e^{i\pi\nu} \frac{\Gamma(-n-\epsilon)}{\Gamma(b)}$. When $\epsilon \sim 0$, we note that ${}_H D_\phi^{\nu} z^n = 0$ by (3.21), since Re $\nu =$ Re $b + n +$ Re $\epsilon > n$. Now as a particular solution of (4.9), we may adopt $g(z) = C(\epsilon) z^{n+\epsilon} - C(\epsilon) z^n$, so that we have

$$\begin{aligned} g(z) &= e^{i\pi\nu} \frac{\Gamma(-n-\epsilon)}{\Gamma(b)} (z^{n+\epsilon} - z^n) \\ &= -e^{i\pi(b+\epsilon)} \frac{\Gamma(1+\epsilon)\Gamma(1-\epsilon)}{\Gamma(b)\Gamma(n+1+\epsilon)} [z^n \log z + O(\epsilon)]. \end{aligned}$$

In the limit of $\epsilon \to 0$, we obtain (4.11). □

4.5. Index law of ${}_{RL}D_0^\nu f(x)$ and distribution theory

We denote the Heaviside step function by $H(x)$. When $f(x)$ is defined for $x \in \mathbb{R}_{>0}$, we assume that

$$f(x)H(x) = \begin{cases} 0, & x \in \mathbb{R}_{\leq 0}, \\ f(x), & x \in \mathbb{R}_{>0}. \end{cases} \quad (4.13)$$

For a function $f(x)$ satisfying $f(x)H(x) \in \mathcal{L}^1_{loc}(\mathbb{R})$, RLfI: ${}_{RL}D_0^{-\lambda} f(x)$ for $\lambda \in \mathbb{C}_+$ is defined by (2.1) and the fD of order $\nu \in \mathbb{C}$ satisfying Re $\nu \geq 0$ is ordinarily defined by ${}_{RL}D_0^\nu f(x) = D^m[{}_{RL}D_0^{\nu-m} f(x)]$, where $m = \lfloor \nu \rfloor + 1$. We now adopt this definition.

Lemma 4.8. *If ${}_C D_0^\nu f(x)$ defined by* (2.2) *exists, then ${}_{RL}D_0^\nu f(x)$ also exists, and ${}_{RL}D_0^\nu f(x) = {}_C D_0^\nu f(x)$.*

P r o o f. This is confirmed by using ${}_C D_0^\nu f(x) = D[{}_C D_0^{\nu-1} f(x)]$ which follows from (2.2). □

For RLfD, the index law does not always hold. In [5], the example given is $f(x) = 1$, when ${}_{RL}D_0^{-1}[{}_{RL}D_0^1 1] = 0 \neq {}_{RL}D_0^1[{}_{RL}D_0^{-1} 1] = 1$, and use of generalized function or distribution is mentioned.

In [10], we define the distributions in the space $\mathcal{D}'_R$ which is dual to the space $\mathcal{D}_R$ of infinitely differentiable functions in $\mathbb{R}$, having a support bounded on the right. Then distributions in $\mathcal{D}'_R$ are regular distributions which belong to $\mathcal{L}^1_{loc}(\mathbb{R})$ and have a support bounded on the left, and their derivatives. We denote the fD of a distribution $h(x)$ by $\tilde{D}^\nu h(x)$.

Let $f(x)H(x) \in \mathcal{L}^1_{loc}(\mathbb{R})$, and let s_0 be the greatest of $s \in \mathbb{R}$ such that ${}_{RL}D^\nu_0 f(x)$ is absolutely continuous on $\mathbb{R}$ for all ν satisfying $\mathrm{Re}\,\nu < s-1$. Then we denote this s_0 by $s_0[f]$. Then

$$\tilde{D}^\nu[f(x)H(x)] = [{}_{RL}D^\nu_0 f(x)]H(x), \qquad \mathrm{Re}\,\nu < s_0[f]. \tag{4.14}$$

If $f(x) = 1$, $s_0[f] = 1$ and we have $\tilde{D}^{-1}[\tilde{D}^1 H(x)] = \tilde{D}^{-1}\delta(x) = H(x)$. In the distribution theory, we have the index law $\tilde{D}^\mu[\tilde{D}^\nu h(x)] = \tilde{D}^{\mu+\nu}h(x)$. Hence if we use $\tilde{D}^\nu[f(x)H(x)]$ in place of ${}_{RL}D^\nu_0 f(x)$, the problem of violation of the index law does not occur.

