VERSITA

Fractional Calculus & Applied Analysis
An International Journal for Theory and Applications
VOLUME 16, NUMBER 3 (2013)
(Print) ISSN 1311-0454
(Electronic) ISSN 1314-2224

DISCUSSION PAPER

CHAOS IN A FRACTIONAL ORDER LOGISTIC MAP

Joakim Munkhammar

Abstract

In this paper we investigate a fractional order logistic map and its discrete time dynamics. After a brief introduction to the discrete-time dynamical systems and fractional dynamics we show some basic properties of the fractional logistic map. We then move on to prove that the special case $\alpha = 1/2$ exhibits a period doubling route to chaos. A bifurcation diagram for the special case of $\alpha = 1/2$ is also included. Finally a discussion concerning the results and open problems is given.

MSC 2010: Primary 26A33; Secondary 37D45, 65P20

Key Words and Phrases: fractional calculus, dynamical systems, fractional dynamical systems, chaotic dynamics

1. Introduction

The concept of fractional order differentiation and integration is nearly as old as the Calculus itself, see e.g. [16, 20, 25, 12]. Yet in the 17th century Leibnitz made remarks on the fractional derivative of order 1/2. Despite many fundamental results and important definitions found more than 150 years ago by Euler and Riemann, among others, the field of applications of fractional calculus has only recently started to draw essential interest, [3, 10, 8, 9, 24, 25, 15, 17]. Fractional calculus applications to physics involve

pp. 511–519 , DOI: 10.2478/s13540-013-0033-8

viscoelastic systems and electro-magnetic waves [9], but also fractional order dynamical systems, fractional mechanics and fractional oscillators have been studied, [1, 17]. Fractional order differential equations have appeared, for example, in fractional quantum mechanics as well as in fractional generalizations of the Einstein field equations and cosmology, [6, 8]. Also fractional order generalizations of various other famous dynamical systems such as the fractional Lorenz-system, the Rössler systems [9] and Chen's system [10] have been investigated. Fractional generalizations of the logistic map have been investigated recently as well, see [5, 7, 14]. Neither of these generalizations have been shown to exhibit chaotic behavior. In [13], a particular type of fractional logistic map was briefly introduced and numerically investigated and a period doubling route to chaos was hinted for the special case of $\alpha = 1/2$. In this paper we show some basic results of this fractional logistic map and prove the period doubling route to chaos for the $\alpha = 1/2$ case.

1.1. Discrete-time dynamical systems

The discrete dynamics of unimodal maps have been extensively studied in [26]. In particular, in population dynamics maps like the Ricker-family [26]:

$$R_{\lambda,\beta}(x) = \lambda x e^{-\beta x},\ \lambda > 1\ \beta > 0, \tag{1.1}$$

or the Hassel-family [26]:

$$H_{\lambda,\beta}(x) = \frac{\lambda x}{(1+x)^{\beta}},\ \lambda > 1\ \beta > 0, \tag{1.2}$$

have been often considered.

However, perhaps the most famous model for population dynamics is the logistic map:

$$Q_{\lambda}(x) = \lambda x(1-x),\ \lambda > 0. \tag{1.3}$$

The logistic map is a unimodal map defined on the half-line $x \in [0, \infty[$, but all of its interesting dynamics takes place on the bounded interval $I = [0, 1]$ for $0 < \lambda \leq 4$. If we consider discrete time evolution by the mapping $Q_{\lambda} : I \to I$, letting the state to be x_n at time n, then we have $x_{n+1} = Q_{\lambda}(x_n)$ at time $n + 1$. In the prescribed interval the parameter λ is increased which results in a period-doubling route to chaos. For more detailed background on unimodal maps and chaos, see e.g. [11, 26].

1.2. Fractional calculus

There are many approaches to define the operators of fractional order differentiation and integration, but perhaps the most commonly used ones are the Riemann-Liouville-type operators and the Caputo-type operator, see [20, 18, 12], etc. The Caputo-type operator is generally favored in cases of integer order initial conditions, better physically interpreted. The fractional logistic map investigated in this paper is based on the Riemann-Liouville fractional operator, as done in [13], defined as follows [20].

