Fractional Calculus & Applied Analysis
An International Journal for Theory and Applications
VOLUME 16, NUMBER 3 (2013)
(Print) ISSN 1311-0454
(Electronic) ISSN 1314-2224

**RESEARCH PAPER**

# EXISTENCE OF SOLUTIONS TO INITIAL VALUE PROBLEMS FOR NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS ON THE SEMI-AXIS

**Tiberiu Trif**

## Abstract

The purpose of the paper is to investigate the global existence of solutions to initial value problems for nonlinear fractional differential equations on the semi-axis. More precisely, it deals with the initial value problem

$$\begin{cases} D^{\alpha}_{0+}x(t) = f(t, x(t)), \quad t \in (0, \infty), \\ \lim_{t\to 0+} t^{1-\alpha}x(t) = x_0, \end{cases} \tag{$*$}$$

where $0 < \alpha < 1$, $D^{\alpha}_{0+}$ denotes the Riemann-Liouville fractional derivative of order $\alpha$, and $f : (0, \infty) \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Unlike all the previous papers dealing with the problem of existence of solutions to (*), this problem is solved here by constructing a special locally convex space which is metrizable and complete. Then Schauder's fixed point theorem enables to provide sufficient conditions on $f$, ensuring that (*) possesses at least one solution. The growth conditions imposed to $f$ are weaker than other similar conditions already used in the literature.

*MSC 2010*: Primary 34A08; Secondary 26A33, 34A12

*Key Words and Phrases*: initial value problems, Riemann-Liouville fractional derivative, existence of solutions, semi-axis


pp. 595–612 , DOI: 10.2478/s13540-013-0038-3

## 1. Introduction

Let $\mathbb{R}^+ := (0, \infty)$ denote the set of all positive real numbers, let $\alpha$ be a real number such that $0 < \alpha < 1$, and let $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ be a continuous function. The purpose of our paper is to provide sufficient conditions on $f$ ensuring the existence of solutions to the nonlinear fractional differential equation

$$D^{\alpha}_{0+} x(t) = f(t, x(t)), \quad t \in \mathbb{R}^+, \tag{1.1}$$

where $D^{\alpha}_{0+}$ is the Riemann-Liouville fractional derivative of order $\alpha$. We are looking for solutions to (1.1) in the space $C_{1-\alpha}(\mathbb{R}^+)$, consisting of all functions $x \in C(\mathbb{R}^+)$ for which the limit $\lim_{t\to 0+} t^{1-\alpha} x(t)$ exists in $\mathbb{R}$. More precisely, we submit every solution $x$ of (1.1) to an initial condition of the form

$$\lim_{t\to 0+} t^{1-\alpha} x(t) = x_0. \tag{1.2}$$

We work under the following hypothesis on $f$:

(H). There exist $\beta \in \mathbb{R}$ such that $0 \leq \beta < \alpha$ as well as two functions $p, q : \mathbb{R}^+ \to [0, \infty)$ such that $p \in C_\beta(\mathbb{R}^+)$, $q \in C_{1-\alpha}(\mathbb{R}^+)$, and

$$|f(t, x)| \leq p(t)|x| + q(t) \quad \text{for all } (t, x) \in \mathbb{R}^+ \times \mathbb{R}. \tag{1.3}$$

We notice that our hypothesis (H) is weaker than that imposed by Kou, Zhou and Yan [3, Eq. (3.5)]. They worked under the assumption that $f$ satisfies (1.3), but required that $p$ and $q$ are bounded. In what follows we do not require the boundedness of neither $p$ nor $q$.

On the other hand, Zhao and Ge [7] worked in their paper under the assumption that $f$ satisfies an inequality of the form

$$|f(t, x)| \leq p(t) \omega \left( \frac{|x|}{1 + t^{\alpha-1}} \right).$$

Although $\omega$ can be an arbitrary continuous nondecreasing function, it is assumed that $p \in L^1([0, \infty))$. In what follows we do not require neither the integrability of $p$ over $[0, \infty)$.

We notice also that Kou, Zhou and Yan [3] did not work on the whole space $C_{1-\alpha}(\mathbb{R}^+)$, but only on its subspace defined by

$$E := \left\{ x \in C_{1-\alpha}(\mathbb{R}^+) \;\middle|\; \lim_{t\to\infty} \frac{t^{1-\alpha} x(t)}{1 + t^2} = 0 \right\}.$$

Endowed with an appropriate norm, $E$ becomes a Banach space. A similar remark is valid for the paper by Zhao and Ge [7], too. Unlike them, in our paper we are working on the whole space $C_{1-\alpha}(\mathbb{R}^+)$. We endow it with the topology induced by a sufficient family of seminorms. With respect to this topology, $C_{1-\alpha}(\mathbb{R}^+)$ becomes a Hausdorff locally convex space which is metrizable and complete.

## 2. Preliminaries

For the reader's convenience, in this section we briefly recall some basic facts on Riemann-Liouville fractional integrals and Riemann-Liouville fractional derivatives (see the monographs by Miller and Ross [4] or Podlubny [6] for further details).

Definition **2.1**. The Riemann-Liouville fractional primitive of order $\alpha$ of a function $x : \mathbb{R}^+ \to \mathbb{R}$ is defined by

$$I_{0+}^{\alpha} x(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s) ds,$$

provided the right side is pointwise defined on $\mathbb{R}^+$.

For instance, $I_{0+}^{\alpha} x(t)$ is defined for each $t \in \mathbb{R}^+$ whenever $x$ belongs to $C(\mathbb{R}^+) \cap L^1_{\text{loc}}(\mathbb{R}^+)$, where $L^1_{\text{loc}}(\mathbb{R}^+)$ is the space of all real-valued functions defined on $\mathbb{R}^+$ which are Lebesgue integrable over every bounded subinterval of $\mathbb{R}^+$.

Definition **2.2**. The Riemann-Liouville fractional derivative of order $\alpha$ of a function $x : \mathbb{R}^+ \to \mathbb{R}$ is defined by

$$D_{0+}^{\alpha} x(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} x(s) ds,$$

provided the right side is pointwise defined on $\mathbb{R}^+$.

Remark **2.1**. ([2, p. 611]) One has

$$D_{0+}^{\alpha} I_{0+}^{\alpha} x = x \quad \text{for all } x \in C(\mathbb{R}^+) \cap L^1_{\text{loc}}(\mathbb{R}^+).$$

Remark **2.2**. ([2, Proposition 2.4]) If $x \in C(\mathbb{R}^+) \cap L^1_{\text{loc}}(\mathbb{R}^+)$ is such that $D_{0+}^{\alpha} x \in C(\mathbb{R}^+) \cap L^1_{\text{loc}}(\mathbb{R}^+)$, then there exists a constant $c \in \mathbb{R}$ for which

$$I_{0+}^{\alpha} D_{0+}^{\alpha} x(t) = x(t) + ct^{\alpha-1} \quad \text{for all } t \in \mathbb{R}^+.$$

Remark **2.3**. If $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying the hypothesis (H), and $x \in C_{1-\alpha}(\mathbb{R}^+)$, then the function

$$\forall\, t \in \mathbb{R}^+ \;\longmapsto\; f(t, x(t)) \in \mathbb{R} \tag{2.1}$$

belongs to $C(\mathbb{R}^+) \cap L^1_{\text{loc}}(\mathbb{R}^+)$.

