VERSITA

Fractional Calculus & Applied Analysis
An International Journal for Theory and Applications
VOLUME 16, NUMBER 3 (2013)
(Print) ISSN 1311-0454
(Electronic) ISSN 1314-2224

RESEARCH PAPER

ON THE ASYMPTOTIC STABILITY OF LINEAR SYSTEM OF FRACTIONAL-ORDER DIFFERENCE EQUATIONS

Raghib Abu-Saris [1], Qasem Al-Mdallal [2]

Abstract

In this paper we investigate the stability of the equilibrium solution of the νth order linear system of difference equations

$$\left(\Delta^{\nu}_{a+\nu-1}\mathbf{y}\right)(t) = \Lambda\mathbf{y}(t+\nu-1); \quad t \in \mathbb{N}_a,\ a \in \mathbb{R},\ \text{and } \Lambda \in \mathbb{R}^{p\times p},$$

subject to the initial condition

$$\mathbf{y}(a+\nu-1) = \mathbf{y}_{-1},$$

where $0 < \nu \leq 1$ and $\mathbf{y}_{-1} \in \mathbb{R}^p$.

MSC 2010: 39A30, 39A05, 39A06, 39A10, 39B12, 26A33, 33B15

Key Words and Phrases: fractional-order difference equations, asymptotic stability, Volterra difference equations, special functions

1. Introduction

There is a voluminous literature on integer- and fractional-order differential equations. See, for instance, [2] – [6], [17], [19] and the book by Podlubny [20]. On the other hand, while there is a sizeable number of textbooks on integer-order difference equations such as [1, 14, 18], fractional-order difference equations are still in their infancy and seem recently to attract widely the attention of the researchers. Furthermore, as we shall see in Section **3**, fractional-order difference equations can be used to model

pp. 613–629 , DOI: 10.2478/s13540-013-0039-2

hereditary systems; i.e., dynamical systems in which the future state depends on the full history of the system states.

Let p be a positive integer, ν be a positive real number, $\mathbb{D}$ be a subset of real numbers, and $\mathbf{F} \in C\left[\mathbb{D}^p, \mathbb{D}^p\right]$; and consider the νth order system of difference equations

$$\left(\Delta^{\nu}_{a+\nu-m}\mathbf{y}\right)(t) = \mathbf{F}\big(\mathbf{y}(t+\nu-m)\big); \quad t \in \mathbb{N}_a, \tag{1.1}$$

subject to the initial condition

$$\Delta^k \mathbf{y}(a+\nu-m) = \mathbf{a}_k; \quad k = 0, 1, ..., m-1, \tag{1.2}$$

where ν is a positive real number, $m = \lceil \nu \rceil$, the ceiling of ν, and $\mathbf{a}_k \in \mathbb{D}^k$.

In the next Section **2** we present definitions and recall/develop a few preliminary results that will be utilized in subsequent sections. Our main results will be presented in Section **3** and supporting numerical examples are given in Section **4**. We conclude our paper in Section **5**, where we discuss and portray directions for future research.

2. Preliminaries

The following definition is standard ([7] - [12], [16]) and is the basis for defining differences of fractional order.

Definition **2.1**. Let $a \in \mathbb{R}$ and $\{\mathbf{y}(t)\}_{t\in\mathbb{N}_a}$ be a sequence of real vectors. Further, suppose ν is a positive real number. Then

$$\left(\Delta_a^{-\nu}\mathbf{y}\right)(t) = \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-s-1)^{(\nu-1)}\ \mathbf{y}(s); \quad t \in \mathbb{N}_{a+\nu}, \tag{2.1}$$

where $x^{(\nu)}$ is the falling factorial power ([18, p. 17], [14, pp. 49-50]) defined by

$$x^{(\nu)} = \frac{\Gamma(x+1)}{\Gamma(x-\nu+1)}$$

and

$$\Gamma(x) = \begin{cases} \displaystyle\int_0^\infty t^{x-1}\ e^{-t}\ dt & \text{if} \quad x > 0 \\[2ex] \dfrac{\Gamma(x+1)}{x} & \text{if} \quad x < 0 \quad \& \quad x \neq -1, -2, -3, ... \end{cases} .$$

Observe that if $n \in \mathbb{N}_0$, then

$$(\nu+n)^{(\nu)} = \frac{\Gamma(\nu+n+1)}{\Gamma(n+1)} = \frac{\Gamma(\nu)\displaystyle\prod_{j=0}^{n}(\nu+j)}{n!}. \tag{2.2}$$

Therefore,

$$\begin{aligned}
\left(\Delta_a^{-\nu}\mathbf{y}\right)(a+\nu+n) &= \frac{1}{\Gamma(\nu)}\sum_{s=a}^{a+n}(a+\nu+n-s-1)^{(\nu-1)}\ \mathbf{y}(s) \\
&= \frac{1}{\Gamma(\nu)}\sum_{s=0}^{n}(\nu+n-s-1)^{(\nu-1)}\ \mathbf{y}(a+s) \\
&= \sum_{s=0}^{n}\frac{\mathbf{y}(a+s)}{(n-s)!}\prod_{j=1}^{n-s}(\nu+j-1). \qquad (2.3)
\end{aligned}$$

In particular,

$$\begin{aligned}
\left(\Delta_a^{-\nu}\mathbf{y}\right)(a+\nu) &= \mathbf{y}(a), \\
\left(\Delta_a^{-\nu}\mathbf{y}\right)(a+\nu+1) &= \mathbf{y}(a+1)+\nu\,\mathbf{y}(a), \quad \text{and} \\
\left(\Delta_a^{-\nu}\mathbf{y}\right)(a+\nu+2) &= \mathbf{y}(a+2)+\nu\,\mathbf{y}(a+1)+\frac{\nu(\nu+1)}{2}\,\mathbf{y}(a).
\end{aligned}$$

The following result was established in [8] and [16, pp. 23-28]. However, the approach we follow here is different.