Definition **1.1**. If $f(x) \in C([a,b])$ and $a < x < b$, $\alpha > 0$, then

$$I_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x-t)^{1-\alpha}}\, dt, \tag{1.4}$$

is called the Riemann-Liouville (R-L) fractional integral of order α. Reasonably, for $\alpha \in]0,1[$ we let

$$D_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x \frac{f(t)}{(x-t)^{\alpha}}\, dt, \tag{1.5}$$

which is called the Riemann-Liouville fractional derivative of order α.

A fractional derivative of order $\alpha = 1/2$ is called also a semi-derivative, and a fractional integral of the same order is called semi-integral, [16]. It is well known that the fractional R-L integral satisfies the important semi-group property: For any $f \in C([a,b])$,

$$I_{a+}^{\alpha} I_{a+}^{\beta} f(x) = I_{a+}^{\alpha+\beta} f(x), \quad \text{with} \quad \alpha > 0, \beta > 0. \tag{1.6}$$

For more information regarding theory of classical fractional calculus and its applications, see [20, 18], etc., in and for the recent history of fractional calculus research, see [25].

2. Fractional dynamics

2.1. Properties of the fractional logistic map

There are a number of approaches to fractional logistic maps, as in [5, 7, 13, 14]. The particular fractional logistic map studied in this paper is a spatial fractional generalization of the logistic map, developed in Munkhammar [13].

Definition **2.1**. For all $x \geq 0$, $\lambda > 0$ and $\alpha \in [0, \infty[$:

$$Q_{\lambda}^{\alpha}(x) \equiv \frac{\lambda}{\Gamma(\alpha+2)} \Big(1 - \frac{2x}{\alpha+2}\Big) x^{1+\alpha}, \tag{2.1}$$

is called the fractional logistic map of order α.

As a motivation for the fractional logistic map, we state the following statement.

Theorem **2.1**. *For $x \geq 0$, $\lambda > 0$ and $\alpha \in]0, \infty[$ the following holds true:* $Q^{\alpha}_{\lambda} = I^{\alpha}_{0+} Q_{\lambda}$.

P r o o f. The fractional integral of a monomial is ([20], etc.):

$$I^{\alpha}_{0+} x^{\beta} = \frac{\Gamma(\beta + 1)}{\Gamma(\alpha + \beta + 1)} x^{\alpha + \beta} \tag{2.2}$$

for at least $x \geq 0$, $\alpha, \beta > 0$, see also [12, p.7]. Thus we get:

$$I^{\alpha}_{0+} Q_{\lambda}(x) = \frac{\lambda}{\Gamma(\alpha + 2)} \Big(1 - \frac{2x}{\alpha + 2}\Big) x^{1+\alpha} = Q^{\alpha}_{\lambda}(x). \tag{2.3}$$

□

The fractional logistic map (2.1) approximates the fractional logistic map for $\alpha \sim 0$. We also give a corollary that for $\alpha \to 0$ the fractional logistic equation equals the logistic equation. Namely,

Corollary **2.1**. *If $\alpha \to 0$, then $Q^{\alpha}_{\lambda} \to Q_{\lambda}$ for at least $x \geq 0$, $\lambda > 0$.*

Corollary **2.1** is self-evident from the definition of the fractional logistic map and Theorem **2.1**. The special case of $\alpha = 1/2$-order fractional logistic map will be called the semi-logistic map, in analogy with semi-fractional integral of order $\alpha = 1/2$ ([16]).

Definition **2.2**. For $x \geq 0$ and $\lambda > 0$ the $\alpha = 1/2$-order fractional logistic map, or the *semi-logistic map*, appears like:

$$Q^{1/2}_{\lambda}(x) = \frac{4\lambda}{3\sqrt{\pi}} \Big(1 - \frac{4x}{5}\Big) x^{3/2}. \tag{2.4}$$

The dynamics of the semi-logistic map was briefly studied numerically in Munkhammar [13], and we give here briefly some numerical results in Section **2.3**.

