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RESEARCH PAPER

A NOTE ON FRACTIONAL BESSEL EQUATION AND ITS ASYMPTOTICS

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Abstract

In this note we propose a fractional generalization of the classical modified Bessel equation. Instead of the integer-order derivatives we use the Riemann-Liouville version. Next, we solve the fractional modified Bessel equation in terms of the power series and provide an asymptotic analysis of its solution for large arguments. We find a leading-order term of the asymptotic formula for the solution to the considered equation. This behavior is verified numerically and shows high accuracy and fast convergence. Our results reduce to the classical formulas when the order of the fractional derivative goes to integer values.

MSC 2010: Primary 26A33; Secondary 34A08, 34B30

Key Words and Phrases: fractional calculus, fractional ordinary differential equation, Bessel function, asymptotic analysis

1. Introduction

Recently, the fractional calculus methods have become increasingly more popular and successful in a large number of physical applications (see for example, [4, 8, 31]). The theoretical and experimental scientists observed that certain materials admit anomalous, fractional behavior, as [6]. There are many systems which evolution is governed by some fractional

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differential equations (for example, [5]). Fractional calculus is nowadays a broad and interesting field of very active research.

In this paper we consider a fractional variant of the classical modified Bessel equation of order ν. As is well-known, the modified Bessel functions have many important applications for example in probability, physics and biology [24, 27, 30]. We investigate a modified Bessel equation in which the integer-order derivatives are replaced by Riemann-Liouville fractional derivatives. The solution to this equation is presented in terms of power series. This generates a new family of special functions indexed by the derivative orders α, β and parameters ν and μ. The problem of solving fractional equations with variable coefficients in terms of the power series was investigated e.g. in in [23], where the authors conducted a thorough analysis. As is very well known, the power series are not suitable for computation for large values of arguments. We find an asymptotic form of the solution to the modified fractional Bessel equation and notice that our approximation provides a very good accuracy even for arguments of order of unity. All the formulas which we obtain reduce to the classical case when the appropriate parameters approach 1^-.

The concept of generalizing some special functions by means of introducing new classes of them provides an interesting ground for fruitful investigations. Fractional calculus gives many tools for making such generalizations. A thorough analysis of the connection between many special functions and fractional calculus operators was given in [13], with the idea that the elementary functions generate generalized special functions by the use of fractional calculus. These special functions include for example the hyper-Bessel, multi-index Mittag-Leffler or Wright (known as Bessel-Maitland in some case) functions, see e.g. [11, 14, 21, 29]. Also, the g-Jacobi and F-Gauss functions have been inspired by the fractional calculus operators, [17], the Laguerre polynomials and the Kummer functions ahve been generalized to fractional indices analogues in [18], and there are many other attempts of generalizations of other special functions (see e.g. [2, 9, 26, 28]). A particular case of the equation we consider has a solution in terms of the generalized Mittag-Leffler function $E_{k,l,m}$ introduced in [12] (see also [7]). A Bessel-type equation with fractional derivatives is investigated also in [25]. However, the form of that equation is different from the equation we analyze here. A simplified version of the fractional modified Bessel equation is applied in modeling the geometry of the human eye's cornea, see in [20].

2. Main result

Consider the following fractional differential equation

$$x^\alpha D_0^\alpha (x^\beta \, D_0^\beta y) = \left(x^{2\mu} + \nu^{2\mu}\right) y, \tag{2.1}$$

where $0 < \alpha, \beta, \mu \leq 1$, and the "order" of the equation (as a notion corresponding to the "order of a Bessel function") is $\nu \geq 0$ (this will not cause any loss of generality). Here D_0^α denotes the Riemann-Liouville fractional derivative defined by

$$D_0^\alpha y := \frac{d}{dx} D_0^{-(1-\alpha)} y, \quad \text{where} \quad (D_0^{-\alpha} y)(x) := \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} y(t) dt \tag{2.2}$$

is the fractional integral operator of order α (see [10]). The fractional derivative operator is a generalization of the classical integer-order one.