Lemma **2.1**. *If $a \in \mathbb{R}$, $\mu, \nu > 0$, then*

$$\Delta_{a+\mu}^{-\nu}(t-a)^{(\mu)} = \mu^{(-\nu)}(t-a)^{(\mu+\nu)} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)}(t-a)^{(\mu+\nu)}; \quad t \in \mathbb{N}_{a+\mu+\nu}. \tag{2.4}$$

P r o o f. By Eq. (2.3),

$$\begin{aligned}
\Delta_{a+\mu}^{-\nu}(t-a)^{(\mu)}\Big|_{t=t_n} &= \sum_{s=0}^{n}\frac{(s+\mu)^{(\mu)}}{(n-s)!}\prod_{j=1}^{n-s}(\nu+j-1) \\
&= \Gamma(\mu+1)\sum_{s=0}^{n}\frac{1}{(n-s)!}\prod_{j=1}^{n-s}(\nu+j-1)\ \frac{1}{s!}\prod_{j=1}^{s}(\mu+j) \\
&= \frac{\Gamma(\mu+1)}{n!}\sum_{s=0}^{n}\binom{n}{s}\prod_{j=1}^{n-s}(\nu+j-1)\ \prod_{j=1}^{s}(\mu+j),
\end{aligned}$$

where $t_n = a+\mu+\nu+n$. But

$$\prod_{j=1}^{n-s}(\nu+j-1) = (-1)^{n-s}\frac{d^{n-s}x^{-\nu}}{dx^{n-s}}\bigg|_{x=1} \quad \text{and} \quad \prod_{j=1}^{s}(\mu+j) = (-1)^{s}\frac{d^{s}x^{-\mu-1}}{dx^{s}}\bigg|_{x=1}.$$

Therefore,

$$\begin{aligned}
&\Delta_{a+\mu}^{-\nu}(t-a)^{(\mu)}\Big|_{t=t_n} \\
= \;& \frac{\Gamma(\mu+1)}{n!}\sum_{s=0}^{n}\binom{n}{s}(-1)^{n-s}\frac{d^{n-s}x^{-\nu}}{dx^{n-s}}\bigg|_{x=1}(-1)^{s}\frac{d^{s}x^{-\mu-1}}{dx^{s}}\bigg|_{x=1} \\
= \;& \frac{\Gamma(\mu+1)}{n!}(-1)^{n}\frac{d^{n}x^{-\nu-\mu-1}}{dx^{n}}\bigg|_{x=1} \\
= \;& \frac{\Gamma(\mu+1)}{n!}(\nu+\mu+1)\cdots(\nu+\mu+1+n-1) \\
= \;& \frac{\Gamma(\mu+1)}{\Gamma(n+1)}\frac{\Gamma(\nu+\mu+n+1)}{\Gamma(\nu+\mu+1)} \\
= \;& \frac{\Gamma(\mu+1)}{\Gamma(\nu+\mu+1)}(\nu+\mu+n)^{(\nu+\mu)} \\
= \;& \frac{\Gamma(\mu+1)}{\Gamma(\nu+\mu+1)}(t-a)^{(\nu+\mu)}\Big|_{t=t_n},
\end{aligned}$$

where $t_n = a+\mu+\nu+n$. □

Following the same lines of reasoning, we derive a formula for the fractional sum of $y(t)=(1+\gamma)^{t-a};\ t\in\mathbb{N}_{a+\nu},\ a,\gamma\in\mathbb{R}$ and $\nu>0$.

By Eq. (2.3),

$$\Delta_a^{-\nu}(1+\gamma)^{t-a}\big|_{t=a+\nu+n} = \sum_{s=0}^{n}\frac{(1+\gamma)^s}{(n-s)!}\prod_{j=1}^{n-s}(\nu+j-1).$$

But

$$(1+\gamma)^s = \frac{(-1)^s}{s!}\frac{d^s g(x)}{dx^s}\bigg|_{x=1};\ g(x) = \frac{1}{1+(1+\gamma)(x-1)}.$$

Therefore,

$$\begin{aligned}
\Delta_a^{-\nu}(1+\gamma)^{t-a}\big|_{t=a+\nu+n} &= \frac{(-1)^n}{n!}\sum_{s=0}^{n}\binom{n}{s}\frac{d^{n-s}x^{-\nu}}{dx^{n-s}}\bigg|_{x=1}\frac{d^s g(x)}{dx^s}\bigg|_{x=1} \\
&= \frac{(-1)^n}{n!}\frac{d^n\left(g(x)x^{-\nu}\right)}{dx^n}\bigg|_{x=1}.
\end{aligned}$$ □

As can be seen, the approach followed above is general and can be employed to derive a compact form for fractional sum. Indeed, we have

THEOREM **2.1**. *If $a \in \mathbb{R}$ and $\nu > 0$, then*

$$\Delta_a^{-\nu} y(t)\big|_{t=a+\nu+n} = \frac{(-1)^n}{n!} \left. \frac{d^n \left(g(x)x^{-\nu}\right)}{dx^n} \right|_{x=1},$$

where $g(x)$ is the generating function defined by

$$g(x) = \sum_{k=0}^{\infty} (-1)^k y(k+a)(x-1)^k; \quad |x-1| < R, \ R > 0.$$

Next, following [7, 11, 12], we define a Caputo-like fractional difference operator. Although a Riemann-Liouville-like definition is possible (see, for instance, [9, 13, 16]), we choose to define fractional differences in the Caputo sense because the constants will have zero fractional difference, among other properties. Indeed, the two definitions are related [7, Definition 6 and Remark 7].