P r o o f. Indeed, clearly the function (2.1) belongs to $C(\mathbb{R}^+)$. By (1.3) we have

$$|f(t,x(t))| \le p(t)|x(t)| + q(t) \quad \text{for all } t \in \mathbb{R}^+.$$

We have $q \in C_{1-\alpha}(\mathbb{R}^+) \subset L^1_{\text{loc}}(\mathbb{R}^+)$. On the other hand, the function

$$\forall\, t \in \mathbb{R}^+ \;\mapsto\; p(t)|x(t)| \in \mathbb{R}$$

lies in $C_{1-(\alpha-\beta)}(\mathbb{R}^+) \subset L^1_{\text{loc}}(\mathbb{R}^+)$ because $p \in C_\beta(\mathbb{R}^+)$ and $x \in C_{1-\alpha}(\mathbb{R}^+)$. Consequently, the function (2.1) belongs to $L^1_{\text{loc}}(\mathbb{R}^+)$. □

Lemma **2.1**. *If $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying the hypothesis* (H), *and $x \in C_{1-\alpha}(\mathbb{R}^+)$, then*

$$\lim_{t\to 0+} t^{1-\alpha} \int_0^t (t-s)^{\alpha-1} f(s,x(s))ds = 0.$$

P r o o f. Let

$$M_1 := \sup_{s\in(0,1]} s^\beta p(s) s^{1-\alpha}|x(s)|, \quad M_2 := \sup_{s\in(0,1]} s^{1-\alpha} q(s),$$

and let $M := \max\{M_1, M_2\}$. For every $t \in (0,1]$ we have

$$\begin{aligned}
t^{1-\alpha} &\left| \int_0^t (t-s)^{\alpha-1} f(s,x(s))ds \right| \\
&\le t^{1-\alpha} \int_0^t (t-s)^{\alpha-1} \Big( p(s)|x(s)| + q(s) \Big) ds \\
&= t^{1-\alpha} \left( \int_0^t (t-s)^{\alpha-1} s^{\alpha-\beta-1} s^\beta p(s) s^{1-\alpha} |x(s)| ds \right. \\
&\qquad\qquad \left. + \int_0^t (t-s)^{\alpha-1} s^{\alpha-1} s^{1-\alpha} q(s) ds \right) \\
&\le M t^{1-\alpha} \left( \int_0^t (t-s)^{\alpha-1} s^{\alpha-\beta-1} ds + \int_0^t (t-s)^{\alpha-1} s^{\alpha-1} ds \right) \\
&= M t^{1-\alpha} \Big( t^{2\alpha-\beta-1} B(\alpha, \alpha-\beta) + t^{2\alpha-1} B(\alpha,\alpha) \Big) \\
&= M B(\alpha, \alpha-\beta)\, t^{\alpha-\beta} + M B(\alpha,\alpha)\, t^\alpha,
\end{aligned}$$

whence the conclusion comes. □

Lemma **2.2**. *Let $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the hypothesis* (H). *A function $x \in C_{1-\alpha}(\mathbb{R}^+)$ is a solution of the initial value problem* (1.1)–(1.2) *if and only if it is a solution of the Volterra* *integral equation*

$$x(t) = x_0 t^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s)) ds, \quad t \in \mathbb{R}^+. \tag{2.2}$$

P r o o f. Suppose first that $x \in C_{1-\alpha}(\mathbb{R}^+)$ satisfies (1.1)–(1.2). Then we have $x \in C(\mathbb{R}^+) \cap L^1_{\text{loc}}(\mathbb{R}^+)$, hence $D^\alpha_{0+} x \in C(\mathbb{R}^+) \cap L^1_{\text{loc}}(\mathbb{R}^+)$ due to (1.1) and Remark **2.3**. By virtue of Remark **2.2** we conclude that there exists a constant $c \in \mathbb{R}$ such that

$$x(t) = ct^{\alpha-1} + I^\alpha_{0+} D^\alpha_{0+} x(t) = ct^{\alpha-1} + I^\alpha_{0+} f(t, x(t)) \quad \text{for all } t \in \mathbb{R}^+. \tag{2.3}$$

By Lemma **2.1**, (1.2), and (2.3) we deduce that $c = x_0$, hence $x$ satisfies (2.2).

Conversely, suppose that $x \in C_{1-\alpha}(\mathbb{R}^+)$ satisfies (2.2). According to Remark **2.3**, the function (2.1) belongs to $C(\mathbb{R}^+) \cap L^1_{\text{loc}}(\mathbb{R}^+)$. Since $D^\alpha_{0+} t^{\alpha-1} = 0$, by (2.2) and Remark **2.1** it follows that

$$D^\alpha_{0+} x(t) = D^\alpha_{0+} I^\alpha_{0+} f(t, x(t)) = f(t, x(t)) \quad \text{for all } t \in \mathbb{R}^+,$$

hence $x$ satisfies (1.1). On the other hand, by (2.2) and Lemma **2.1** it follows that $x$ satisfies (1.2), too. □

## 3. The locally convex space $C_{1-\alpha}(\mathbb{R}^+)$

On the linear space $X := C_{1-\alpha}(\mathbb{R}^+)$ consider the family $P := (p_n)_{n \in \mathbb{N}}$ of semi-norms $p_n : X \to [0, \infty)$, defined by

$$p_n(x) := \sup_{t \in (0,n]} t^{1-\alpha} |x(t)|, \quad x \in X.$$

Let $\mathcal{T}$ denote the topology induced by $P$ on $X$. Then $(X, \mathcal{T})$ is a Hausdorff locally convex space. The family

$$\mathcal{B} := \{ B_{r,n} \mid r \in \mathbb{R}^+, \ n \in \mathbb{N} \}$$

consisting of balls

$$B_{r,n} := \{ x \in X \mid p_n(x) \le r \}$$

is a neighbourhood-base at the origin of $X$ with respect to $\mathcal{T}$. Moreover, since $\{ B_{1/m,n} \mid m, n \in \mathbb{N} \}$ is a countable neighbourhood-base at the origin of $X$, the topology $\mathcal{T}$ is metrizable. It can be proved that the topology $\mathcal{T}_\rho$ induced on $X$ by the metric $\rho$, defined by

$$\rho(x, y) := \sum_{n=1}^\infty \frac{1}{2^n} \frac{p_n(x-y)}{1 + p_n(x-y)}, \quad x, y \in X,$$

coincides with $\mathcal{T}$.