5. fD of Cosine and Sine Functions

In this section, we consider ${}_LD^\nu_\phi f(z)$ for $\phi = 0$ and $\phi = -\pi$. For $\phi = 0$, we give ${}_WD^\nu_0 f(z)$, and then ${}_LD^\nu_0 f(z)$ is obtained by

$${}_LD^\nu_0 f(z) = e^{-i\pi\nu} \cdot {}_WD^\nu_0 f(z). \tag{5.1}$$

The equation given below for ${}_LD^\nu_\phi f(z)$ and ${}_WD^\nu_0 f(z)$ are valid for $\nu \in {}_-\mathbb{C}$. The corresponding equations where subscript H appears in place of L are valid for all $\nu \in \mathbb{C}$.

When $a \in {}_+\mathbb{C}$ and $x \in \mathbb{R}$, we have ${}_WD^\nu_0 e^{-ax} = a^\nu e^{-ax}$ and ${}_LD^\nu_{-\pi} e^{ax} = a^\nu e^{ax}$, by (3.8). Here we do not consider the fD of the sum $e^{ax} + e^{-ax}$.

Let $b, c, r, \psi \in \mathbb{R}_{>0}$, $x \in \mathbb{R}$, and let $b \pm ic = re^{\pm i\psi}$. Then we have

$$\begin{aligned}
&{}_WD^\nu_0 e^{-(b\pm ic)x} = r^\nu e^{-bx\mp i(cx-\nu\psi)}, \quad {}_LD^\nu_{-\pi} e^{(b\pm ic)x} = r^\nu e^{bx\pm i(cx+\nu\psi)},\\
&{}_WD^\nu_0 (e^{-bx}\cos cx) = r^\nu e^{-bx}\cos(cx-\nu\psi),\\
&{}_LD^\nu_{-\pi}(e^{bx}\cos cx) = r^\nu e^{bx}\cos(cx+\nu\psi).
\end{aligned} \tag{5.2}$$

${}_LD^\nu_\phi e^{\mp az}$ are given for $\mathrm{Re}\,(\pm ae^{i\phi}) > 0$. The case where $\pm ae^{i\phi}$ is pure imaginary, is excluded from our study, because of Assumption B. In that case, we can consider the limit of $\mathrm{Re}\,(\pm ae^{i\phi}) \to 0+$. By taking the limit of $b \to 0$ in (5.2), we obtain

$${}_WD^\nu_0 \cos cx = c^\nu \cos(cx - \frac{\pi}{2}\nu), \quad {}_LD^\nu_{-\pi}\cos cx = c^\nu\cos(cx + \frac{\pi}{2}\nu), \tag{5.3}$$

and these equations with cos replaced by sin.

The above derivation of the two expressions in (5.3) follows Liouville's argument for $\nu \in {}_-\mathbb{C}$, given in Lützen's review [8, p.327].

In the original paper by Weyl [18, 17], the fD is given by $W^\nu f(x) = {}_WD^\nu_0 f(x)$, and it is applied to a periodic function of period 1 of the form $f(x) = \sum_{n=-\infty}^\infty c_n e^{2\pi inx}$, and its fD is given by $W^\nu f(t) = \sum_{n=-\infty}^\infty (2\pi in)^\nu c_n \times e^{2\pi int}$, when $c_0 = 0$ and $\sum_{n=-\infty}^\infty |n^\nu c_n|^2$ converges. We can confirm this in a similar way as above. In [1, p. 426], a proof of the above formula is given for the case when $W^{\nu}f(t)$ is defined by $W^{\nu}f(t) = {}_{L}D^{\nu}_{-\pi}f(t)$.