DEFINITION **2.2**. Let $a \in \mathbb{R}$ and $\{\mathbf{y}(t)\}_{t \in \mathbb{N}_a}$ be a sequence of real vectors. Further, suppose ν is a positive real number. The νth order fractional difference of $\mathbf{y}$ in Caputo's sense is defined by

$$\left(\Delta_a^{\nu} \mathbf{y}\right)(t) = \Delta_a^{-(m-\nu)} \Delta^m \mathbf{y}(t); \quad t \in \mathbb{N}_{a+m-\nu}, \tag{2.5}$$

where $m - 1 < \nu \leq m$, and Δ^m is the whole-order difference operator defined by

$$\Delta^m \mathbf{y}(t) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} \mathbf{y}(t+m-k).$$

In particular, if $0 < \nu \leq 1$, then

$$\left(\Delta_a^{\nu} \mathbf{y}\right)(t) = \Delta_a^{-(1-\nu)} \Delta \mathbf{y}(t); \quad t \in \mathbb{N}_{a+1-\nu}. \tag{2.6}$$

For the sake of illustration, we present two examples. The fist is straight forward and its proof will be omitted.

LEMMA **2.2**. *If $\nu > 0$ and $a \in \mathbb{R}$, then*

$$\Delta_a^{\nu} \mathbf{c} = 0; \quad c \text{ is a constant vector.}$$

More generally,

$$\Delta_a^{\nu} \left(\sum_{j=0}^{k} t^j \mathbf{c_j} \right) = 0 \quad \text{if} \quad k < \lceil \nu \rceil, \ \text{a nonnegative integer.}$$

Lemma **2.3**. *If $a \in \mathbb{R}$, $\nu > 0$, and $\mu > m = \lceil \nu \rceil$; then*

$$\Delta^{\nu}_{a+\mu}(t-a)^{(\mu)} = \mu^{(\nu)}(t-a)^{(\mu-\nu)} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\nu+1)}(t-a)^{(\mu-\nu)}; \quad t \in \mathbb{N}_{a+\mu+m-\nu}, \tag{2.7}$$

where $m = \lceil \nu \rceil$.

Proof. First, since $\Delta(t-a)^{\mu} = \mu(t-a)^{(\mu-1)}$ [18, p. 19: Th. 2.3]; by induction, we have

$$\begin{aligned} \Delta^m(t-a)^{(\mu)} &= \mu(\mu-1)\cdots(\mu-(m-1))(t-a)^{(\mu-m)} \\ &= \frac{\Gamma(\mu+1)}{\Gamma(\mu-m+1)}(t-a)^{(\mu-m)} \end{aligned}$$

Now,

$$\begin{aligned} \Delta^{\nu}_{a+\mu}(t-a)^{(\mu)} &= \Delta^{-(m-\nu)}_{a+\mu}\Delta^m(t-a)^{(\mu)} \\ &= \frac{\Gamma(\mu+1)}{\Gamma(\mu-m+1)}\Delta^{-(m-\nu)}_{a+\mu}(t-a)^{(\mu-m)} \\ &= \frac{\Gamma(\mu+1)}{\Gamma(\mu-m+1)}\frac{\Gamma(\mu-m+1)}{\Gamma(\mu-\nu+1)}(t-a)^{(\mu-\nu)}; \quad t \in \mathbb{N}_{a+\nu-m+\mu}. \end{aligned}$$

□

The following two results were established in [8] and [16, pp. 28-29], and in [9] and [16, pp. 33-35], respectively. Both results play an important role in the sequel.

Theorem **2.2**. *Let $a \in \mathbb{R}$, $\mathbf{y} : \mathbb{N}_a \to \mathbb{R}^p$, and $\mu, \nu > 0$. Then,*

$$\Delta^{-\nu}_{a+\mu}\Delta^{-\mu}_a \mathbf{y}(t) = \Delta^{-\mu-\nu}_a \mathbf{y}(t) = \Delta^{-\mu}_{a+\nu}\Delta^{-\nu}_a \mathbf{y}(t); \quad t \in \mathbb{N}_{a+\mu+\nu}. \tag{2.8}$$

Theorem **2.3**. *For $\nu > 0$ and $m \in \mathbb{N}$, the following equality holds:*

$$\Delta^m\Delta^{-\nu}_a \mathbf{y}(t) = \Delta^{-\nu}_a\Delta^m \mathbf{y}(t) + \sum_{k=0}^{m-1} \frac{(t-a)^{(\nu-m+k)}}{\Gamma(\nu+k-m+1)}\, \Delta^k \mathbf{y}(a); \quad t \in \mathbb{N}_{a+\nu}. \tag{2.9}$$