We call (2.1) as the modified fractional Bessel equation of order ν, in analogy with the classical case. Note that when $\alpha, \beta, \mu \to 1^-$, equation (2.1) reduces to the classical Bessel equation. In what follows we find a solution to the fractional Bessel equation and its asymptotic form for large arguments. As shown below, the leading-order asymptotic behavior of the solution to (2.1) does not depend on its order ν.

We seek for a solution to (2.1) in the form of power series, that is

$$y(x) = \sum_{n=0}^{\infty} a_n x^{\mu n + b}, \tag{2.3}$$

where a_n and b have to be determined. Substituting this power series into (2.1) and noting that $(D_0^\alpha x^{\mu n+b})(x) = \Gamma(\mu n+b+1)/\Gamma(\mu n+b+1-\alpha) x^{\mu n+b-\alpha}$ (and similarly for D_0^β) we get

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{\Gamma(\mu n + b + 1)^2}{\Gamma(\mu n + b + 1 - \alpha)\Gamma(\mu n + b + 1 - \beta)} a_n x^{\mu n + b} \\ &\qquad = \sum_{n=2}^{\infty} a_{n-2} x^{\mu n + b} + \sum_{n=0}^{\infty} \nu^{2\mu} a_n x^{\mu n + b}. \end{aligned} \tag{2.4}$$

Equating the coefficients, we obtain recurrence formulas for coefficients a_n, as follows:

$$a_0\left(\frac{\Gamma(b+1)^2}{\Gamma(b+1-\alpha)\Gamma(b+1-\beta)}-\nu^{2\mu}\right)=0,$$
$$a_1\left(\frac{\Gamma(\mu+b+1)^2}{\Gamma(\mu+b+1-\alpha)\Gamma(\mu+b+1-\beta)}-\nu^{2\mu}\right)=0, \tag{2.5}$$
$$a_n=\left(\frac{\Gamma(\mu n+b+1)^2}{\Gamma(\mu n+1-\alpha)\Gamma(\mu n+1-\beta)}-\nu^{2\mu}\right)^{-1}a_{n-2},\quad n\geq 2.$$

From this, we must take either a_0 or a_1 equal to zero and we choose $a_1=0$ analogously as in the classical case. By the ratio test we see that the series is absolutely convergent. The unknown b is determined as a solution to the following equation

$$\frac{\Gamma(b+1)^2}{\Gamma(b+1-\alpha)\Gamma(b+1-\beta)}=\nu^{2\mu}. \tag{2.6}$$

This equation cannot be solved analytically except when $\alpha,\beta=0,1$ and in the general case has to be handled numerically or by approximate methods. The plot of the left-hand side of (2.6) as a function of b is depicted on Fig. 1. In the classical case this would reduce to the quadratic, while for general α and β the graph becomes skewed to the left. Notice that for $\nu=0$ we have two solutions: $b=\alpha-1$ or $b=\beta-1$, which reduce to $b=0$ in the classical case of $\alpha=\beta=1$. This means that in the fractional setting we can always find two linearly independent solutions to the fractional Bessel equation (2.1). We will denote these two solutions of (2.6) by b^+ and b^-, and the corresponding Bessel functions – as $I^{\alpha,\beta}_{\nu^+,\mu}$ and $I^{\alpha,\beta}_{\nu^-,\mu}$.

Now, let $b^\pm$ be either b^+ or b^- and notice that from (2.5) and since $a_1=0$, we have all odd-numbered coefficients vanishing, while

$$a_{2n}=a_0\left(\prod_{i=1}^{n}\frac{\Gamma(2\mu i+b^\pm+1)^2}{\Gamma(2\mu i+b^\pm+1-\alpha)\Gamma(2\mu i+b^\pm+1-\beta)}-\nu^{2\mu}\right)^{-1}. \tag{2.7}$$

We also have to choose a_0, since because of homogeneity of (2.1) it is an arbitrary constant. To be consistent with classical case, we choose $a_0=1/\Gamma(1+\nu)$. Taking into account (2.7) we obtain a solution to the fractional Bessel equation (2.1):

$$I^{\alpha,\beta}_{\nu^\pm,\mu}(x)=$$
$$\sum_{n=0}^{\infty}a_0\left(\prod_{i=1}^{n}\frac{\Gamma(2\mu i+b^\pm+1)^2}{\Gamma(2\mu i+b^\pm+1-\alpha)\Gamma(2\mu i+b^\pm+1-\beta)}-\nu^{2\mu}\right)^{-1}x^{2\mu n+b^\pm}, \tag{2.8}$$