6. Remarks on the Definition of ${}_{M}D^{\nu}_{\phi}f(z)$

In Section **3.3**, ${}_{M}D^{\nu}_{\phi}f(z)$ is defined for $f(z)$ satisfying Condition A or B. Here we give the latter condition and also another condition.

We now assume that $f_1(z)$ is analytic on a neighborhood of $P_{\phi}(z)$, and is a one-valued analytic function on a neighborhood of infinity, except a possible isolated singularity at infinity.

We choose $\tilde{C}$ shown in Fig. 5 such that $f_1(z)$ is analytic in the regions given by $\min(X, Y) \le |z| < \infty$ and $0 < |z| \le \max(\delta, \delta_1)$, and also in a region enclosing the pathes a, b, c and d.

Condition B. $f(z)$ is expressed as $f(z) = z^{\gamma} f_1(z)$ where $\gamma \in \mathbb{R}$, so that $f(ze^{2i\pi}) = e^{2i\pi\gamma} f(z)$.

Condition C. There exists an $m \in \mathbb{Z}_{>0}$ for which $f^{(m)}(z)$ satisfies Condition B.

Definition 6.1. Let ${}_{M}D^{\nu}_{\phi}f(z)$ defined by Definition 3.4 with Condition B be denoted by ${}_{\tilde{M}}D^{\nu}_{\phi}f(z)$. Then when Condition C is satisfied, we put ${}_{M}D^{\nu}_{\phi}f(z) = {}_{\tilde{M}}D^{\nu-m}_{\phi}f^{(m)}(z)$.

The following lemma and two theorems follow from Lemma 3.4, Theorems **3.3** and **3.5**, respectively.

Lemma 6.1. *${}_{M}D^{\nu}_{\phi}f(z)$ defined by Definition 6.1 is analytic as a function of $\nu \in \mathbb{C}$, if $\nu - m \notin \mathbb{Z}_{<0}$ and $\gamma - \nu \notin \mathbb{Z}$.*

Theorem **6.1**. *For ${}_{M}D^{\nu}_{\phi}f(z)$ defined by Definition 6.1, Theorem* **3.3** *holds valid.*

THEOREM **6.2**. *Let ${}_MD^\nu_\phi f(z)$ defined by Definition 6.1 exist for $n \in \mathbb{Z}$ satisfying $n \geq m$. Then ${}_MD^n_\phi f(z) = f^{(n)}(z)$.*

LEMMA 6.2. *Let $n \in \mathbb{Z}_{\geq 0}$, and let $\phi \in \mathbb{R}$ and $z \in \mathbb{C}$ satisfy $|\phi - \arg z| < \pi$. Then $s_2[z^n \log z] = n$, and for $\nu \in (\mathbb{C}\backslash\mathbb{Z}) \cup \{m \in \mathbb{Z} | m > n\}$,*

$$ {}_MD^\nu_\phi(z^n \log z) = n!\, e^{-i\pi(\nu-n-1)}\Gamma(\nu - n) z^{-\nu+n}. \tag{6.1}$$

This follows from Theorems **4.2**, Lemma 4.4 and Theorem **6.1**.

Appendix A: Proofs of Theorems 3.1 ∼ 3.4 and 4.1

Let $\delta,\ r,\ R \in \mathbb{R}_{>0}$ satisfy $\delta \leq r \leq R$, and

$$I_{\delta,R} = \int_\delta^R \eta^{\lambda-1} f(z + \eta e^{i\phi}) d\eta, \quad J_r = \int_0^{2\pi} (re^{i\theta})^\lambda f(z + re^{i\theta} e^{i\phi}) i d\theta. \tag{A.1}$$