Using the aforementioned theorems, we have:

$$\begin{aligned}
\mathbf{y}(t) &= \Delta^m \Delta_a^{-m} \mathbf{y}(t); \quad t \in \mathbb{N}_{a+m} \\
&= \Delta_a^{-m} \Delta^m \mathbf{y}(t) + \sum_{k=0}^{m-1} \frac{(t-a)^{(k)}}{\Gamma(k+1)} \, \Delta^k \mathbf{y}(a) \\
&= \Delta_{a+m-\nu}^{-\nu} \Delta_a^{-m+\nu} \Delta^m \mathbf{y}(t) + \sum_{k=0}^{m-1} \frac{(t-a)^{(k)}}{k!} \, \Delta^k \mathbf{y}(a) \\
&= \Delta_{a+m-\nu}^{-\nu} \Delta_a^{\nu} \mathbf{y}(t) + \sum_{k=0}^{m-1} \frac{(t-a)^{(k)}}{k!} \, \Delta^k \mathbf{y}(a) \\
&= \frac{1}{\Gamma(\nu)} \sum_{s=a+m-\nu}^{t-\nu} (t-s-1)^{(\nu-1)} \Delta_a^{\nu} \mathbf{y}(s) + \sum_{k=0}^{m-1} \frac{(t-a)^{(k)}}{k!} \, \Delta^k \mathbf{y}(a).
\end{aligned}$$

Thus, we have established a modified version ($a \in \mathbb{R}$) of Theorem 8, from [7].

THEOREM **2.4**. *Suppose* $\mathbf{y}$ *is defined on* $\mathbb{N}_a$ *with "a" a real number. Then, for* $\nu > 0$, *non-integer and* $m = \lceil \nu \rceil$,

$$\mathbf{y}(t) = \sum_{k=0}^{m-1} \frac{(t-a)^{(k)}}{k!} \Delta^k \mathbf{y}(a) + \frac{1}{\Gamma(\nu)} \sum_{s=a+m-\nu}^{t-\nu} (t-s)^{(\nu-1)} \Delta_a^{\nu} \mathbf{y}(s); \quad t \in \mathbb{N}_{a+m}.$$

3. Main Results

In a recent paper [11], Chen established several sufficient conditions for the attractivity/asymptotic stability of nonlinear difference systems of fractional order. However, while it is a mission next-to-impossible to establish necessary and sufficient conditions for general nonlinear systems, investigating the asymptotic behavior of nonlinear autonomous systems begins by linearizing about an equilibrium solution. As such, we focus our attention on the asymptotic stability of linear systems of difference equations of fractional order.

To this end, by Theorem **2.4**, the solution of Equation (1.1) is given by

$$
\begin{aligned}
\mathbf{y}(t) = \sum_{k=0}^{m-1} \frac{(t-a-\nu+m)^{(k)}}{k!} \Delta^k \mathbf{y}(a+\nu-m) \\
+ \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-s-1)^{(\nu-1)} \Delta^{\nu}_{a+\nu-m} \mathbf{y}(s); \quad t \in \mathbb{N}_{a+\nu} \\
= \sum_{k=0}^{m-1} \frac{(t-a-\nu+m)^{(k)}}{k!} \mathbf{a}_k + \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-s-1)^{(\nu-1)} \mathbf{F}\left(\mathbf{y}(s+\nu-m)\right).
\end{aligned}
$$

Thus, for $t = a+\nu+n,\ n = 0,1,2,...$, Eq. (3.1) simplifies to

$$
\begin{aligned}
&\mathbf{y}(a+\nu+n) \qquad (3.1) \\
&= \sum_{k=0}^{m-1} \mathbf{a}_k \frac{(n+m)^{(k)}}{k!} + \frac{1}{\Gamma(\nu)} \sum_{s=a}^{a+n} (a+\nu+n-s-1)^{(\nu-1)} \mathbf{F}\left(\mathbf{y}(s+\nu-m)\right) \\
&= \sum_{k=0}^{m-1} \mathbf{a}_k \frac{(n+m)^{(k)}}{k!} + \frac{1}{\Gamma(\nu)} \sum_{s=0}^{n} (\nu+n-s-1)^{(\nu-1)} \mathbf{F}\big(\mathbf{y}(a+s+\nu-m)\big) \\
&= \sum_{k=0}^{m-1} \mathbf{a}_k \frac{(n+m)^{(k)}}{k!} + \sum_{s=0}^{n} \frac{1}{(n-s)!} \prod_{k=1}^{n-s} (\nu+k-1)\ \mathbf{F}\big(\mathbf{y}(a+s+\nu-m)\big).
\end{aligned}
$$

Observe that Eq. (3.2) gives the solution iteratively. As such, $\mathbf{y}$ can be determined for any n as long as the previous values remain in the domain of $\mathbf{F}$. Furthermore, when $0 < \nu \leq 1$ and $\mathbf{F}(\mathbf{x}) = \Lambda \mathbf{x}$, Eq. (3.2) reduces to

$$
\begin{aligned}
\mathbf{y}(a+\nu+n) &= \mathbf{y}_{-1} + \Lambda \sum_{s=0}^{n} \frac{1}{(n-s)!} \left(\prod_{k=1}^{n-s} (\nu+k-1) \right) \mathbf{y}(a+\nu+s-1), \\
&= \mathbf{y}_{-1} + \sum_{s=0}^{n} B(n-s)\ \mathbf{y}(a+\nu+s-1), \qquad (3.2)
\end{aligned}
$$

for $n = -1, 0, 2, \cdots$. Here, the kernel $B(n-s)$, is given by

$$
B(n-s) = \frac{\Lambda}{(n-s)!} \prod_{k=1}^{n-s} (\nu+k-1).
$$

Notice that Eq. (3.2) is a nonhomogeneous Volterra difference equation of the convolution type. Moreover, the aforementioned equation will be used heavily in the sequel.