THEOREM **3.1**. *A sequence $(x_k)$ of functions in $X$ converges to* 0 *with respect to $\mathcal{T}$ if and only if it satisfies the following conditions:*

(i) $(x_k)$ *converges uniformly to* 0 *on every compact set $C \subset \mathbb{R}^+$;*

(ii) $\Big(t^{1-\alpha}x_k(t)\Big)_{k\in\mathbb{N}}$ *converges to* 0 *when $t \to 0+$ and $k \to \infty$, uniformly with respect to $t$, i.e., for every $\varepsilon > 0$ there exists $\delta > 0$ and $k_0 \in \mathbb{N}$ such that*

$$t^{1-\alpha}|x_k(t)| \le \varepsilon \quad \text{for all } k \ge k_0 \text{ and all } t \in (0,\delta). \tag{3.1}$$

P r o o f. *Necessity*. Suppose that $(x_k)$ is a sequence of functions in $X$, converging to 0 with respect to $\mathcal{T}$. In order to prove (i), let $C$ be a nonempty compact subset of $\mathbb{R}^+$, and let $a := \min C$, $b := \max C$. Then we have

$$0 < a \le b < \infty \quad \text{and} \quad C \subseteq [a,b].$$

Next let $\varepsilon > 0$, let $\varepsilon' := \varepsilon a^{1-\alpha}$, and let $n$ be a natural number such that $n \ge b$. Since $B_{\varepsilon',n}$ is a neighbourhood of 0 with respect to $\mathcal{T}$, there exists $k_0 \in \mathbb{N}$ such that $x_k \in B_{\varepsilon',n}$ for all $k \ge k_0$. Then for all $k \ge k_0$ and all $t \in C$ we have

$$a^{1-\alpha}|x_k(t)| \le t^{1-\alpha}|x_k(t)| \le p_n(x_k) \le \varepsilon',$$

whence

$$|x_k(t)| \le \varepsilon' a^{\alpha-1} = \varepsilon.$$

Therefore, $(x_k)$ converges uniformly to 0 on $C$.

In order to prove (ii), let $\varepsilon > 0$. Since $B_{\varepsilon,1}$ is a neighbourhood of 0 with respect to $\mathcal{T}$, there exists $k_0 \in \mathbb{N}$ such that $p_1(x_k) \le \varepsilon$ for all $k \ge k_0$. Then for all $k \ge k_0$ and all $t \in (0,1)$ we have

$$t^{1-\alpha}|x_k(t)| \le p_1(x_k) \le \varepsilon.$$

*Sufficiency*. Suppose now that $(x_k)$ is a sequence of functions in $X$ satisfying (i) and (ii). In order to prove that $(x_k)$ converges to 0, let $V$ be an arbitrary neighbourhood of 0 with respect to $\mathcal{T}$. Choose $\varepsilon > 0$ and $n \in \mathbb{N}$ such that $B_{\varepsilon,n} \subseteq V$. By (ii) it follows that there exists $\delta > 0$ and $k_0 \in \mathbb{N}$ such that (3.1) holds.

If $\delta > n$, then we have $p_n(x_k) \le \varepsilon$, hence $x_k \in B_{\varepsilon,n} \subseteq V$ for all $k \ge k_0$.

If $\delta \le n$, then set $C := [\delta, n]$ and $\varepsilon' := \varepsilon/n^{1-\alpha}$. By (i) it follows that there exists $k_1 \in \mathbb{N}$ such that

$$|x_k(t)| \le \varepsilon' \quad \text{for all } k \ge k_1 \text{ and all } t \in C.$$

From this inequality we deduce that

$$t^{1-\alpha}|x_k(t)| \le \varepsilon' t^{1-\alpha} \le \varepsilon' n^{1-\alpha} = \varepsilon \quad \text{for all } k \ge k_1 \text{ and all } t \in C. \tag{3.2}$$

By (3.1) and (3.2) we conclude that $p_n(x_k) \leq \varepsilon$, i.e., $x_k \in B_{\varepsilon,n} \subseteq V$ for all $k \geq \max\{k_0, k_1\}$. Consequently, $(x_k)$ converges to 0 with respect to $\mathcal{T}$. □

THEOREM **3.2**. *The metrizable locally convex space $(X, \mathcal{T})$ is complete.*

P r o o f. Let $(x_k)$ be a Cauchy sequence in $(X, \mathcal{T})$. We show first that $(x_k(t))$ is a Cauchy sequence of real numbers for every $t \in \mathbb{R}^+$ arbitrarily chosen. Let $t \in \mathbb{R}^+$, and let $\varepsilon > 0$. Set $\varepsilon' := \varepsilon t^{1-\alpha}$ and select a natural number $n$ such that $n \geq t$. Since $B_{\varepsilon',n}$ is a neighbourhood of 0 with respect to $\mathcal{T}$, there exists $k_0 \in \mathbb{N}$ such that $x_k - x_\ell \in B_{\varepsilon',n}$ for all $k, \ell \geq k_0$. Then for all $k, \ell \geq k_0$ we have

$$t^{1-\alpha}|x_k(t) - x_\ell(t)| \leq p_n(x_k - x_\ell) \leq \varepsilon',$$

whence $|x_k(t) - x_\ell(t)| \leq \varepsilon'/t^{1-\alpha} = \varepsilon$. In conclusion, $(x_k(t))$ is a Cauchy sequence in $\mathbb{R}$ for every $t \in \mathbb{R}^+$. Let $x : \mathbb{R}^+ \to \mathbb{R}$ be the function defined by

$$x(t) := \lim_{k\to\infty} x_k(t), \quad t \in \mathbb{R}^+.$$

We claim that

$$(x_k - x)_{k\in\mathbb{N}} \text{ converges uniformly to 0 on every compact set } C \subset \mathbb{R}^+. \tag{3.3}$$

Indeed, let $C$ be any nonempty compact subset of $\mathbb{R}^+$, and let $a := \min C$, $b := \max C$. Then we have

$$0 < a \leq b < \infty \quad \text{and} \quad C \subseteq [a, b].$$

Next let $\varepsilon > 0$, let $\varepsilon' := \varepsilon a^{1-\alpha}$, and let $n \in \mathbb{N}$ be such that $n \geq b$. Since $B_{\varepsilon',n}$ is a neighbourhood of 0 with respect to $\mathcal{T}$, there exists $k_0 \in \mathbb{N}$ such that $x_k - x_\ell \in B_{\varepsilon',n}$ for all $k, \ell \geq k_0$. Then for all $k, \ell \geq k_0$ and all $t \in C$ we have

$$a^{1-\alpha}|x_k(t) - x_\ell(t)| \leq t^{1-\alpha}|x_k(t) - x_\ell(t)| \leq p_n(x_k - x_\ell) \leq \varepsilon',$$

whence

$$|x_k(t) - x_\ell(t)| \leq \varepsilon'/a^{1-\alpha} = \varepsilon.$$

Letting $\ell \to \infty$ we conclude that

$$|x_k(t) - x(t)| \leq \varepsilon \quad \text{for all } k \geq k_0 \text{ and all } t \in C.$$

Therefore (3.3) holds, as claimed. In particular, by (3.3) it follows that $x \in C(\mathbb{R}^+)$.