We use I_{H^*} to denote the integral which is obtained from the one in (3.12) for $\nu = -\lambda$, by replacing C_H by C_{H^*} shown in Fig. 3. Then

$$I_{H^*} = -I_{\delta,X} + J_\delta + e^{2\pi i\lambda} I_{\delta,Y}. \tag{A.2}$$

Let I_L be the integral in (3.6), and I_H and $\tilde{I}$ be those in (3.12) and (3.15), respectively, for $\nu = -\lambda$. Hence

$$I_L = \lim_{\delta\to 0} \lim_{X\to\infty} I_{\delta,X}, \quad I_H = \lim_{X\to\infty} \lim_{Y\to\infty} I_{H^*}. \tag{A.3}$$

P r o o f o f Theorem **3.1**. When I_L exists, by taking the limits of $\delta \to 0$, $X, Y \to \infty$ in (A.2), we obtain

$$I_H = -I_L + e^{2i\pi\lambda} I_L = e^{i\pi\lambda} \cdot 2i \cdot \sin(\pi\lambda) \cdot I_L. \tag{A.4}$$

We then use $\sin(\pi\lambda) = \frac{\pi}{\Gamma(\lambda)\Gamma(1-\lambda)}$. Note that this relation between I_L and I_H is the same as the relation (3.11) between the integrals in (3.4) and (3.11) for $\lambda = z$. □

P r o o f o f Theorem **3.2**. Now $\nu = -\lambda = n \in \mathbb{Z}_{\geq 0}$. When I_H exists, by taking the limits of $X, Y \to \infty$ in (A.2), we obtain

$$I_H = J_\delta, \quad J_\delta = \frac{2\pi i}{n!} e^{i\phi n} f^{(n)}(z), \tag{A.5}$$

We use these in (3.12). □

The contour $\tilde{C}$ shown in Fig. 5 consists of $\{a, \infty^+, b, 0^+, c, \infty^-, d, 0^-\}$, in this order. We divide the contour into two parts

$$\tilde{C}^{(1)} := \{\infty^-, d, 0^-, a\}, \quad \tilde{C}^{(2)} := \{\infty^+, b, 0^+, c\}. \tag{A.6}$$

Their contributions are

$$\tilde{I}^{(1)} := e^{-2\pi i\gamma} J_Y - I_{H^*}, \quad \tilde{I}^{(2)} := e^{-2\pi i\lambda - 2\pi i\gamma}(-J_X + I_{H^*}). \tag{A.7}$$

P r o o f o f Theorem **3.3**. From (A.7), we have

$$\tilde{I} = -(1 - e^{-2\pi i\lambda - 2\pi i\gamma})I_{H^*} - e^{-2\pi i\lambda - 2\pi i\gamma}J_X + e^{-2\pi i\gamma}J_Y. \quad (A.8)$$

When $s_2[f] < \infty$ and I_H exists, in the limit of $X, Y \to \infty$, we have

$$\tilde{I} = -(1 - e^{-2\pi i\lambda - 2\pi i\gamma})I_H = -2ie^{-i\pi\lambda - i\pi\gamma}\sin(\pi(\lambda + \gamma))I_H. \quad (A.9)$$

We then use $\sin(\pi\lambda) = \frac{\pi}{\Gamma(\lambda)\Gamma(1-\lambda)}$. When I_L exists, we have a relation between I_L and $\tilde{I}$ by using (A.4) in the righthand side of (A.9). Note that this relation between I_L and $\tilde{I}$ is the same as the relation between the integrals in (3.10) and (3.14) if $\gamma = -\lambda - \kappa$. □

LEMMA A.1. *Let $f(z)$ be an entire function. Then*
(i) $\tilde{I} = \tilde{I}^{(1)} = \tilde{I}^{(2)} = 0$, *and* (ii) *if* $s_2[f] < \infty$ *and* $\mathrm{Re}\,\nu > s_2[f]$, $I_H = 0$.