3.1. Scalar fractional-order difference equations

In this section, we focus our attention on the scalar case. With this in mind, let $y(a+\nu+s-1) = y_{s-1}$. Then, with $\Lambda = \lambda$, a scalar; Eq. (3.2) simplifies to

$$y_n = y_{-1} + \lambda \sum_{s=0}^{n} \frac{1}{(n-s)!} \left(\prod_{k=1}^{n-s} (\nu + k - 1) \right) y_{s-1}; \quad n = -1, 0, 2, ... \tag{3.3}$$

Furthermore, by critically examining Eq. (3.3), one can see that $y_n = p_n y_{-1}$, where p_n satisfies (3.3) with $p_{-1} = 1$. Thus, without loss of generality, we shall take $y_{-1} = 1$.

If $\lambda = 0$, then $y_n = y_{-1}$ for $n = 0, 1, ...$ Furthermore, if $\lambda > 0$, then $y_n > 1$ for $n = 0, 1, 2, ...$ Indeed, as the next lemma asserts, y_n diverges to infinity.

Lemma **3.1**. *Suppose $y_{-1} = 1$. Then the solution $\{y_n\}$, of Eq. (3.3) diverges to infinity.*

P r o o f. Since $y_n > 1$ for $n \geq 0$,

$$y_n > 1 + \lambda \sum_{s=0}^{n} \frac{1}{(n-s)!} \left(\prod_{k=1}^{n-s} (\nu + k - 1) \right) = 1 + \lambda \sum_{s=0}^{n} \frac{1}{s!} \left(\prod_{k=1}^{s} (\nu + k - 1) \right).$$

But

$$\frac{\prod_{k=1}^{s} (\nu + k - 1)/s!}{1/(\nu + s)} = \frac{\prod_{k=0}^{s} (\nu + k)}{s!} = \nu \frac{\prod_{k=1}^{s} (\nu + k)}{s!} > \nu.$$

Since $\prod_{k=1}^{s} (\nu + k)/s!$ is monotonic increasing; by the limit comparison test, the desired result follows. $\square$

Remark **3.1**. It is worth mentioning that

$$\sum_{s=0}^{n} \frac{1}{s!} \left(\prod_{k=1}^{s} (\nu + k - 1) \right) = \frac{1}{n!} \left(\prod_{k=1}^{n} (\nu + k) \right).$$

The above identity can be established by the principle of mathematical induction.

Hence, from now and on, we assume that $\lambda < 0$. To this end, let $\tilde{y}(z) = \mathcal{Z}(\{y_n\})$ be the unilateral z-transform of the sequence, $\{y_n\}$. Then

$$\tilde{y} = \left(1 - z^{-1}\right)^{-1} + \lambda \left(1 - z^{-1}\right)^{-\nu} \left(1 + z^{-1} \tilde{y}\right); \quad |z| > R \geq 1.$$

Solving for $\tilde{y}$ yields

$$\tilde{y} = \frac{\left(1 - z^{-1}\right)^{-1} + \lambda \left(1 - z^{-1}\right)^{-\nu}}{1 - \lambda z^{-1} \left(1 - z^{-1}\right)^{-\nu}} = \frac{\frac{z}{z-1} + \lambda \left(\frac{z}{z-1}\right)^{\nu}}{1 - \lambda \frac{1}{z} \left(\frac{z}{z-1}\right)^{\nu}}. \tag{3.4}$$

That says

$$y_n = \int_C z^{n-1} \tilde{y}(z)\, dz,$$

where C, see Figure 3.1, is any positively-oriented simple-closed contour in the analyticity region of $\tilde{y}$ that encircles all singular points of $\tilde{y}(z)$.

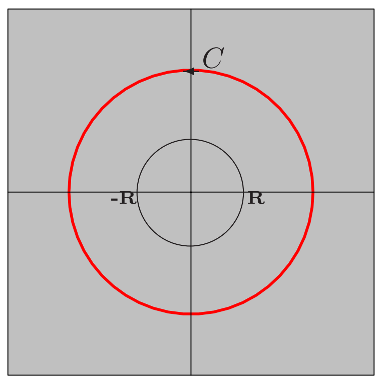
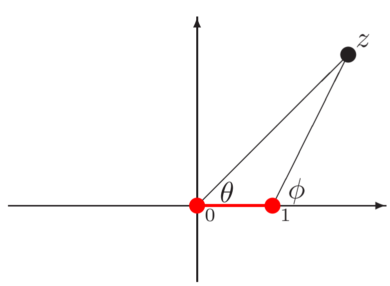


Fig. 3.1: Region of analyticity of $\tilde{y}$ and the contour C

The "function" $(z/(z-1))^{\nu} = z^{\nu}/(z-1)^{\nu}$ is multi-valued. As such, we introduce the branch cut shown in Figure 3.2.



$|z| > 0, \ -\pi \leq \theta < \pi$ and $|z-1| > 0, \ -\pi \leq \phi < \pi$

Fig. 3.2: Branch cut of $(z/(z-1))^{\nu}$

With that in mind, we consider the contour C_ρ, depicted in Figure 3.3 below. The inner circles are of radius ρ which we take it small enough so that all isolated singularities of $\tilde{y}$ are inside the C_ρ.