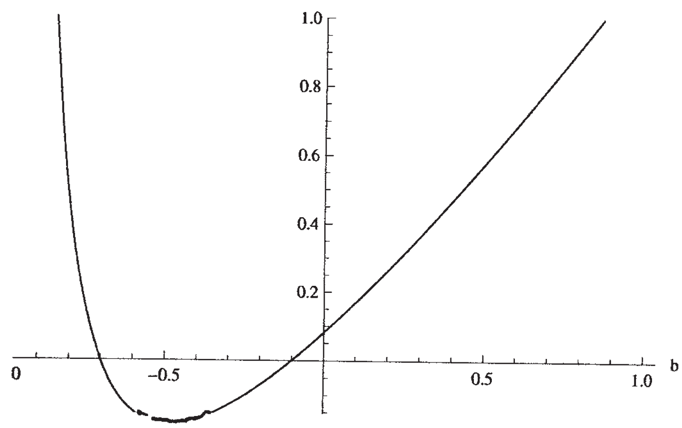

FIGURE 1. Plot of the left-hand side of (2.6) with $\alpha = 0.9$ and $\beta = 0.3$.

where $0 < \alpha, \beta, \mu \leq 1$ and with the convention that $\prod_{i=1}^{0} = 1$. This modified fractional Bessel function is a generalization of the classical Bessel function since when $\alpha \to 1^-$ we have $I_{\nu,1}^{\alpha,1} \to I_0$, which is a classical modified Bessel function. This can be seen directly from series representation (2.8). Additionally, we can see that $I_{0,1}^{0,1} = \exp(x^2/2)$ which solves (2.1) when we treat D_0^0 as the identity operator. The series coefficients of (2.8) are somehow similar to the coefficients of the generalized Mittag-Leffler function $E_{k,l,m}$. Indeed, notice that for $\beta \to 0^+$ and $\nu = 0$ this equation reduces to $D_0^\alpha y = x^{2\mu-\alpha} y$, which has a solution in terms of this special function (see [10], formula 4.1.33).

The modified fractional Bessel function $I_{\nu,\mu}^{\alpha,\beta}$ defined in (2.8) has a very interesting behavior as $x \to +\infty$. As we will see, it is a generalization of the classical results from the asymptotic theory. First, we formulate an elementary result as a lemma. It can be proved by a straightforward application of the Laplace transform to the power function.

LEMMA **2.1**. *Let $F_\mu(x) := \int_0^1 t^\mu e^{-xt} dt$, then F_μ has the following leading order behavior*

$$F_\mu(x) \sim x^{-(\mu+1)} \Gamma(\mu + 1) \tag{2.9}$$

as $x \to +\infty$.

The proof of the following theorem is based on the classical Laplace method for asymptotic integrals, see for example [19]. The main point is that we are dealing with an integro-differential equation which introduces some complications.

THEOREM **2.1**. *The solution to the modified fractional Bessel equation (2.1) has the following asymptotic leading-order behavior as $x \to +\infty$:*

$$I^{\alpha,\beta}_{\nu^\pm,\mu}(x) \sim C^{\alpha,\beta}_{0^\pm,\mu} x^{-\frac{1}{1+\alpha}\left(\alpha(2\mu-\alpha)+\left(1-\frac{2\mu}{\alpha+\beta}\right)\left(\frac{1}{2}\alpha(\alpha+1)+1-\beta\right)\right)} \exp\left(\frac{\alpha+\beta}{2\mu} x^{\frac{2\mu}{\alpha+\beta}}\right), \tag{2.10}$$

where $0 < \alpha, \beta, \mu \leq 1$ and $C^{\alpha,\beta}_{\nu^\pm,\mu}$ is some constant.