Now we establish some properties of the fractional logistic map.

Theorem **2.2**. *For $\alpha \in [0, 1]$:*

$$Q^{\alpha}_{\lambda}(0) = Q^{\alpha}_{\lambda}(1 + \frac{1}{2}\alpha) = 0, \tag{2.5}$$

holds for all $\lambda > 0$.

P r o o f. $Q^{\alpha}_{\lambda}(0) = 0$ follows directly from Definition **2.1**. We also find that $Q^{\alpha}_{\lambda}(1 + \frac{1}{2}\alpha) = 0$ by taking $x = 1 + \frac{1}{2}\alpha$:

$$Q_\lambda^\alpha(1+\frac{1}{2}\alpha) = \frac{\lambda}{\Gamma(\alpha+2)}\Big(1-\frac{2+\alpha}{\alpha+2}\Big)\Big(1+\frac{1}{2}\alpha\Big)^{1+\alpha} = 0. \tag{2.6}$$

Note that $\lambda > 0$ is a part of the definition of the fractional logistic map (Definition **2.1**). $\square$

The dynamics of a discrete dynamical system is defined through its fixed points and in order to find fixed points of the fractional logistic map the following identity needs to be resolved:

$$Q_\lambda^\alpha(x) = x. \tag{2.7}$$

This identity leads to the following equation:

$$x^{1+\alpha} - \frac{\alpha+2}{2}x^\alpha + \frac{\Gamma(\alpha+3)}{2\lambda} = 0, \tag{2.8}$$

that has non-fractional solutions for $\alpha \in \mathbb{N}$ corresponding to simple integer integrations of the logistic map.

2.2. Chaos

In this section we show that the $\alpha = 1/2$ fractional logistic map exhibits chaos. In order to prove this we first need to give the definition of the Schwarzian derivative.

Definition **2.3**. For a function $f \in C^3$ we define

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left[\frac{f''(x)}{f'(x)}\right]^2 \tag{2.9}$$

as the Schwarzian derivative of f.

We also need a theorem regarding chaos and the Schwarzian derivative, [11, 21].

Theorem **2.3**. *Any map $f(x) \in C^3$ defined for $x \geq 0$ with a negative Schwarzian derivative Sf on $x > 0$ exhibits a period-doubling route to chaos.*

The proof of this theorem can be found in Singer [22]. For more details on unimodal maps and the Schwarzian derivative, see [11, 21]. The following hold for the fractional logistic map:

$$\frac{d}{dx}Q_\lambda^\alpha = \frac{\lambda(\alpha - 2x + 1)}{\Gamma(\alpha + 2)}x^\alpha, \tag{2.10}$$

$$\frac{d^2}{dx^2}Q_\lambda^\alpha = \frac{\lambda(\alpha - 2x)}{\Gamma(\alpha + 1)}x^{\alpha-1}, \tag{2.11}$$

$$\frac{d^2}{dx^2}Q_\lambda^\alpha = \frac{\lambda(\alpha - 2x - 1)}{\Gamma(\alpha)}x^{\alpha-2}. \tag{2.12}$$

This brings the following Schwarzian derivative (**2.3**) of the fractional logistic map $Q_\lambda^\alpha(x)$ (**2.2**):

$$SQ_\lambda^\alpha(x) = \alpha\frac{(\alpha - 2x - 1)}{(\alpha - 2x)}x^{-2+\alpha} - \frac{3}{2}\left[\frac{(\alpha - 2x)}{(\alpha - 2x + 1)}(\alpha + 1)x^{-1+\alpha}\right]^2. \tag{2.13}$$

The general solution of (2.13) is non-trivial. But in Munkhammar [13] the chaos was numerically shown to emerge from the special case of the semi-logistic map (**2.2**). Namely,