Next we claim that

$$\begin{cases} \text{for every } \varepsilon > 0 \text{ there exists } k_0 \in \mathbb{N} \text{ such that} \\ t^{1-\alpha}|x_k(t) - x(t)| \leq \varepsilon \text{ for all } k \geq k_0 \text{ and all } t \in (0, 1). \end{cases} \tag{3.4}$$

Indeed, let $\varepsilon > 0$. Since $B_{\varepsilon,1}$ is a neighbourhood of 0 with respect to $\mathcal{T}$, there exists $k_0 \in \mathbb{N}$ such that $x_k - x_\ell \in B_{\varepsilon,1}$ for all $k, \ell \geq k_0$. Then we have

$$t^{1-\alpha}|x_k(t) - x_\ell(t)| \leq \varepsilon \quad \text{for all } k, \ell \geq k_0 \text{ and all } t \in (0,1).$$

Letting $\ell \to \infty$ we conclude that

$$t^{1-\alpha}|x_k(t) - x(t)| \leq \varepsilon \quad \text{for all } k \geq k_0 \text{ and all } t \in (0,1),$$

hence (3.4) holds.

Finally, we claim that $x \in C_{1-\alpha}(\mathbb{R}^+)$. In order to show that the limit $\lim_{t\to 0+} t^{1-\alpha}x(t)$ exists in $\mathbb{R}$ it suffices to show that for every $\varepsilon > 0$ there exist $a > 0$ and $\delta > 0$ such that the inequality

$$|t^{1-\alpha}x(t) - s^{1-\alpha}x(s)| \leq \varepsilon$$

holds for all $t, s \in (0, a)$ with $|t - s| < \delta$. So let $\varepsilon > 0$ arbitrarily chosen. By (3.4) it results that there exists $k_0 \in \mathbb{N}$ such that

$$t^{1-\alpha}|x_k(t) - x(t)| \leq \frac{\varepsilon}{3} \quad \text{for all } k \geq k_0 \text{ and all } t \in (0,1).$$

Select a natural number $k$ such that $k \geq k_0$. Since $\lim_{t\to 0+} t^{1-\alpha}x_k(t)$ exists in $\mathbb{R}$, there exist $a > 0$ and $\delta > 0$ such that

$$|t^{1-\alpha}x_k(t) - s^{1-\alpha}x_k(s)| \leq \frac{\varepsilon}{3} \quad \text{for all } t, s \in (0,a) \text{ with } |t - s| < \delta.$$

Without losing the generality we may assume that $a \leq 1$. Then for all $t, s \in (0,1)$ with $|t - s| < \delta$ we have

$$\begin{aligned}
&|t^{1-\alpha}x(t) - s^{1-\alpha}x(s)| \\
&\quad \leq t^{1-\alpha}|x(t) - x_k(t)| + |t^{1-\alpha}x_k(t) - s^{1-\alpha}x_k(s)| + s^{1-\alpha}|x_k(s) - x(s)| \\
&\quad \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

Consequently, the limit $\lim_{t\to 0+} t^{1-\alpha}x(t)$ exists in $\mathbb{R}$ and $x \in C_{1-\alpha}(\mathbb{R}^+)$, as claimed.

By (3.3) and (3.4), based on Theorem **3.1**, it follows that $(x_k - x)$ converges to 0, i.e., $(x_k)$ converges to $x$ with respect to $\mathcal{T}$. □

THEOREM **3.3**. *Let $Y$ be a subset of the metrizable locally convex space $(X, \mathcal{T})$ which satisfies the following conditions:*

(i) *$Y$ is pointwise bounded on $\mathbb{R}^+$;*
(ii) *$Y$ is equicontinuous on $\mathbb{R}^+$;*
(iii) *$Y$ is equiconvergent at $0+$, i.e., for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $y \in Y$ and all $t \in (0, \delta)$ one has*

$$\left| t^{1-\alpha}y(t) - \lim_{t\to 0+} t^{1-\alpha}y(t) \right| \leq \varepsilon.$$

*Then $Y$ is relatively compact in $(X, \mathcal{T})$.*

P r o o f. Let $\mathbb{R}_0^+ := [0, \infty)$. On the linear space $\widetilde{X} := C(\mathbb{R}_0^+)$ we consider the compact convergence topology $\widetilde{\mathcal{T}}$, induced by the family $\widetilde{P} := (\widetilde{p}_n)_{n\in\mathbb{N}}$ of semi-norms $\widetilde{p}_n : \widetilde{X} \to [0, \infty)$, defined by

$$\widetilde{p}_n(\widetilde{x}) := \max_{t\in[0,n]} |\widetilde{x}(t)|, \quad \widetilde{x} \in \widetilde{X}.$$

It is well-known that $(\widetilde{X}, \widetilde{\mathcal{T}})$ is a locally convex space which is metrizable and complete.

Associate with every function $x \in C_{1-\alpha}(\mathbb{R}^+)$ the function $\widetilde{x} : \mathbb{R}_0^+ \to \mathbb{R}$, defined by

$$\widetilde{x}(t) := \begin{cases} t^{1-\alpha}x(t) & \text{if } t \in \mathbb{R}^+ \\ \lim\limits_{t\to 0+} t^{1-\alpha}x(t) & \text{if } t = 0. \end{cases}$$

By (i), (ii), and (iii) it follows that $\widetilde{Y} := \{\widetilde{y} \mid y \in Y\}$ is pointwise bounded and equicontinuous on $\mathbb{R}_0^+$. By Ascoli's theorem it follows that $\widetilde{Y}$ is relatively compact in $(\widetilde{X}, \widetilde{\mathcal{T}})$.