REMARK **2.1**. We see that for $\alpha, \beta, \mu \to 1^-$ the asymptotic form (2.10) reduces to a well known formula $I^{1,1}_{\nu,1}(x) = I_0(x) \sim 1/\sqrt{2\pi} x^{-1/2} \exp(x)$ (see e.g. [1], p. 377, formula 9.7.1).

P r o o f. We use the rule for composition of fractional integral $D_0^{-\alpha}$ and the Riemann-Liouville fractional derivative ([22], formula 2.113)

$$D_0^{-\alpha} D_0^{\alpha}(x^\beta D_0^\beta y(x))(x) = x^\beta D_0^\beta y(x) - D_0^{\alpha-1}(x^\beta D_0^\beta y)(0)\frac{x^{\alpha-1}}{\Gamma(\alpha)}, \tag{2.11}$$

where $0 < \alpha \leq 1$. Since we are looking for the exponential asymptotic behavior, the second term at the right-hand side is irrelevant to our considerations and we can neglect it. By this remark, we transform (2.1) into an asymptotic Volterra integro-differential equation

$$\begin{aligned} D_0^\beta y(x) &\sim x^{-\beta}\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\left(t^{2\mu-\alpha} - \nu^{2\mu} t^{-\alpha}\right) y(t)dt \\ &\sim x^{-\beta}\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} t^{2\mu-\alpha} y(t)dt, \end{aligned} \tag{2.12}$$

where we immediately noted that $t^{-\alpha} = o(t^{2\mu-\alpha})$ as $x \to +\infty$, and thus neglected it. Since we are looking for the behavior of (2.12) for large x we change the variable $s = t/x$ to obtain the constant limits of integration

$$D_0^\beta y(x) \sim \frac{x^{2\mu-\beta}}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} s^{2\mu-\alpha} y(sx)ds. \tag{2.13}$$

First, we will find the exponential behavior of y and then an algebraic factor of the leading order term. To determine the exponential, seek for $y(x) = \exp\left(x^\lambda/\lambda\right)$, where λ is to be determined. We have

$$D_0^\beta y(x) \sim \frac{x^{2\mu-\beta}}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} s^{2\mu-\alpha} e^{\frac{x^\lambda}{\lambda}s^\lambda} ds. \tag{2.14}$$

Notice that for large x the integrand in (2.14) is dominated by the exponential term which has its maximum at $s = 1$. To move this maximum to the lower limit we substitute again $t = 1 - s$ and obtain

$$D_0^\beta y(x) \sim \frac{x^{2\mu-\beta}}{\Gamma(\alpha)} \int_0^1 (1-t)^{2\mu-\alpha} t^{\alpha-1} e^{\frac{x^\lambda}{\lambda}(1-t)^\lambda} dt. \tag{2.15}$$

As in the Laplace method for asymptotic integrals (see e.g. [3]) for $x \to +\infty$ the greatest contribution to the integral (2.12) comes from the neighborhood of the maximum of exponential term, that is when t is close to 0. To extract the leading-order behavior we approximate $(1-t)^{2\mu-\alpha} \approx 1$ and $(1-t)^\lambda \approx 1 - \lambda t$. It is easy to calculate that further approximations introduce higher-order terms which do not contribute to the exponential behavior. By these approximations and Lemma **2.1**, we obtain

$$\begin{aligned} D_0^\beta y(x) &\sim \frac{x^{2\mu-\beta} e^{\frac{x^\lambda}{\lambda}}}{\Gamma(\alpha)} \int_0^1 t^{\alpha-1} e^{-x^\lambda t} dt \\ &= \frac{x^{2\mu-\alpha} e^{\frac{x^\lambda}{\lambda}}}{\Gamma(\alpha)} F_{\alpha-1}(x^\lambda) \sim x^{2\mu-\beta-\alpha\lambda} e^{\frac{x^\lambda}{\lambda}}. \end{aligned} \tag{2.16}$$