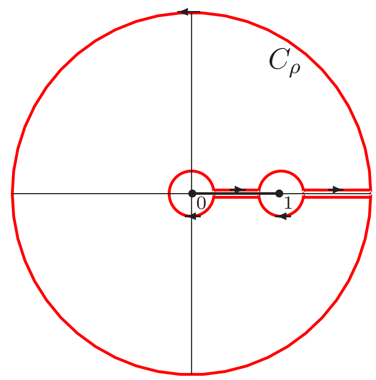


Fig. 3.3: The contour C_ρ

Since the line integrals of $z^{n-1}\tilde{y}(z)$ over the inner circles in Figure 3.3 tend to zero as ρ tends to zero, we have

$$\int_C z^{n-1}\ \tilde{y}(z)\ dz = \lim_{\rho\to 0}\int_{C_\rho} z^{n-1}\ \tilde{y}(z)\ dz.$$

But, by the Residue Theorem,

$$\int_{C_\rho} z^{n-1}\ \tilde{y}(z)\ dz = 2\pi i\sum_i \mathrm{Res}\left(z^{n-1}\tilde{y}(z), z_i\right) = 2\pi i\sum_i z_i^{n-1}\mathrm{Res}\left(\tilde{y}(z), z_i\right),$$

where the z_i's are the isolated singularities $\tilde{y}(z)$. Indeed, the singularities of $\tilde{y}(z)$ are simple poles given by the zeros of

$$Q(z) = 1 - \lambda\frac{1}{z}\left(\frac{z}{z-1}\right)^\nu.$$

To see this, suppose z_i is a zero of $Q(z)$. Then

$$\lambda\left(\frac{z_i}{z_i-1}\right)^\nu = z_i,$$

and so

$$Q'(z_i) = \frac{1-\nu-z_i}{z_i(1-z_i)} \neq 0$$

because $Q(1-\nu)\neq 0$ if $0<\nu<1$. Furthermore, with $P(z) = z/(z-1) + \lambda(z/(z-a))^\nu$,

$$P(z_i) = \frac{z_i}{z_i-1} + z_i \neq 0.$$

Finally, we arrive at the following capstone result.

THEOREM **3.1**. *Suppose* $0<\nu<1$ *and* $\lambda<0$. *Then the zero solution of Eq. (1.1) is asymptotically stable if and only if the isolated zeroes, off the nonnegative real axis, of* $Q(z)=1-\lambda\frac{1}{z}\left(\frac{z}{z-1}\right)^{\nu}$ *lie inside the unit disk.*

P r o o f. Given the branch depicted in Figure 3.2, let

$$z=|z|e^{i\theta} \quad \text{and} \quad z-1=|z-1|e^{i\phi},$$

where

$$|z|,|z-1|>0 \quad \text{and} \quad -\pi\le\theta,\ \phi<\pi.$$

That says,

$$\begin{aligned} 0=Q(z) &\Leftrightarrow \lambda\frac{|z|^{\nu-1}}{|z-1|^{\nu}}\ e^{-i[(1-\nu)\theta+\nu\phi]}=1\\ \Leftrightarrow\quad (1-\nu)\theta+\nu\phi=-\pi \quad &\& \quad \lambda\frac{|z|^{\nu-1}}{|z-1|^{\nu}}=-1. \end{aligned}$$

Furthermore, since $-\pi\le\theta,\ \phi<\pi$, then $\theta=\phi=-\pi$ and, consequently, $|z-1|=|z|+1$.

In view of the above argument, there are finitely many poles inside C_ρ and the residue of each is finite. Hence the result. □

COROLLARY **3.1**. *If* $\nu=1/2$, *then the zero solution of Eq. (1.1) is asymptotically stable if and only if* $-\sqrt{2}<\lambda<0$.

3.2. Systems of fractional-order difference equations

Following the same lines of reasoning employed in Subsection 3.1 and applying the z-transform to Eq. (3.2), one obtains

$$\tilde{\mathbf{y}}(z)=\left(1-z^{-1}\right)^{-1}\mathbf{y}_{-1}+\Lambda\left(1-z^{-1}\right)^{-\nu}\left(\mathbf{y}_{-1}+z^{-1}\tilde{\mathbf{y}}(z)\right).$$

Re-arranging the terms and solving for $\tilde{\mathbf{y}}(z)$, one gets

$$\tilde{\mathbf{y}}(z)=\left(I_p-z^{-1}\left(1-z^{-1}\right)^{-\nu}\Lambda\right)^{-1}\left(\left(1-z^{-1}\right)^{-1}+\left(1-z^{-1}\right)^{-\nu}\Lambda\right)\mathbf{y}_{-1},$$

where I_p is the identity matrix of order p.

Hence, analogously to Theorem **3.1**, we formulate the following fundamental result.

THEOREM **3.2**. *Suppose* $0<\nu<1$. *Then the zero solution of Eq. (1.1) is asymptotically stable if and only if the isolated zeroes, off the nonnegative real axis, of* $\det\left(I_p-z^{-1}\left(1-z^{-1}\right)^{-\nu}\Lambda\right)$ *lie inside the unit disk.*

Remark **3.2**. It is worth emphasizing that our results degenerate to the corresponding ones in the integer-order case. Indeed, it is well-known that the equilibrium solution (i.e., $x(0) = 0$) of $\Delta x(n) = \lambda x(n)$ is asymptotically stable if and only if $|1+\lambda| < 1$. Similarly, the equilibrium solution of $\Delta x(n) = \Lambda x(n)$ is asymptotically stable if and only if the eigenvalues of $I + \Lambda$ lie inside the unit disk.