P r o o f. In (A.6), the contour is divided into two. In the domain enclosed by either of them, the integrand of the integral $\tilde{I}$ is analytic. Hence we have (i). By (A.7), $\tilde{I}^{(1)} = J_Y - I_{H^*} = 0$. We conclude (ii) from this. □

P r o o f o f Theorem **3.4**. In this case, we use (3.16) for $n = 0$. We then note that $z^\gamma f(z)$ converges uniformly to $f(z)$ on $\tilde{C}$ as $\gamma \to 0$, and that, when $\gamma = 0$, $\tilde{I} = 0$, as shown in (i) of Lemma A.1. (ii) follows from (i) and Theorem **3.3** or from (ii) of Lemma A.1 and Lemma 3.3. □

P r o o f o f Theorem **3.5**. Now $\nu = -\lambda = n \in \mathbb{Z}_{\geq 0}$. We first assume that Condition A or B is satisfied and $\gamma \notin \mathbb{Z}$. Then by putting $X = Y$ in (A.8) and (A.2), we obtain $I_{H^*} = J_\delta$ and

$$\tilde{I} = -(1 - e^{-2\pi i\gamma})J_\delta. \quad (A.10)$$

Using this with J_δ given in (A.5), in (3.15), we finish the proof. When $r = m \in \mathbb{Z}$, we use (3.16) with n replaced by m. □

P r o o f o f Theorem **4.1**. Using (3.6), we write ${}_LD_\phi^{-\lambda}[{}_LD_\phi^{-\kappa}f(z)]$ as

$$e^{i(\phi+\pi)\lambda}\frac{1}{\Gamma(\lambda)}\int_0^\infty t^{\lambda-1}\Big[e^{i(\phi+\pi)\kappa}\frac{1}{\Gamma(\kappa)}\int_t^\infty (x-t)^{\kappa-1}f(z+xe^{i\phi})dx\Big]dt.$$

We confirm that this is equal to ${}_LD_\phi^{-\lambda-\kappa}f(z)$, by exchanging the order of integrations and using $\int_0^x t^{\lambda-1}(x-t)^{\kappa-1}dt = x^{\lambda+\kappa-1}B(\lambda,\kappa)$. □

Appendix B: Application of Nonstandard Analysis

Formula (3.17) shows that ${}_MD^{\nu}_{\phi}z^a$ is an analytic function of $\nu \in \mathbb{C}$ if $a \notin \mathbb{Z}_{\geq 0}$, with poles at ν satisfying $a - \nu \in \mathbb{Z}_{\geq 0}$, and the equation in (3.17) is valid for $a \notin \mathbb{Z}_{\geq 0}$. Regarding $n \in \mathbb{Z}_{\geq 0}$ as $n + \epsilon$ with ϵ satisfying $\epsilon \simeq 0$, we obtain (3.16) in Definition 3.4. This results in (3.20).

In nonstandard analysis [15], we consider the set of real numbers, which we denote by $\mathbb{R}^{n*}$, in addition to $\mathbb{R}$. In the set $\mathbb{R}^{n*}$, there exists the infinitesimal neighborhood of $a \in \mathbb{R}$, which we denote by $\mathbb{R}^{n*}_{\simeq a}$. If $x \in \mathbb{R}^{n*}_{\simeq a}$, $0 < |x - a| < \delta$ for all $\delta \in \mathbb{R}_{>0}$. We also consider the corresponding sets of complex numbers, $\mathbb{C}^{n*} = \{x + iy | x, y \in \mathbb{R}^{n*}\}$ and $\mathbb{C}^{n*}_{\simeq a} = \{a + x + iy | x, y \in \mathbb{R}^{n*}_{\simeq 0}\}$ for $a \in \mathbb{C}$. The fact that $a_1 \in \mathbb{R}^{n*}_{\simeq a}$ or $a_1 \in \mathbb{C}^{n*}_{\simeq a}$, is denoted by $a_1 \simeq a$, $a_1 - a \simeq 0$, and $a = \mathrm{St}[a_1]$. We assume that, if either $a \in \mathbb{R}^{n*}$ or $a \in \mathbb{C}^{n*}$ and $a \neq 0$, then $1/a \in \mathbb{R}^{n*}$ or $1/a \in \mathbb{C}^{n*}$, accordingly. When $a \simeq 0$, we say that a is infinitesimal and $1/a$ is unlimited (or, "illimited" as the notion used by Robert [15]).