Now, for the left-hand side of (2.16) notice that

$$D_0^\beta y(x) = \frac{d}{dx} D_0^{-(1-\beta)} y(x), \tag{2.17}$$

and thus to obtain its leading behavior we must analyze the fractional integral $D_0^{-(1-\beta)}$. But these considerations are exactly analogous to the ones previously done for the $D_0^{-\alpha}$. Having that in mind we obtain

$$\begin{aligned} D_0^\beta y(x) &= \frac{d}{dx} D_0^{-(1-\beta)} e^{\frac{x^\lambda}{\lambda}} \sim \frac{d}{dx} \left(x^{(1-\beta)(1-\lambda)} e^{\frac{x^\lambda}{\lambda}} \right) \\ &= x^{(1-\beta)(1-\lambda)-1} \left((1-\beta)(1-\lambda) + x^\lambda \right) e^{\frac{x^\lambda}{\lambda}}. \end{aligned} \tag{2.18}$$

Comparing equations (2.16) and (2.18) we note that for the leading terms to balance we must have $2\mu - \beta - \alpha\lambda = (1-\beta)(1-\lambda) - 1 + \lambda$. Which gives us

$$\lambda := \frac{2\mu}{\alpha + \beta}. \tag{2.19}$$

This is the sought asymptotic order of solution to (2.1).

Having found the exponential part of y, we now turn to the algebraic factor. To this end we seek for $y(x) = f(x)\exp(x^\lambda/\lambda)$, where λ is as in (2.19). We want to expand the integrand in (2.15) in the Taylor series at $t = 0$ and collect terms up to the next order than before. These are the only terms necessary to obtain a polynomial order factor. Further powers of x

would introduce higher orders which would not contribute to f as can be seen by a straightforward calculation. Writing $f(x(1-t)) \approx f(x) - xf'(x)t$ we can make an approximation valid for large x,

$$D_0^\beta y(x) \sim \frac{x^{2\mu-\beta}}{\Gamma(\alpha)} \left(f(x) \int_0^1 (1-t)^{2\mu-\alpha} t^{\alpha-1} e^{\frac{x^\lambda}{\lambda}(1-t)^\lambda} dt \right. \\ \left. -xf'(x) \int_0^1 (1-t)^{2\mu-\alpha} t^\alpha e^{\frac{x^\lambda}{\lambda}(1-t)^\lambda} dt \right). \tag{2.20}$$

In the second integral in (2.20) we approximate $(1-t)^{2\mu-\alpha} \approx 1$ and $(1-t)^\lambda \approx 1-\lambda t$, since only these terms are necessary to obtain the desired order of x. In the first integral in (2.20) we approximate further $(1-t)^{2\mu-\alpha} \approx 1-(2\mu-\alpha)t$ and $(1-t)^\lambda \approx 1-\lambda t+\lambda(\lambda-1)/2t^2$. As before, this is needed to have the next order of x than we had before in the exponential case. All these approximations are justified since the whole mass of the integral is concentrated near $t = 0$ and the rest terms are subdominant to the ones expressed by us. Now, the integral (2.20) has the form

$$D_0^\beta y(x) \sim \frac{x^{2\mu-\alpha}}{\Gamma(\alpha)} e^{\frac{x^\lambda}{\lambda}} \left(f(x) \int_0^1 (1-(2\mu-\alpha)t) \left(1+\frac{\lambda-1}{2} x^\lambda t^2 \right) t^{\alpha-1} e^{-x^\lambda t} dt \right. \\ \left. -xf'(x) \int_0^1 t^\alpha e^{-x^\lambda t} dt \right), \tag{2.21}$$

where we expanded $\exp\left((\lambda-1)/2x^\lambda t^2\right) \approx 1+(\lambda-1)/2x^\lambda t^2$. Using Lemma **2.1**, we can write

$$D_0^\beta y(x) \sim \\ \frac{x^{2\mu-\beta} e^{\frac{x^\lambda}{\lambda}}}{\Gamma(\alpha)} \left[f(x) \left(F_{\alpha-1}(x^\lambda) - (2\mu-\alpha) F_\alpha(x^\lambda) + \frac{\lambda-1}{2} x^\lambda F_{\alpha+1}(x^\lambda) \right) \right. \\ \left. -xf'(x) F_\alpha(x^\lambda) \right] \\ \sim x^{2\mu-\beta-\lambda(\alpha+1)+1} e^{\frac{x^\lambda}{\lambda}} \left[\left(x^{\lambda-1} - \alpha \left((2\mu-\alpha) - \frac{(\alpha+1)(\lambda-1)}{2} \right) x^{-1} \right) f(x) \right. \\ \left. -\alpha f'(x) \right], \tag{2.22}$$

where we used the formula $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$. Now, we expand the β fractional derivative at the left-hand side of (2.22) just as we did before in