On the other hand, if $\nu \to 1^-$, then the result of Theorem **3.1** simplifies to the isolated zeroes, off the nonnegative real axis,

$$Q(z) = 1 - \lambda \frac{1}{z}\left(\frac{z}{z-1}\right) = \frac{z-(1+\lambda)}{z-1}$$

lie inside the unit disk. Furthermore, the result of Theorem **3.2** simplifies to the isolated zeroes, off the nonnegative real axis, of

$$\begin{aligned} \det\left(I_p - z^{-1}\left(1-z^{-1}\right)^{-1}\Lambda\right) &= \det\left(I_p - \frac{1}{z-1}\Lambda\right) \\ &= \det\left(\frac{1}{z-1}\left[zI_p - (I_p+\Lambda)\right]\right) \end{aligned}$$

lie inside the unit disk.

An immediate consequence of Theorem **3.2** is the following corollary.

Corollary **3.2**. *If $\nu = 1/2$ and Λ is a triangular matrix with diagonal elements $\lambda_i,\ i = 1, .., p$, then the zero solution of Eq. (1.1) is asymptotically stable if and only if $-\sqrt{2} < \lambda_i < 0$ for $i = 1, ..., p$.*

Remark **3.3**. It is worth mentioning that the main results obtained in this section are parallel to those reported by Elaydi, in his progress report [15]. However, in his results Elaydi assumes that the kernel $\{B(n-s)\}$ belongs to ℓ^1. In other words, our results can be viewed as extensions of Elaydi's results.

4. Numerical Results

To demonstrate the results established in the previous section, we consider the following two examples.

Example **4.1**. Consider the following 1/2-order difference equation:

$$\left(\Delta^{1/2}_{a-1/2} y\right)(t) = \lambda y(t-1/2); \quad t \in \mathbb{N}_a,\ a \in \mathbb{R},\ \text{and}\ \lambda \in \mathbb{R}, \tag{4.1}$$

subject to the initial condition

$$y(a-1/2) = 1. \tag{4.2}$$

Recall that Corollary **3.1** asserts that the zero solution of Eq. (4.1) is asymptotically stable if $-\sqrt{2} < \lambda < 0$. This, as can be seen, is evident in Figure 4.1, which depicts the solution $\{y_n\}$ for different values of λ.

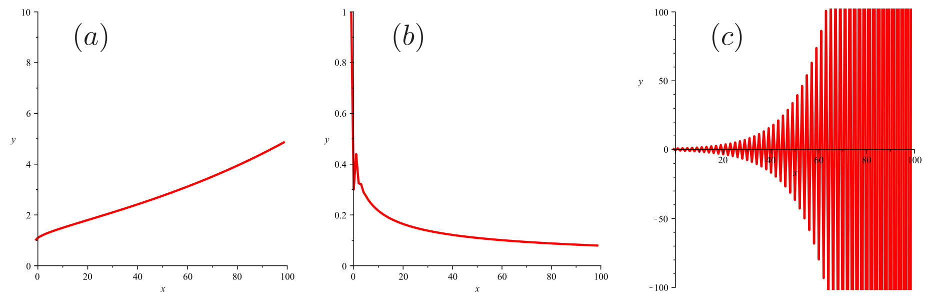


Fig. 4.1: The first 100 iterates of the solution $\{y_n\}$ of Example 4.1, for (a) $\lambda = 0.1$, (b) $\lambda = -0.7$, and (c) $\lambda = -1.5$

Example **4.2**. Consider the following 1/2-order system of difference equations:

$$\left(\Delta_{a-1/2}^{1/2}\mathbf{y}\right)(t) = \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_2 \end{bmatrix} \mathbf{y}(t-1/2); \quad t \in \mathbb{N}_a,\ a \in \mathbb{R},\ \text{and } \lambda_1, \lambda_2 \in \mathbb{R}, \tag{4.3}$$

subject to the initial condition

$$\mathbf{y}(a-1/2) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{4.4}$$

Recall that Corollary **3.2** asserts that the zero solution of Eq. (4.3) is asymptotically stable if $-\sqrt{2} < \lambda_1,\ \lambda_2 < 0$. This, for different values of λ, is confirmed in Figure 4.2 below. The figure depicts the 2-norm of the solution $\{\mathbf{y}_n\}$ for the first 100 iterates.

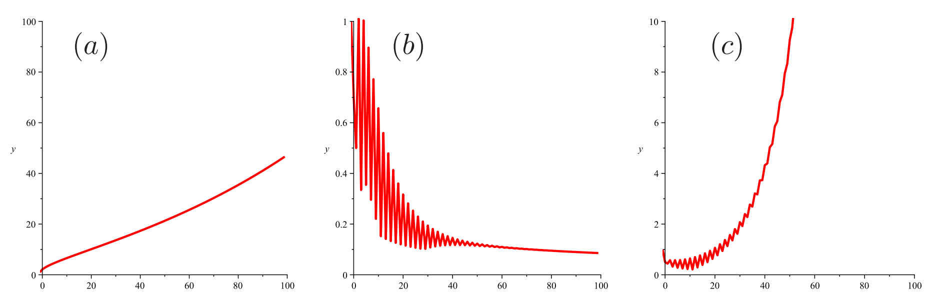


Fig. 4.2: The first 100 iterates of 2-norm of the solution $\{y_n\}$ of Example 4.2, for (a) $\lambda_1 = 0.1,\ \lambda_2 = 0.01$, (b) $\lambda_1 = -1.3,\ \lambda_2 = -1$, and (c) $\lambda_1 = -1.5,\ \lambda_2 = -0.7$