THEOREM **2.4**. *The semi-logistic map* (**2.2**)*:*

$$Q_\lambda^{1/2}(x) = \frac{4\lambda}{3\sqrt{\pi}}\Big(1 - \frac{4x}{5}\Big)x^{3/2}, \tag{2.14}$$

exhibits a period-doubling route to chaos with bifurcation parameter λ*.*

P r o o f. Let $\alpha = 1/2$ in (2.13) we get:

$$DQ_\lambda^{1/2}(x) = -\frac{23}{8}x^{-2}, \tag{2.15}$$

that is,

$$\forall x > 0 : DQ_\lambda^{1/2}(x) < 0. \tag{2.16}$$

By Theorem **2.3** this shows that the semi-logistic map experiences a chaotic period doubling route to chaos. □

Theorem **2.4** shows that at least for a special case the fractional logistic map experiences a period doubling route to chaos. A brief numerical study of the semi-logistic map is presented in next Section **2.3**.

2.3. Numerical simulations

The semi-logistic map was investigated numerically and a bifurcation diagram as λ is increasing from 4.5 to 6.0 is shown in Figure 1. The numerical simulation shows a period-doubling route to chaos, proven in Theorem **2.4**.

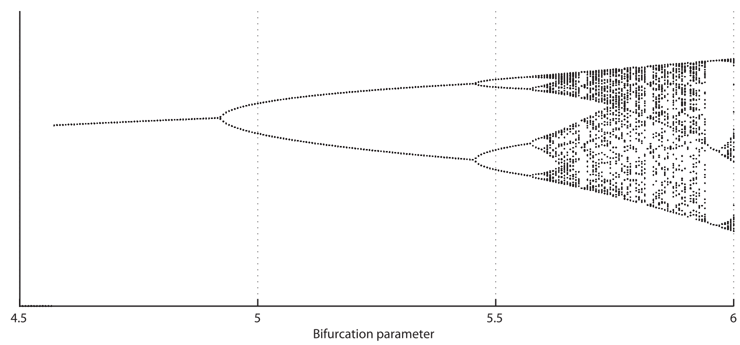

FIGURE 1. A bifurcation diagram for the fractional logistic map for $\alpha = 1/2$ as λ is increased from 4.5 to 6.0.

The appearance of this bifurcation diagram in Figure 1 is similar to that observed for the logistic map, by Thunberg [26].

3. Discussion

In this paper we discuss the discrete time evolution in a particular form of fractional logistic map. We mention some basic properties of the fractional logistic map and show that for the special case $\alpha = 1/2$ it experiences a period doubling route to chaos. The proof for the general case of a period doubling route to chaos for the fractional logistic map is left as an open problem. In addition to this, the fractional logistic map is numerically simulated and a bifurcation diagram of the semi-logistic map $\alpha = 1/2$ is given. The generalization of fractional logistic maps, including this one, largely remains open for investigation.

Further, let us mention that investigating the general connection between this map and various applications in fractional calculus could be of interest, especially since most fractional dynamical systems involve time and not space as the fractional parameter. The chaos theory for discrete maps is well understood, but how it is related to fractional calculus phenomena is perhaps less clarified and need a further investigation.

Acknowledgment

I would like to thank the FCAA reviewers for some valuable comments.

References

[1] B.N.N. Achar, J.W. Hanneken, T. Enck, T. Clarke, Dynamics of the fractional oscilator. *Physica A* **297** (2001), 361–367.

[2] G. Calcagni, Geometry and field theory in multi-fractional spacetime. *J. of High Energy Physics* **2012** (2012), Paper # 65 ; DOI: 10.1007/JHEP01(2012)065; at http://link.springer.com/article/10.1007/JHEP01(2012)065

[3] R.C. Cascaval, E.C. Eckstein, C.L. Frota, J.A. Goldstein, Fractional telegraph equations. *J. Math. Anal. Appl.* **276** (2002), 145–159.

[4] R.L. Devaney, *An Introduction to Chaotic Dynamical Systems.* 2nd Ed., Westview Press, 2003.