We propose to adopt (3.16) in Definition 3.4. In the language of nonstandard analysis, it is stated as follows. We assume that (3.17) is used for $a \in \mathbb{C}^{n*} \backslash \mathbb{Z}$, and adopt the following definition.

Definition B.1. *For $n \in \mathbb{Z}_{\geq 0}$ and $\nu \in \mathbb{C}$, we define ${}_MD^{\nu}_{\phi}z^n$ by ${}_MD^{\nu}_{\phi}z^n = \mathrm{St}[{}_MD^{\nu}_{\phi}z^a]$, if $a \simeq n$.*

References

[1] P.L. Butzer and R.J. Nessel, *Fourier Analysis and Application, Vol. I: One-Dimensional Theory.* Birkhäuser Verlag, Basel (1971).

[2] L.M.B.C. Campos, On a Concept of Derivative of Complex Order with Applications to Special Functions. *IMA J. Appl. Math.* **33** (1984), 109–133.

[3] L.M.B.C. Campos, Rules of derivation with complex order for analytic and branched functions. *Portugaliae Mathematica* **43** (1986), 347–376.

[4] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, *Theory and Applications of Fractional Differential Equations.* Elsevier, Amsterdam (2006).

[5] J.L. Lavoie, R. Tremblay and T.J. Osler, Fundamental properties of fractional derivatives via Pochhammer integrals. In: *Fractional Calculus and its Applications* (Ed. B. Ross), *Lecture Notes in Mathematics* **457** (1975), 327–356.

[6] J.L. Lavoie, T.J. Osler and R. Tremblay, Fractional derivatives and special functions. *SIAM Review* **18** (1976), 240–268.

[7] J. Liouville, Mémoire sur quelque Question de Géométrie et de Mécanique, et sur un genre de calcul pour résoudre ces questions. *J. de L'Ecole Polyt.* **13** (1832), 1–69.

[8] J. Lützen, *Joseph Liouville 1809–1882: Master of Pure and Applied Mathematics*. Springer-Verlag, New York (1990).

[9] K.S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*. John Wiley, New York (1993).

[10] T. Morita and K. Sato, Solution of fractional differential equation in terms of distribution theory, *Interdiscipl. Inform. Sc.* **12** (2006), 71–83.

[11] K. Nishimoto, *Fractional Calculus, I.* Descartes Press, Koriyama (1989).

[12] K. Nishimoto, *An Essence of Nishimoto's Fractional Calculus.* Descartes Press, Koriyama (1991).

[13] M.D. Ortigueira, *Fractional Calculus for Scientists and Engineers.* Springer, Dordrecht, Heidelberg etc. (2011); DOI: 10.1007/978-94-007-0747-7.

[14] I. Podlubny, *Fractional Differential Equations*. Academic Press, San Diego (1999).

[15] A.M. Robert, *Nonstandard Analysis.* Dover, Mineola (2003).

[16] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives, Theory and Applications*. Gordon and Breach Sc. Publ., Amsterdam (1993).

[17] K. Sato, Fractional calculus (Definitions and developments) − Connection with fractals (in Japanese). In: The 11th Symposium on History of Mathematics, *Reports of Institute of Mathematics and Computer Sciences, Tsuda College* **22** (2001), 22–51.

[18] H. Weyl, In: *Hermann Weyl Gesammelte Abhandlung, Band I.* Springer-Verlag, Berlin (1968), 663–669.

[19] E.T. Whittaker and G.N. Watson, *A Course of Modern Analysis.* Cambridge Univ. Press, Cambridge (1935).

[20] D.V. Widder, *The Laplace Transform.* Princeton Univ. Press, Princeton (1941).

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