(2.18). We thus have

$$D_0^{\beta} y(x) = \frac{d}{dx} D_0^{-(1-\beta)} \left(f(x) e^{\frac{x^\lambda}{\lambda}} \right)$$
$$\sim x^{(1-\beta)(1-\lambda)} e^{\frac{x^\lambda}{\lambda}} \left(f'(x) + (1-\beta)(1-\lambda) f(x) x^{-1} + f(x) x^{\lambda-1} \right). \tag{2.23}$$

Finally, we compare the two behaviors (2.22) and (2.23). First, by (2.19) note that $(1-\beta)(1-\lambda) = 2\mu - \beta - \lambda(\alpha+1) + 1$, thus the leading factors cancel. Then, by canceling the $x^{\lambda-1}$ term and by a little of algebra we obtain an asymptotic differential equation for f

$$\frac{f'}{f} = -\frac{1}{1+\alpha} \left[\alpha(2\mu - \alpha) + \frac{1}{2}\alpha(\alpha+1)(1-\lambda) + (1-\beta)(1-\lambda) \right] \frac{1}{x} \tag{2.24}$$

which has the solution

$$f(x) = C x^{-\frac{1}{1+\alpha}\left(\alpha(2\mu-\alpha) + \left(1 - \frac{2\mu}{\alpha+\beta}\right)\left(\frac{1}{2}\alpha(\alpha+1) + 1 - \beta\right)\right)}. \tag{2.25}$$

As we see, f is indeed an algebraic function. This finishes the proof. □

For a numerical verification of the obtained result we used the symbolic environment *MATHEMATICA*. The summation of series (2.8) is of course very slow, especially for large arguments. On the top of Fig. 2 we can see plots of $I_{0^+,1}^{\alpha,1}/y_{asym}$, where y_{asym} is as in (2.10). The bottom of the Fig. 2 presents a more general case, where $\alpha = 0.6$, $\beta = 0.9$ and $\mu = 0.7$ for two linearly independent functions $I_{0^\pm,\mu}^{\alpha,\beta}$. We can see that for every considered case we have convergence to the constant $C_{0^+,\mu}^{\alpha,\beta}$. This constant with respect to α for $\beta = \mu = 1$, obtained by numerical simulation, is presented on the Fig. 3. We note, that the numerical analysis confirms previously known values of $C_{0^+,1}^{0,1} = 1$ and $C_{0^+,1}^{1,1} = 1/\sqrt{2\pi} \approx 0.4$.

Also, we can see that the convergence of asymptotic approximation (2.10) of the exact solution to (2.1) is very fast. This approximation becomes very good just as soon as $x \geq 5$. The numerical values of $C_{0^+,1}^{\alpha,1}$ are presented in Table 1.

α	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$C^{\alpha,1}_{0^+,1}$	1.	0.90	0.82	0.74	0.67	0.62	0.56	0.52	0.47	0.43	0.40

TABLE 1. Numerical values for $C^{\alpha,1}_{0^+,1}$.