[5] S. Das, P.K. Gupta, K. Vishal, Approximate approach to the Das Model of Fractional Logistic Population Growth. *Applications and Appl. Math.* **5**, No 10 (2010), 1702–1708.

[6] A. El-Nabulsi, Fractional unstable Euclidean universe. *Electr. J. Theor. Phys.* **2**, No 8 (2005), 1–11.

[7] A.M.A. El-Sayad, A.E.M. El-Mesiry, H.A.A. El-Saka, On the fractional order logistic equation. *Appl. Math. Let.* **20** (2007), 817–823.

[8] N. Laskin, Fractional Schrödinger equation. *Phys. Rev. E* **66** (2002), 056108.

[9] C. Li, G. Chen, Chaos and hyper chaos in the fractional order Rössler equations. *Physica A* **341** (2004), 55–61.

[10] C. Li, G. Peng, Chaos in Chen's system with a fractional order. *Chaos Sol. Fractals* **22** (2004), 443–450.

[11] W. de Melo, Bifurcation of unimodal maps. *Qualitative Theory of Dynamical Systems* **4**, No 2 (2004), 413–424.

[12] J.D. Munkhammar, Fractional calculus and the Taylor-Riemann series, *RHIT Undergrad. Math. J.* **6**, No 1 (2005), at http://www.rose-hulman.edu/mathjournal/archives/2005/vol6-n1/.

[13] J.D. Munkhammar, Numerical chaos in fractional order logistic equation. *arXiv:1011.2389v1* [math.GM] (2010).

[14] A. Nagai, Fractional logistic map. *arXiv:nlin/0206018v1* [nlin.SI] (2002).

[15] R.R. Nigmatullin, D. Baleanu, The derivation of the generalized functional equations describing self-similar processes. *Fract. Calc. Appl. Anal.* **15**, No 4 (2012), 718–740; DOI: 10.2478/s13540-012-0049-5; at http://link.springer.com/journal/13540/

[16] K. Oldham, J. Spanier, *The Fractional Calculus.* Acad. Press, New York-London (1974).

[17] L. Oparnica, Generalized fractional calculus with applications in mechanics. *Matematicki Vesnik* **53** (2001), 151–158.

[18] I. Podlubny, *Fractional Differential Equations*. Academic Press, San Diego (1999).

[19] O.E. Rössler, An equation for continuous chaos. *Phys. Lett. A* **57** (1976), 397–399.

[20] S.G. Samko, A.A. Kilbas, O.I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*. Gordon and Breach, Amsterdam (1993).

[21] G.H. Schuster, *Deterministic Chaos: An Introduction*. Wiley-VCH (2005).

[22] D. Singer, Orbits and bifurcation of maps of the interval. *SIAM J. Appl. Math.* **35** (1978), 260–267.

[23] S.H. Strogatz, *Non-linear Dynamics and Chaos*. Perseus Books Publ. (1994).

[24] J.A. Tenreiro Machado, And I say to myself: "What a fractional world!". *Fract. Calc. and Appl. Anal.* **14**, No 4 (2011), 635–654; DOI: 10.2478/s13540-011-0037-1; at http://link.springer.com/journal/13540/.

[25] J. A. Tenreiro Machado, V. Kiryakova, F. Mainardi, Recent history of fractional calculus. *Commun. Nonlinear Sci. Numer. Simulat.* **16**, No 3 (2011), 1140–1153; doi:10.1016/j.cnsns.2010.05.027.

[26] H. Thunberg, Periodicity versus chaos in one-dimensional dynamics. *SIAM Review* **43**, No 1 (2001), 3–30.

Department of Engineering Sciences
Uppsala University – SWEDEN

Corr. address: (First) Studentstaden 23:230
752 33 Uppsala, SWEDEN
Received: May 4, 2012
e-mail: joakim.munkhammar@gmail.com
Revised: 16 November 2012

Please cite to this paper as published in:

Fract. Calc. Appl. Anal., Vol. **16**, No 3 (2013), pp. 511–519;
DOI:10.2478/s13540-013-0033-8