In order to prove that $Y$ is relatively compact in $(X, \mathcal{T})$, let $(y_k)$ be any sequence in $Y$, and let $(\widetilde{y}_k)$ be the corresponding sequence in $\widetilde{Y}$. Since $\widetilde{Y}$ is relatively compact, there exist $\widetilde{y} \in \widetilde{X}$ as well as a subsequence $(\widetilde{y}_{k_j})$ of $(\widetilde{y}_k)$ such that $(\widetilde{y}_{k_j})$ converges to $\widetilde{y}$ with respect to $\widetilde{\mathcal{T}}$. Then $(\widetilde{y}_{k_j} - \widetilde{y})_{j\in\mathbb{N}}$ converges uniformly to 0 on every compact set $C \subset \mathbb{R}_0^+$. Let $y : \mathbb{R}^+ \to \mathbb{R}$ be the function defined by $y(t) := \widetilde{y}(t)/t^{1-\alpha}$ for all $t \in \mathbb{R}^+$. Obviously, we have $y \in C_{1-\alpha}(\mathbb{R}^+)$. By a reasoning similar to that already used in the proof of claim (3.3) one can easily deduce that

$$(y_{k_j} - y)_{j\in\mathbb{N}} \text{ converges uniformly to 0 on every compact set } C \subset \mathbb{R}^+. \tag{3.5}$$

Let $\varepsilon > 0$ arbitrarily chosen. Since $(\widetilde{y}_{k_j} - \widetilde{y})_{j\in\mathbb{N}}$ converges uniformly to 0 on $[0, 1]$, there exists $j_0 \in \mathbb{N}$ such that

$$|\widetilde{y}_{k_j}(t) - \widetilde{y}(t)| \le \varepsilon \quad \text{for all } j \ge j_0 \text{ and all } t \in [0, 1],$$

i.e.,

$$t^{1-\alpha}|y_{k_j}(t) - y(t)| \le \varepsilon \quad \text{for all } j \ge j_0 \text{ and all } t \in (0, 1), \tag{3.6}$$

By (3.5) and (3.6), based on Theorem **3.1**, it follows that $(y_{k_j} - y)_{j\in\mathbb{N}}$ converges to 0, i.e., $(y_{k_j})_{j\in\mathbb{N}}$ converges to $y$ with respect to $\mathcal{T}$. Consequently, $Y$ is relatively compact in $(X, \mathcal{T})$. □

## 4. Main result

THEOREM **4.1**. *If $\beta$ is a real number such that $0 \le \beta < \alpha$, and $p \in C_\beta(\mathbb{R}^+)$ and $q \in C_{1-\alpha}(\mathbb{R}^+)$ are nonnegative functions, then for every $y_0 \in \mathbb{R}$ the linear integral equation*

$$y(t) = y_0 t^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \big(p(s)y(s) + q(s)\big) ds \tag{4.1}$$

*has a unique solution $y \in C_{1-\alpha}(\mathbb{R}^+)$.*

P r o o f. Obviously, it suffices to show that (4.1) has a unique solution in $C_{1-\alpha}([0,h])$ for each $h > 0$, where $C_{1-\alpha}([0,h])$ is the linear space consisting of all functions $y \in C((0,h])$ for which the limit $\lim_{t\to 0+} t^{1-\alpha} y(t)$ exists in $\mathbb{R}$. When endowed with the norm

$$\|y\|_{1-\alpha} := \sup_{t\in(0,h]} t^{1-\alpha}|y(t)|,$$

$C_{1-\alpha}([0,h])$ becomes a Banach space.

Let $h > 0$, and let $T : C_{1-\alpha}([0,h]) \to C_{1-\alpha}([0,h])$ be the operator defined by

$$\big(Ty\big)(t) := y_0 t^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \big(p(s)y(s) + q(s)\big) ds$$

for all $y \in C_{1-\alpha}([0,h])$ and all $t \in (0,h]$. By Lemma **2.1** it follows that $T$ is well defined. We claim that for all $y, z \in C_{1-\alpha}([0,h])$, all $t \in (0,h]$, and all $n \in \mathbb{N}$ one has

$$t^{1-\alpha} \left| \big(T^n y\big)(t) - \big(T^n z\big)(t) \right| \le P^n K_n t^{n(\alpha-\beta)} \|y-z\|_{1-\alpha}, \tag{4.2}$$

where $T^n$ denotes the $n$th iterate of $T$,

$$P := \|p\|_\beta = \sup_{t\in(0,h]} t^\beta p(t) \quad \text{and} \quad K_n := \prod_{k=1}^{n} \frac{\Gamma(k(\alpha-\beta))}{\Gamma(\alpha + k(\alpha-\beta))}.$$

We proceed by induction on $n$. Let $y, z \in C_{1-\alpha}([0,h])$, and let $t \in (0,h]$. In order to prove that (4.2) holds for $n = 1$, we notice that

$$\begin{aligned}
t^{1-\alpha}\left|\big(Ty\big)(t)-\big(Tz\big)(t)\right| &\leq \frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}p(s)|y(s)-z(s)|ds \\
&= \frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t s^{\beta}p(s)s^{1-\alpha}|y(s)-z(s)|s^{\alpha-\beta-1}(t-s)^{\alpha-1}ds \\
&\leq \frac{Pt^{1-\alpha}}{\Gamma(\alpha)}\,\|y-z\|_{1-\alpha}\int_0^t s^{\alpha-\beta-1}(t-s)^{\alpha-1}ds \\
&= P\frac{\Gamma(\alpha-\beta)}{\Gamma(2\alpha-\beta)}\,t^{\alpha-\beta}\|y-z\|_{1-\alpha}.
\end{aligned}$$

Therefore (4.2) holds indeed for $n=1$. Assuming that (4.2) holds for some $n\in\mathbb{N}$, let us prove that it holds for $n+1$, too. We have

$$\begin{aligned}
t^{1-\alpha}&\left|\big(T^{n+1}y\big)(t)-\big(T^{n+1}z\big)(t)\right| = t^{1-\alpha}\left|T\big(T^n y\big)(t)-T\big(T^n z\big)(t)\right| \\
&\leq \frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}p(s)\left|\big(T^n y\big)(s)-\big(T^n z\big)(s)\right|ds \\
&= \frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t s^{\beta}p(s)s^{1-\alpha}\left|\big(T^n y\big)(s)-\big(T^n z\big)(s)\right|s^{\alpha-\beta-1}(t-s)^{\alpha-1}ds \\
&\leq \frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t P^{n+1}K_n s^{n(\alpha-\beta)}\|y-z\|_{1-\alpha}s^{\alpha-\beta-1}(t-s)^{\alpha-1}ds \\
&= P^{n+1}K_n\frac{\Gamma((n+1)(\alpha-\beta))}{\Gamma(\alpha+(n+1)(\alpha-\beta))}\,t^{(n+1)(\alpha-\beta)}\|y-z\|_{1-\alpha} \\
&= P^{n+1}K_{n+1}t^{(n+1)(\alpha-\beta)}\|y-z\|_{1-\alpha}.
\end{aligned}$$

This shows that (4.2) holds for $n+1$. By (4.2) we deduce that

$$\|T^n y-T^n z\|_{1-\alpha}\leq P^n K_n h^{n(\alpha-\beta)}\|y-z\|_{1-\alpha}$$

for all $n\in\mathbb{N}$ and all $y,z\in C_{1-\alpha}([0,h])$.