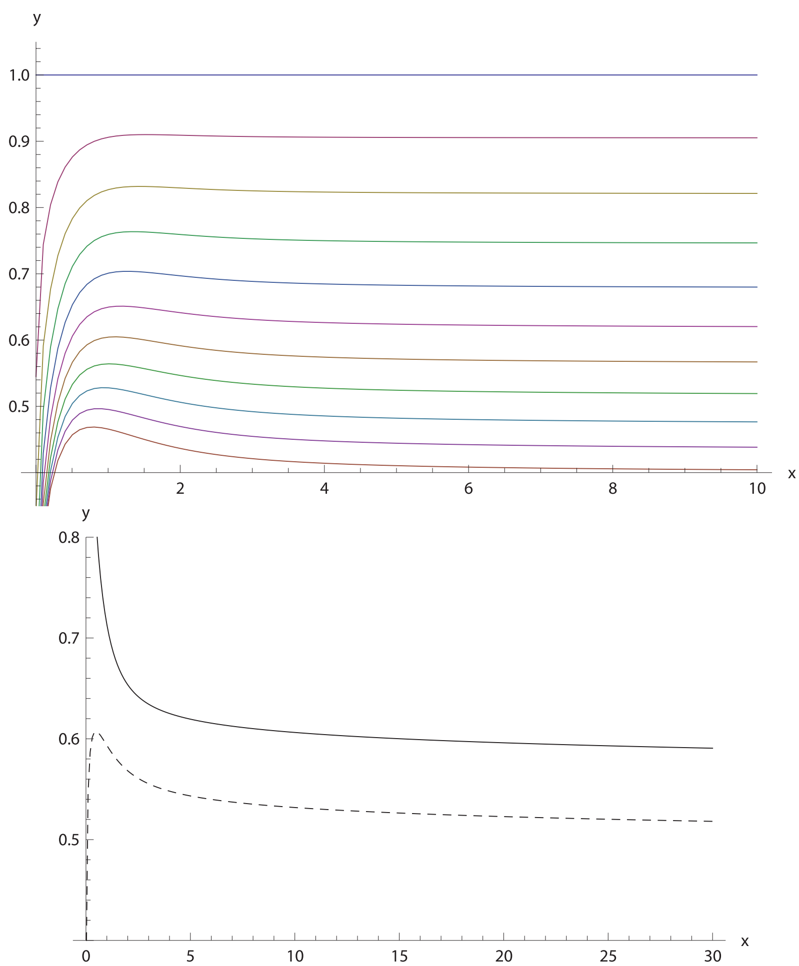

FIGURE 2. On the top: plots of $I_{0^+,1}{}^{\alpha}, 1/y_{asym}$ for $\alpha = 0$ (upper) to $\alpha = 1$ (lower) going by 0.1. On the bottom: plots of $I^{\alpha,\beta}_{0^+,\mu}/y_{asym}$ (dashed line) and $I^{\alpha,\beta}_{0^-,\mu}$ (solid line) for $\alpha = 0.6$, $\beta = 0.9$ and $\mu = 0.7$.

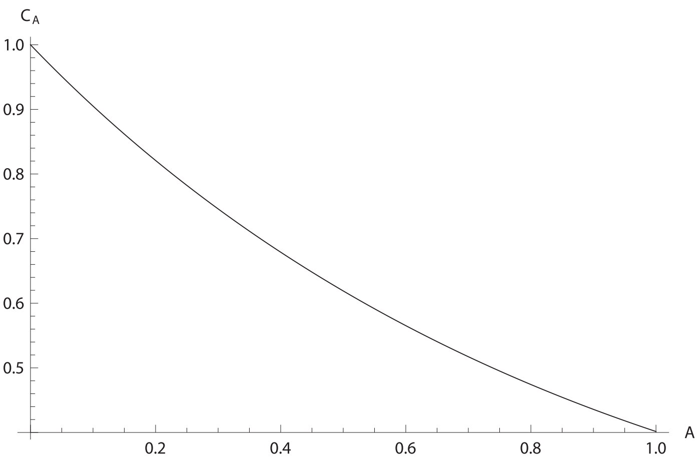

FIGURE 3. Dependence of $C^{\alpha,1}_{0^+,1}$.

3. Conclusion

In this paper we considered a generalization of the modified Bessel equation. This generalization is done by changing integer-order derivatives into the fractional versions. That is, we proposed a fractional modified Bessel equation of order ν. We found its solution as a power series and derived an asymptotic expansion when $x \to +\infty$. The asymptotic behavior was also verified by numerical analysis which showed that the approximation is good even for small values of the argument. Moreover, it can be clearly seen that the formula for asymptotic behavior reduces to the classical case when $\alpha, \beta, \mu \to 1^-$. Thus, the fractional modified Bessel function is indeed a generalization of the classical case.

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