5. Conclusions

In this paper, we investigated the stability of the equilibrium solution of system of linear fractional-order difference equations of order ν, where $0 < \nu \leq 1$. To achieve this target, the well-known unilateral $z-$transform was successfully employed. Theorems which entail tractable necessary and sufficient conditions for the stability of the equilibrium solution were established. In addition, numerical examples that further support the established results were presented.

It is our hope that the results established here will ignite further interest in investigating the asymptotic behavior of (non-autonomous) linear and nonlinear difference equations of fractional order. Furthermore, while approaching fractional-order difference equations quantitatively has merit, we advocate qualitative analysis in studying the long term behavior of the solutions of such equations.

Acknowledgements

The authors would like to express their appreciation for the valuable comments of the reviewers which improved the exposition of this article.

References

[1] R.P. Agarwal, *Difference Equations and Inequalities* (2nd edition). Marcel Dekker, New York (2000).

[2] I. Ali, V. Kiryakova, S.L. Kalla, Solutions of fractional multi-order integral and differential equations using a Poisson-type transform. *J. of Mathematical Analysis and Applications* **269,** No 1 (2002), 172–199.

[3] Q.M. Al-Mdallal, An efficient method for solving fractional Sturm-Liouville problems. *Chaos, Solitons and Fractals* **40**, No 1 (2009), 183–189.

[4] Q.M. Al-Mdallal, On the numerical solution of fractional Sturm-Liouville problems. *International J. of Computer Mathematics* **87**, No 12 (2010), 2837–2845.

[5] Q.M. Al-Mdallal, M.I. Syam, M.N. Anwar, A collocation-shooting method for solving fractional boundary value problems. *Communications in Nonlinear Science and Numerical Simulation* **15** (2010), 3814–3822.

[6] Q.M. Al-Mdallal, M.I. Syam, An efficient method for solving non-linear singularly perturbed two points boundary-value problems of fractional order. *Communications in Nonlinear Science and Numerical Simulations* **17**, No 6 (2011), 2299–2308.

[7] G.A. Anastassiou, Discrete fractional calculus and inequalities. *arXiv:0911.3370v1, 17* (2009), 11 pages.

[8] F.M. Atici, P.L. Eloe, A transform method in discrete fractional calculus. *International J. of Difference Equations* **2** (2007), 165–176.

[9] F.M. Atici, P.L. Eloe, Initial value problems in discrete fractional calculus. *Proc. of the American Mathematical Society* **137**, No 3 (2009), 981–989.

[10] F.M. Atici, P.L. Eloe, Linear system of fractional nabla difference equations. *Rocky Mountain J. of Mathematics* **41**, No 2 (2011), 353–370.

[11] F. Chen, Fixed points and asymptotic stability of nonlinear fractional difference equations. *Electronic J. of Qualitative Theory of Differential Equations* **39** (2011), 1–18.

[12] F. Chen, X. Luo, Y. Zhou. Existence results for nonlinear fractional order difference equation. *Advances in Difference Equations* **2011** (2011), Article ID 713201, 12 pages.

[13] J.F. Cheng, Y.M. Chu, On the fractional difference equations of order $(2, q)$. *Advances in Difference Equations* **2011** (2011), Article ID 497259, 16 pages.

[14] S.N. Elaydi, *An Introduction to Difference Equations* (2nd edition). Springer-Verlag, New York (1999).

[15] S.N. Elaydi, Stability and asymptoticity of Volterra difference equations: A progress report. *Journal of Computational and Applied Mathematics* **228**, (2009), 504–513.

[16] M.T. Holm, *The Theory of Discrete Fractional Calculus: Development and Application*. Ph.D. Thesis, University of Nebraska - Lincoln (2011).

[17] R. Gorenflo, A. Iskenderov, Y. Luchko, Mapping between solutions of fractional diffusion-wave equations. *Fractional Calculus and Applied Analysis* **3**, No 1 (2000), 75–86.

[18] W.G. Kelley, A.C. Peterson, *Difference Equations: An Introduction with Applications* (2nd edition). Harcourt Academic Press, New York (2001).

[19] V. Kiryakova, A brief story about the operators of the generalized fractional calculus. *Fractional Calculus and Applied Analysis* **11**, No 2 (2008), 203–220; available at http://www.math.bas.bg/∼fcaa.

[20] I. Podlubny, *Fractional Differential Equations*. Academic Press, New York (1999).

[1] *Richard W. Riley College of Education and Leadership*
Walden University
155, 5th Ave S.
Minneapolis, 55401 MN – USA
e-mail: rabusaris@yahoo.com

[2] *Department of Mathematical Sciences, College of Science*
United Arab Emirates University
Al Ain, PO Box 17551, UAE
e-mail: q.almdallal@uaeu.ac.ae

Received: September 28, 2012
Revised: April 14, 2013

Please cite to this paper as published in:
Fract. Calc. Appl. Anal., Vol. **16**, No 3 (2013), pp. 613–629;
DOI:10.2478/s13540-013-0039-2