Choose $n_0\in\mathbb{N}$ such that $n_0(\alpha-\beta)\geq 2$. Since $\Gamma$ is increasing on $[2,\infty)$, for every $n>n_0$ one has

$$K_n = K_{n_0}\prod_{k=n_0+1}^{n}\frac{\Gamma(k(\alpha-\beta))}{\Gamma(\alpha+k(\alpha-\beta))} < K_{n_0}\prod_{k=n_0+1}^{n}\frac{\Gamma(k(\alpha-\beta))}{\Gamma((k+1)(\alpha-\beta))}\,,$$

whence

$$K_n < \frac{K_{n_0}\Gamma((n_0+1)(\alpha-\beta))}{\Gamma((n+1)(\alpha-\beta))}\,.$$

Therefore, for all $n>n_0$ and all $y,z\in C_{1-\alpha}([0,h])$ one has

$$\|T^n y-T^n z\|_{1-\alpha}\leq \frac{P^n h^{n(\alpha-\beta)}K_{n_0}\Gamma((n_0+1)(\alpha-\beta))}{\Gamma((n+1)(\alpha-\beta))}\,\|y-z\|_{1-\alpha}.$$

Since

$$\lim_{n\to\infty} \frac{P^n h^{n(\alpha-\beta)} K_{n_0} \Gamma((n_0+1)(\alpha-\beta))}{\Gamma((n+1)(\alpha-\beta))} = 0,$$

it follows that for sufficiently large $n$ the operator $T^n$ is a contraction. By virtue of a well known generalization of Banach's contraction principle (see, for instance, Agarwal, Meehan and O'Regan [1, p. 9] or Ortega and Rheinboldt [5, 12.1.1, pp. 383–385]) $T$ has a unique fixed point in $C_{1-\alpha}([0,h])$. In other words, (4.1) has a unique solution in $C_{1-\alpha}([0,h])$. □

We are now in the position to state and prove the main result of our paper.

THEOREM **4.2**. *If* $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ *is a continuous function satisfying the hypothesis* (H) *and the parameters* $\alpha$ *and* $\beta$ *are such that* $2\alpha - \beta > 1$, *then for every* $x_0 \in \mathbb{R}$ *the initial value problem* (1.1)–(1.2) *has at least one solution* $x \in C_{1-\alpha}(\mathbb{R}^+)$.

Proof. **(a)** According to Lemma **2.2**, it suffices to show that the Volterra integral equation (2.2) has at least one solution in $X := C_{1-\alpha}(\mathbb{R}^+)$. Let $T : X \to X$ be the operator defined by

$$\bigl(Tx\bigr)(t) := x_0 t^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s))ds$$

for all $x \in X$ and all $t \in \mathbb{R}^+$. By Lemma **2.1** it follows that $T$ is well defined. We need to prove that $T$ has at least one fixed point $x \in X$.

Further, let $y_0 := 1 + |x_0|$, and let $y \in X$ be the unique solution to the linear integral equation (4.1). Then we have $y(t) > 0$ for all $t \in \mathbb{R}^+$. Indeed, otherwise it would exist $t_0 \in \mathbb{R}^+$ such that $y(t_0) = 0$ and $y(t) > 0$ for all $t \in (0, t_0)$. But then, according to (4.1) we would have

$$0 = y(t_0) = y_0 t_0^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^{t_0} (t_0 - s)^{\alpha-1} \bigl(p(s)y(s) + q(s)\bigr) ds > 0,$$

which is a contradiction.

In the complete Hausdorff locally convex space $(X, \mathcal{T})$, introduced in Section **3**, consider the set defined by

$$Z := \{x \in X \mid |x(t)| \le y(t) \text{ for all } t \in \mathbb{R}^+\}.$$

Since the convergence of a sequence in $X$ with respect to $\mathcal{T}$ implies the pointwise convergence of that sequence (see Theorem **3.1**), it is immediately seen that $Z$ is a nonempty closed convex subset of $X$.

We claim that $T(Z) \subseteq Z$. Indeed, for every $z \in Z$ and all $t \in \mathbb{R}^+$ one has

$$\begin{aligned} \big|(Tz)(t)\big| &\leq |x_0|t^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,z(s))|ds \qquad (4.3)\\ &\leq y_0 t^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\big(p(s)|z(s)| + q(s)\big)ds \\ &\leq y_0 t^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\big(p(s)y(s) + q(s)\big)ds \\ &= y(t), \end{aligned}$$

whence $Tz \in Z$.

**(b)** Next we claim that $T$ is continuous on $Z$. To this end, let $x$ be any point in $Z$, and let $V$ be an arbitrary neighbourhood of $Tx$ in $X$ with respect to $\mathcal{T}$. Choose $r > 0$ and $n \in \mathbb{N}$ such that $Tx + B_{r,n} \subseteq V$. Set

$$M_1 := \sup_{s\in(0,1]} s^\beta p(s) s^{1-\alpha} y(s), \quad M_2 := \sup_{s\in(0,1]} s^{1-\alpha} q(s),$$

and $M := \max\{M_1, M_2\}$. Further, let $z \in Z$ and $t \in (0,1]$ be arbitrarily chosen. Since

$$|f(s,z(s))| \leq p(s)|z(s)| + q(s) \leq p(s)y(s) + q(s)$$

for all $s \in \mathbb{R}^+$, a computation similar to that performed in the proof of Lemma **2.1** shows that

$$\int_0^t (t-s)^{\alpha-1}|f(s,z(s))|ds \leq M\big(B(\alpha,\alpha-\beta)t^{2\alpha-\beta-1} + B(\alpha,\alpha)t^{2\alpha-1}\big). \qquad (4.4)$$

Pick $\delta \in (0,1)$ such that

$$\frac{2Mn^{1-\alpha}}{\Gamma(\alpha)}\big(B(\alpha,\alpha-\beta)\delta^{2\alpha-\beta-1} + B(\alpha,\alpha)\delta^{2\alpha-1}\big) \leq \frac{r}{2}. \qquad (4.5)$$

After that, put $A := \max_{t\in[\delta,n]} y(t)$. Since $f$ is uniformly continuous on the compact set $[\delta,n]\times[-A,A]$, there exists $p > 0$ such that

$$|f(s,u) - f(s,v)| \leq \frac{r\alpha\Gamma(\alpha)}{2n}$$

for all $s \in [\delta,n]$ and all $u,v \in [-A,A]$ with $|u-v| \leq p$.

Set $q := p\,\delta^{1-\alpha}$ and $U := \big(x + B_{q,n}\big)\cap Z$. Clearly, $U$ is a neighbourhood of $x$ in $Z$ with respect to $\mathcal{T}$. It remains to prove that $T(U) \subseteq V$ in order to establish the continuity of $T$ at $x$. Note first that for every $z \in U$ we have

$z - x \in B_{q,n}$, hence $s^{1-\alpha}|z(s) - x(s)| \le q$ for all $s \in (0, n]$. Therefore, for all $s \in [\delta, n]$ one has $|z(s) - x(s)| \le q/s^{1-\alpha} \le q/\delta^{1-\alpha} = p$, whence

$$|f(s, z(s)) - f(s, x(s))| \le \frac{r\alpha\Gamma(\alpha)}{2n}\,. \tag{4.6}$$

Now let $z \in U$ and $t \in (0, n]$ be arbitrarily chosen. If $t \in (0, \delta]$, then due to (4.4) we have

$$\begin{aligned}
\big|\big(Tz\big)(t) - \big(Tx\big)(t)\big| &\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, z(s)) - f(s, x(s))| ds \\
&\le \frac{1}{\Gamma(\alpha)} \left( \int_0^t (t-s)^{\alpha-1} |f(s, z(s))| ds + \int_0^t (t-s)^{\alpha-1} |f(s, x(s))| ds \right) \\
&\le \frac{2M}{\Gamma(\alpha)} \big( B(\alpha, \alpha-\beta) t^{2\alpha-\beta-1} + B(\alpha, \alpha) t^{2\alpha-1} \big) \\
&\le \frac{2M}{\Gamma(\alpha)} \big( B(\alpha, \alpha-\beta) \delta^{2\alpha-\beta-1} + B(\alpha, \alpha) \delta^{2\alpha-1} \big),
\end{aligned}$$

whence

$$\begin{aligned}
&t^{1-\alpha} \big|\big(Tz\big)(t) - \big(Tx\big)(t)\big| \\
&\qquad \le \frac{2Mn^{1-\alpha}}{\Gamma(\alpha)} \big( B(\alpha, \alpha-\beta) \delta^{2\alpha-\beta-1} + B(\alpha, \alpha) \delta^{2\alpha-1} \big) \le \frac{r}{2}\,,
\end{aligned} \tag{4.7}$$

by virtue of (4.5). If $t \in [\delta, n]$, then we have

$$\begin{aligned}
&\big|\big(Tz\big)(t) - \big(Tx\big)(t)\big| \\
&\qquad \le \frac{1}{\Gamma(\alpha)} \int_0^\delta (t-s)^{\alpha-1} |f(s, z(s)) - f(s, x(s))| ds \\
&\qquad\quad + \frac{1}{\Gamma(\alpha)} \int_\delta^t (t-s)^{\alpha-1} |f(s, z(s)) - f(s, x(s))| ds.
\end{aligned} \tag{4.8}$$

Note that

$$\begin{aligned}
&\int_0^\delta (t-s)^{\alpha-1} |f(s, z(s)) - f(s, x(s))| ds \\
&\qquad \le \int_0^\delta (\delta-s)^{\alpha-1} \left( \frac{\delta-s}{t-s} \right)^{1-\alpha} \big( |f(s, z(s))| + |f(s, x(s))| \big) ds \\
&\qquad \le \int_0^\delta (\delta-s)^{\alpha-1} |f(s, z(s))| ds + \int_0^\delta (\delta-s)^{\alpha-1} |f(s, z(s))| ds,
\end{aligned}$$

whence

$$\int_0^\delta (t-s)^{\alpha-1}|f(s,z(s))-f(s,x(s))|ds \tag{4.9}$$
$$\leq 2M\big(B(\alpha,\alpha-\beta)\delta^{2\alpha-\beta-1}+B(\alpha,\alpha)\delta^{2\alpha-1}\big),$$

by virtue of (4.4). On the other hand, due to (4.6) we have

$$\int_\delta^t (t-s)^{\alpha-1}|f(s,z(s))-f(s,x(s))|ds \tag{4.10}$$
$$\leq \frac{r\alpha\Gamma(\alpha)}{2n}\cdot\frac{(t-\delta)^\alpha}{\alpha} < \frac{r\Gamma(\alpha)}{2n}\, n^\alpha = \frac{r\Gamma(\alpha)}{2n^{1-\alpha}}\,.$$

From (4.8), (4.9) and (4.10) we get

$$t^{1-\alpha}\left|\big(Tz\big)(t)-\big(Tx\big)(t)\right| \leq n^{1-\alpha}\left|\big(Tz\big)(t)-\big(Tx\big)(t)\right|$$
$$\leq \frac{2Mn^{1-\alpha}}{\Gamma(\alpha)}\left(B(\alpha,\alpha-\beta)\delta^{2\alpha-\beta-1}+B(\alpha,\alpha)\delta^{2\alpha-1}\right)+\frac{r}{2}\,,$$

whence

$$t^{1-\alpha}\left|\big(Tz\big)(t)-\big(Tx\big)(t)\right| \leq r \tag{4.11}$$

by virtue of (4.5). Now due to (4.7) and (4.11) we may conclude that

$$p_n(Tz-Tx)=\sup_{t\in(0,n]} t^{1-\alpha}\left|\big(Tz\big)(t)-\big(Tx\big)(t)\right| \leq r,$$

whence $Tz-Tx\in B_{r,n}$. Therefore $Tz\in Tx+B_{r,n}\subseteq V$. Since $z\in U$ was arbitrarily chosen, we deduce that $T(U)\subseteq V$, hence $T$ is continuous at $x$.

**(c)** Finally, we claim that the image $T(Z)$ is relatively compact in $X$. To prove this, we show that $T(Z)$ satisfies the conditions (i), (ii), and (iii) in Theorem **3.3**. From (4.3) it follows that $T(Z)$ is pointwise bounded on $\mathbb{R}^+$. In order to establish the equicontinuity of $T(Z)$ on $\mathbb{R}^+$, let $t_0\in\mathbb{R}^+$ be arbitrarily chosen. Select real numbers $a$ and $b$ such that $0<a<t_0<b$ and set

$$M_1 := \sup_{s\in(0,b]} s^\beta p(s)s^{1-\alpha}y(s), \quad M_2 := \sup_{s\in(0,b]} s^{1-\alpha}q(s),$$

and $M:=\max\{M_1,M_2\}$. Further, let $z\in Z$ and $t\in[a,b]$. Since

$$|f(s,z(s))|\leq p(s)|z(s)|+q(s)\leq p(s)y(s)+q(s) \quad \text{for all } s\in\mathbb{R}^+,$$

for all $t \geq t_0$ one has

$$\left|\int_{t_0}^{t}(t-s)^{\alpha-1}f(s,z(s))ds\right| \leq \int_{t_0}^{t}(t-s)^{\alpha-1}\big(p(s)y(s)+q(s)\big)ds$$
$$\leq M\int_{t_0}^{t}(t-s)^{\alpha-1}\big(s^{\alpha-\beta-1}+s^{\alpha-1}\big)ds$$
$$\leq M\big(a^{\alpha-\beta-1}+a^{\alpha-1}\big)\int_{t_0}^{t}(t-s)^{\alpha-1}ds,$$

whence

$$\left|\int_{t_0}^{t}(t-s)^{\alpha-1}f(s,z(s))ds\right| \leq \frac{M\big(a^{\alpha-\beta-1}+a^{\alpha-1}\big)}{\alpha}\,|t-t_0|^{\alpha}. \qquad (4.12)$$

We have also

$$\left|\int_{0}^{t_0}\left((t-s)^{\alpha-1}-(t_0-s)^{\alpha-1}\right)f(s,z(s))ds\right|$$
$$\leq \int_{0}^{t_0}\left((t_0-s)^{\alpha-1}-(t-s)^{\alpha-1}\right)\big(p(s)y(s)+q(s)\big)ds$$
$$\leq M\int_{0}^{t_0}\left((t_0-s)^{\alpha-1}-(t-s)^{\alpha-1}\right)\big(s^{\alpha-\beta-1}+s^{\alpha-1}\big)ds$$
$$\leq M\big(a^{\alpha-\beta-1}+a^{\alpha-1}\big)\int_{0}^{t_0}\left((t_0-s)^{\alpha-1}-(t-s)^{\alpha-1}\right)ds,$$

whence

$$\left|\int_{0}^{t_0}\left((t-s)^{\alpha-1}-(t_0-s)^{\alpha-1}\right)f(s,z(s))ds\right| \qquad (4.13)$$
$$\leq \frac{M\big(a^{\alpha-\beta-1}+a^{\alpha-1}\big)}{\alpha}\big(|t-t_0|^{\alpha}-|t^{\alpha}-t_0^{\alpha}|\big).$$

Since

$$\big|\big(Tz\big)(t)-\big(Tz\big)(t_0)\big|$$
$$\leq |x_0|\cdot|t^{\alpha-1}-t_0^{\alpha-1}|+\frac{1}{\Gamma(\alpha)}\left|\int_{t_0}^{t}(t-s)^{\alpha-1}f(s,z(s))ds\right|$$
$$+\frac{1}{\Gamma(\alpha)}\left|\int_{0}^{t_0}\left((t-s)^{\alpha-1}-(t_0-s)^{\alpha-1}\right)f(s,z(s))ds\right|,$$

by taking into consideration (4.12) and (4.13) we deduce that

$$\big|\big(Tz\big)(t)-\big(Tz\big)(t_0)\big| \leq |x_0|\cdot|t^{\alpha-1}-t_0^{\alpha-1}|+\frac{2M\big(a^{\alpha-\beta-1}+a^{\alpha-1}\big)}{\Gamma(\alpha+1)}\,|t-t_0|^{\alpha} \qquad (4.14)$$

for all $z \in Z$ and all $t \in [t_0, b]$. A similar reasoning shows that (4.14) holds also for all $z \in Z$ and all $t \in [a, t_0]$. By (4.14) it follows that $T(Z)$ is equicontinuous at $t_0$.

It remains to establish the equiconvergence of $T(Z)$ at $0+$. Note first that for all $z \in Z$ one has

$$\lim_{t \to 0+} t^{1-\alpha}\big(Tz\big)(t) = x_0$$

due to Lemma **2.1**. Therefore, for each $z \in Z$ and each $t \in \mathbb{R}^+$ one has

$$\begin{aligned} &\left| t^{1-\alpha}\big(Tz\big)(t) - \lim_{t \to 0+} t^{1-\alpha}\big(Tz\big)(t) \right| \\ &\quad \le \frac{t^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, z(s))| ds \\ &\quad \le \frac{t^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \big(p(s)y(s) + q(s)\big) ds. \end{aligned} \tag{4.15}$$

But Lemma **2.1** ensures also that

$$\lim_{t \to 0+} t^{1-\alpha} \int_0^t (t-s)^{\alpha-1} \big(p(s)y(s) + q(s)\big) ds = 0. \tag{4.16}$$

By (4.15) and (4.16) it follows immediately that $T(Z)$ satisfies the condition (iii) in Theorem **3.3**.

In conclusion, $Z$ is a closed convex subset of the complete locally convex space $(X, \mathcal{T})$, while $T : Z \to Z$ is a continuous operator with the property that $T(Z)$ is relatively compact in $X$. By virtue of Schauder's fixed point theorem (see, for instance, [1, Theorem 8.3]) $T$ has at least one fixed point in $Z$. In other words, the initial value problem (1.1)–(1.2) has at least one solution $x \in Z$. □

We conclude our paper with an example illustrating the applicability of Theorem **4.2**.

Example **4.1**. Consider the initial value problem for the fractional differential equation

$$\begin{cases} D_{0+}^{2/3} x(t) = \left(\frac{1}{t^{1/4}} + \ln t\right) x(t)^{1/3} \ln\left(1 + x(t)^{2/3}\right) + \frac{1}{t^{1/3}}, \quad t \in \mathbb{R}^+, \\ \lim_{t \to 0+} t^{1/3} x(t) = 1. \end{cases} \tag{4.17}$$

Here $\alpha = 2/3$, while

$$f(t, x) = \left(\frac{1}{t^{1/4}} + \ln t\right) x^{1/3} \ln\left(1 + x^{2/3}\right) + \frac{1}{t^{1/3}}.$$

Choosing $p(t) := \frac{1}{t^{1/4}} + \ln t$ and $q(t) := 1/t^{1/3}$, we see that $p \in C_{1/4}(\mathbb{R}^+)$, $q \in C_{1/3}(\mathbb{R}^+)$ and (1.3) holds. Therefore, the hypothesis (H) is satisfied with $\beta = 1/4$. Since $2\alpha - \beta - 1 > 0$, Theorem **4.2** guarantees that the initial value problem (4.17) has at least one solution $x \in C_{1/3}(\mathbb{R}^+)$.

We point out that neither Theorem 3.1 in the paper by Kou, Zhou and Yan [3], nor Theorem 4.1 in the paper by Zhao and Ge [7] can be applied to (4.17).

## References

[1] R.P. Agarwal, M. Meehan, D. O'Regan, *Fixed Point Theory and Applications*. Cambridge University Press, Cambridge (2001).

[2] D. Delbosco, L. Rodino, Existence and uniqueness for a nonlinear fractional differential equation. *J. Math. Anal. Appl.* **204** (1996), 609–625.

[3] C. Kou, H. Zhou, Y. Yan, Existence of solutions of initial value problems for nonlinear fractional differential equations on the half-axis. *Nonlinear Anal.* **74** (2011), 5975–5986.

[4] K.S. Miller, B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*. John Wiley, N. York (1993).

[5] J.M. Ortega, W.C. Rheinboldt, *Iterative Solution of Nonlinear Equations in Several Variables*. Academic Press, N. York and London (1970). SIAM, 1987

[6] I. Podlubny, *Fractional Differential Equations*. Academic Press, San Diego etc. (1999).

[7] X. Zhao, W. Ge, Unbounded solutions for a fractional boundary value problem on the infinite interval. *Acta Appl. Math.* **109** (2010), 495–505.

*Babeş-Bolyai University*
*Faculty of Mathematics and Computer Science*
*Kogălniceanu Str., no. 1*
*Cluj-Napoca – 400084, ROMANIA*

*e-mail: ttrif@math.ubbcluj.ro*
*Received: September 24, 2012*

---

Please cite to this paper as published in:

*Fract. Calc. Appl. Anal.*, Vol. **16**, No 3 (2013), pp. 595–612;
DOI:10.2478/s13540-013-0038-3